Robust Nonlinear 3D Control of an Inverted Pendulum Balanced on a Quadrotor: Supplementary Material

Weiming Yang, Gan Yu, Joel Reis, and Carlos Silvestre

Abstract

This is a complementary document to the paper presented in [1]. Here we fully disclose all the detailed derivations and equations that are essential to implement the controller reported therein. The main paper addresses the nonlinear control problem of balancing an inverted pendulum on a flying underactuated unmanned aerial vehicle. To simultaneously tackle the system's slow and fast transients, a novel error transformation approach is considered where no linearization method whatsoever is used, making this controller suitable beyond the scope of trim maneuvers. Furthermore, the controller features an adaptive bounded law that is used to compensate for unknown external disturbances. Our Lyapunov-based control design is rooted in an integral backstepping process, wherein the origin of the closed-loop total system error is shown to be almost globally asymptotically stable.

I. Computation of auxiliary variables

A flowchart representing the overall control structure is depicted in Fig. 1. Equation numbers correspond to numbering presented in [1].

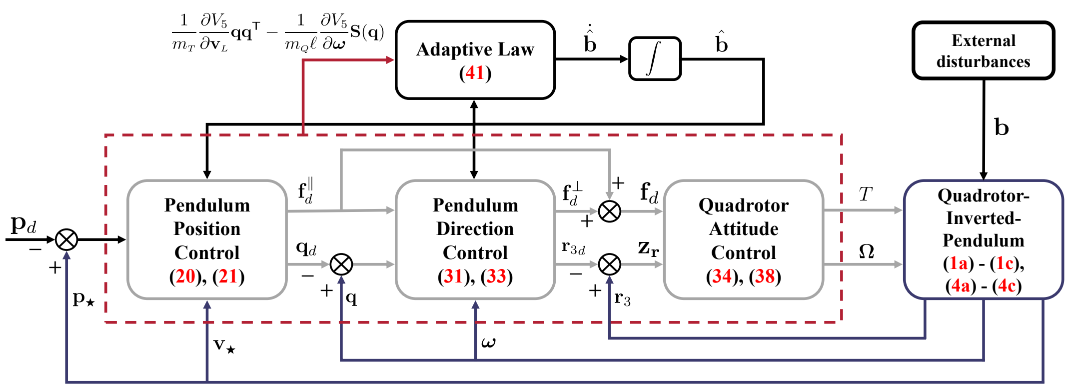

Figure 1. Block diagram of composite feedback controller and adaptive law.

We start by noting that (4a) may be rewritten as

$$\dot{\mathbf{v}}_\star = \hat{\dot{\mathbf{v}}}_\star|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{m_T}(\mathbf{q}\mathbf{q}^\mathsf{T}\mathbf{f} - \mathbf{f}_d^{\parallel}) + \frac{1}{m_T}\mathbf{q}\mathbf{q}^\mathsf{T}\tilde{\mathbf{b}}, \tag{SM 1}$$

where

$$\hat{\dot{\mathbf{v}}}_\star|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = g\mathbf{e}_3 - \frac{1}{m_T}\mathbf{q}\mathbf{q}^\mathsf{T}\boldsymbol{\xi}. \tag{SM 2}$$

All the time derivatives that explicitly feature $\dot{\mathbf{v}}_\star$ can be divided into three terms: one related to $\mathbf{f}_d^{\parallel}$ and $\hat{\mathbf{b}}$, one related to the error $\mathbf{q}\mathbf{q}^\mathsf{T}\mathbf{f} - \mathbf{f}_d^{\parallel}$, and another related to $\tilde{\mathbf{b}}$.

Similarly, we can rearrange (8) as

$$\dot{\mathbf{z}}_v = \hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{m_T}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_\star}(\mathbf{q}\mathbf{q}^\mathsf{T}\mathbf{f} - \mathbf{f}_d^{\parallel}) + \frac{1}{m_T}\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_\star}\mathbf{q}\mathbf{q}^\mathsf{T}\tilde{\mathbf{b}}, \tag{SM 3}$$

where

$$\hat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} = \hat{\dot{\mathbf{v}}}_\star|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \ddot{\mathbf{p}}_d \tag{SM 4}$$

and

$$\frac{\partial \mathbf{z}_v}{\partial \mathbf{v}_\star} = \mathbf{I}. \tag{SM 5}$$

The authors are with the Department of Electrical and Computer Engineering, Faculty of Science and Technology, University of Macau, Macau, China (e-mail:{weiming.yang,ganyu,joelreis,csilvestre}@um.edu.mo)

C. Silvestre is on leave from Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal.

We start by presenting the expression for $\widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^\parallel}$, first shown in (29). We have

$$\begin{aligned}\widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^\parallel} &= \mathbf{S}(\dot{\mathbf{q}})\boldsymbol{\omega} - \frac{k_q}{h_q}\left(\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q})\mathbf{q}_d + \mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})\mathbf{q}_d + \mathbf{S}^2(\mathbf{q})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^\parallel}\right) + \mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q}_d)\frac{1}{\|\boldsymbol{\xi}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel} \\ &\quad - \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^\parallel} + \mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)\left(\frac{\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}}{\|\boldsymbol{\xi}\|} - \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}\frac{\boldsymbol{\xi}^\mathsf{T}}{\|\boldsymbol{\xi}\|^3}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}\right) \\ &\quad + \frac{\epsilon}{h_q m_T}\|\boldsymbol{\xi}\|\left((\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q}) + \mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}}))\,\mathbf{e} + \frac{1}{\|\boldsymbol{\xi}\|^2}\mathbf{S}^2(\mathbf{q})\mathbf{e}\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel} + \mathbf{S}^2(\mathbf{q})\widehat{\dot{\mathbf{e}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}\right), \end{aligned} \tag{SM 6}$$

where

$$\widehat{\dot{\mathbf{e}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel} = \epsilon^2\mathbf{K}_\mathbf{p}\mathbf{z}_v + \epsilon\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^\parallel}, \tag{SM 7}$$

and

$$\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel} = m_T\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel} - \ddot{\mathbf{p}}_d\right), \tag{SM 8}$$

with

$$\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel} = \epsilon^2(\mathbf{I}+\mathbf{K}_\mathbf{v}\mathbf{K}_\mathbf{p})\mathbf{z}_v + \epsilon(\mathbf{K}_\mathbf{p}+\mathbf{K}_\mathbf{v})\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^\parallel}, \tag{SM 9}$$

and where

$$\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^\parallel} = \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}, \tag{SM 10}$$

$$\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^\parallel}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^\parallel} = -\frac{1}{m_T}\left[(\dot{\mathbf{q}}\mathbf{q}^\mathsf{T} + \mathbf{q}\dot{\mathbf{q}}^\mathsf{T})\boldsymbol{\xi} + \mathbf{q}\mathbf{q}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}\right] - \dddot{\mathbf{p}}_d, \tag{SM 11}$$

$$\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^\parallel} = \epsilon^2(\mathbf{I}+\mathbf{K}_\mathbf{v}\mathbf{K}_\mathbf{p})\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^\parallel} + \epsilon(\mathbf{K}_\mathbf{p}+\mathbf{K}_\mathbf{v})\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^\parallel}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}, \tag{SM 12}$$

and, finally,

$$\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^\parallel} = m_T\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^\parallel} - \ddddot{\mathbf{p}}_d\right). \tag{SM 13}$$

The expression for $\partial\mathbf{z}_\omega/\partial\mathbf{v}_\star$, also first shown in (29), is given by

$$\begin{aligned}\frac{\partial\mathbf{z}_\omega}{\partial\mathbf{v}_\star} &= -\frac{k_q}{h_q}\mathbf{S}^2(\mathbf{q})\frac{\partial\mathbf{q}_d}{\partial\mathbf{v}_\star} - \mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}}{\|\boldsymbol{\xi}\|})\frac{\partial\mathbf{q}_d}{\partial\mathbf{v}_\star} - \frac{\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)}{\|\boldsymbol{\xi}\|^2}\left(\|\boldsymbol{\xi}\|\frac{\partial\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}}{\partial\mathbf{v}_\star} + \frac{\dot{\boldsymbol{\xi}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}\boldsymbol{\xi}^\mathsf{T}}{\|\boldsymbol{\xi}\|}\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{v}_\star}\right) \\ &\quad + \frac{\epsilon}{h_q m_T\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q})\mathbf{e}\boldsymbol{\xi}^\mathsf{T}\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{v}_\star} + \frac{\epsilon\|\boldsymbol{\xi}\|}{h_q m_T}\mathbf{S}^2(\mathbf{q})\frac{\partial\mathbf{e}}{\partial\mathbf{v}_\star}, \end{aligned} \tag{SM 14}$$

where

$$\frac{\partial\mathbf{e}}{\partial\mathbf{v}_\star} = \epsilon\frac{\partial\mathbf{z}_v}{\partial\mathbf{v}_\star}, \tag{SM 15}$$

and

$$\frac{\partial\mathbf{q}_d}{\partial\mathbf{v}_\star} = \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q}_d)\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{v}_\star}, \tag{SM 16}$$

with

$$\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{v}_\star} = m_T\frac{\partial\boldsymbol{\zeta}}{\partial\mathbf{v}_\star}, \tag{SM 17}$$

and

$$\frac{\partial\boldsymbol{\zeta}}{\partial\mathbf{v}_\star} = \epsilon(\mathbf{K}_\mathbf{p}+\mathbf{K}_\mathbf{v})\frac{\partial\mathbf{z}_v}{\partial\mathbf{v}_\star}, \tag{SM 18}$$

and where

$$\frac{\partial\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^\parallel}}{\partial\mathbf{v}_\star} = -\frac{1}{m_T}\mathbf{q}\mathbf{q}^\mathsf{T}\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{v}_\star}, \tag{SM 19}$$

$$\frac{\partial\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^\parallel}}{\partial\mathbf{v}_\star} = \epsilon^2(\mathbf{I}+\mathbf{K}_\mathbf{v}\mathbf{K}_\mathbf{p})\frac{\partial\mathbf{z}_v}{\partial\mathbf{v}_\star} + \epsilon(\mathbf{K}_\mathbf{p}+\mathbf{K}_\mathbf{v})\frac{\partial\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^\parallel}}{\partial\mathbf{v}_\star}, \tag{SM 20}$$

$$\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star} = m_T \frac{\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star}, \tag{SM 21}$$

and, finally,

$$\frac{\partial \widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star} = \frac{1}{\|\boldsymbol{\xi}\|}\left(\mathbf{q}_d \widehat{\dot{\boldsymbol{\xi}}}\Big|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}^{\top} - \widehat{\dot{\boldsymbol{\xi}}}\Big|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \mathbf{q}_d^{\top}\right)\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_\star} - \frac{\mathbf{S}^2(\mathbf{q}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^{\top}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_\star}}{\|\boldsymbol{\xi}\|^3} + \frac{\mathbf{S}^2(\mathbf{q}_d)\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star}}{\|\boldsymbol{\xi}\|}. \tag{SM 22}$$

In the following we present some useful partial derivatives of the Lyapunov function candidates with respect to $\mathbf{v}_\star$ and $\boldsymbol{\omega}$. They are given by

$$\frac{\partial V_3}{\partial \mathbf{v}_\star} = \mathbf{e}^{\top}\frac{\partial \mathbf{e}}{\partial \mathbf{v}_\star} - h_q \mathbf{z}_q^{\top}\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_\star}, \tag{SM 23}$$

$$\frac{\partial V_4}{\partial \mathbf{v}_\star} = \frac{\partial V_3}{\partial \mathbf{v}_\star} + h_\omega \mathbf{z}_\omega^{\top}\frac{\partial \mathbf{z}_\omega}{\partial \mathbf{v}_\star}, \tag{SM 24}$$

$$\frac{\partial V_5}{\partial \mathbf{v}_\star} = \frac{\partial V_4}{\partial \mathbf{v}_\star} - h_r \mathbf{r}_{3d}^{\top}\mathbf{R}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\top}\mathbf{S}(\mathbf{r}_{3d})\frac{\partial \mathbf{r}_{3d}}{\partial \mathbf{v}_\star}, \tag{SM 25}$$

and

$$\frac{\partial V_5}{\partial \boldsymbol{\omega}} = h_\omega \mathbf{z}_\omega^{\top}\frac{\partial \mathbf{z}_\omega}{\partial \boldsymbol{\omega}} - h_r \mathbf{r}_{3d}^{\top}\mathbf{R}\mathbf{S}(\mathbf{e}_3)\mathbf{R}^{\top}\mathbf{S}(\mathbf{r}_{3d})\frac{\partial \mathbf{r}_{3d}}{\partial \boldsymbol{\omega}}, \tag{SM 26}$$

where

$$\frac{\partial \mathbf{z}_\omega}{\partial \boldsymbol{\omega}} = \mathbf{S}(\mathbf{q}), \tag{SM 27}$$

$$\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}} = -\mathbf{S}(\mathbf{q}), \tag{SM 28}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}} = -\frac{1}{m_T}\left(\mathbf{q}^{\top}\boldsymbol{\xi} + \mathbf{q}\boldsymbol{\xi}^{\top}\right)\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}, \tag{SM 29}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_\star} = -\frac{(\dot{\mathbf{q}}\mathbf{q}^{\top} + \mathbf{q}\dot{\mathbf{q}}^{\top})\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_\star} + \mathbf{q}\mathbf{q}^{\top}\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star}}{m_T}, \tag{SM 30}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}} = \epsilon(\mathbf{K}_\mathbf{p} + \mathbf{K}_\mathbf{v})\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}}, \tag{SM 31}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_\star} = \epsilon^2(\mathbf{I} + \mathbf{K}_\mathbf{v}\mathbf{K}_\mathbf{p})\frac{\partial \widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star} + \epsilon(\mathbf{K}_\mathbf{p} + \mathbf{K}_\mathbf{v})\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_\star}, \tag{SM 32}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}} = m_T\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}}, \tag{SM 33}$$

$$\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_\star} = m_T\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_\star}, \tag{SM 34}$$

$$\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star} = -\frac{k_q}{h_q}\left([\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q})+\mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})]\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_\star}+\mathbf{S}^2(\mathbf{q})\frac{\partial \widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star}\right)+\frac{\epsilon}{h_q m_T}\frac{1}{\|\boldsymbol{\xi}\|}\Bigg[\mathbf{S}^2(\mathbf{q})(\boldsymbol{\xi}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}})\frac{\partial \mathbf{e}}{\partial \mathbf{v}_\star}$$
$$+\frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q})\mathbf{e}\left(\|\boldsymbol{\xi}\|\widehat{\dot{\boldsymbol{\xi}}}|^\top_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_\star}-\frac{1}{\|\boldsymbol{\xi}\|}\boldsymbol{\xi}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_\star}\right)+\mathbf{S}^2(\mathbf{q})\mathbf{e}\boldsymbol{\xi}^\top\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star}+\mathbf{S}^2(\mathbf{q})\widehat{\dot{\mathbf{e}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_\star}$$
$$+[\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q})+\mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})]\left(\mathbf{e}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_\star}+\|\boldsymbol{\xi}\|^2\frac{\partial \mathbf{e}}{\partial \mathbf{v}_\star}\right)+\epsilon\|\boldsymbol{\xi}\|^2\mathbf{S}^2(\mathbf{q})\frac{\partial \widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star}\Bigg]-\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\boldsymbol{\xi}\|})\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_\star}$$
$$+\frac{\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q}_d)+\mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}})}{\|\boldsymbol{\xi}\|}\left(\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star}-\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_\star}}{\|\boldsymbol{\xi}\|^2}\right)-\mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\boldsymbol{\xi}\|})\frac{\partial \widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star}$$
$$-\mathbf{S}(\mathbf{q})\mathbf{S}(\frac{\|\boldsymbol{\xi}\|^2\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}-\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\boldsymbol{\xi}\|^3})\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_\star}+\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)\Bigg(-\frac{\boldsymbol{\xi}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star}-\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\top\frac{\partial \widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star}}{\|\boldsymbol{\xi}\|^3}$$
$$\frac{\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \mathbf{v}_\star}\|\boldsymbol{\xi}\|-\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_\star}}{\|\boldsymbol{\xi}\|^2}+\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\left(\|\boldsymbol{\xi}\|^3\widehat{\dot{\boldsymbol{\xi}}}|^\top_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_\star}-3\|\boldsymbol{\xi}\|\boldsymbol{\xi}^\top\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_\star}\right)}{\|\boldsymbol{\xi}\|^6}\Bigg), \tag{SM 35}$$

and, at last,

$$\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \boldsymbol{\omega}} = -\mathbf{S}(\boldsymbol{\omega})\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}+\mathbf{S}(\dot{\mathbf{q}})-\frac{k_q}{h_q}(\mathbf{q}\mathbf{q}_d^\top-\mathbf{q}_d\mathbf{q}^\top)\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}+\frac{k_q}{h_q}\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}+\frac{\|\boldsymbol{\xi}\|\epsilon}{h_q m_T}\left((\mathbf{q}\mathbf{e}^\top-\mathbf{e}\mathbf{q}^\top)\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}\right.$$
$$\left.-\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{e})\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}\right)+\frac{\mathbf{q}_d\widehat{\dot{\boldsymbol{\xi}}}|^\top_{\mathbf{f}=\mathbf{f}_d^{\parallel}}-\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\mathbf{q}_d^\top}{\|\boldsymbol{\xi}\|}\frac{\partial \dot{\mathbf{q}}}{\partial \boldsymbol{\omega}}+\frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)\frac{\partial \frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}}{\partial \boldsymbol{\omega}}, \tag{SM 36}$$

Other important partial derivatives used in the derivations are the following:

$$\frac{\partial \mathbf{f}_d}{\partial \mathbf{v}_\star} = -\mathbf{q}\mathbf{q}^\top\frac{\partial \boldsymbol{\xi}}{\partial \mathbf{v}_\star}+m_Q\ell\mathbf{S}^2(\mathbf{q})\left(\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \mathbf{v}_\star}+\frac{h_q}{h_\omega}\frac{\partial \mathbf{q}_d}{\partial \mathbf{v}_\star}\right), \tag{SM 37}$$

$$\frac{\partial \mathbf{f}_d}{\partial \boldsymbol{\omega}} = 2m_Q\ell\mathbf{q}\boldsymbol{\omega}^\top\mathbf{I}+m_Q\ell\mathbf{S}^2(\mathbf{q})\left(\frac{\partial \widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\partial \boldsymbol{\omega}}+\frac{k_\omega}{h_\omega}\mathbf{S}(\mathbf{q})\right), \tag{SM 38}$$

$$\frac{\partial \mathbf{r}_{3d}}{\partial \boldsymbol{\omega}} = -\frac{\|\mathbf{f}_d\|\mathbf{I}-\frac{1}{\|\mathbf{f}_d\|}\mathbf{f}_d\mathbf{f}_d^\top}{\|\mathbf{f}_d\|^2}\frac{\partial \mathbf{f}_d}{\partial \boldsymbol{\omega}}, \tag{SM 39}$$

and

$$\frac{\partial \mathbf{r}_{3d}}{\partial \mathbf{v}_\star} = -\frac{\|\mathbf{f}_d\|\mathbf{I}-\frac{1}{\|\mathbf{f}_d\|}\mathbf{f}_d\mathbf{f}_d^\top}{\|\mathbf{f}_d\|^2}\frac{\partial \mathbf{f}_d}{\partial \mathbf{v}_\star}. \tag{SM 40}$$

We also need to compute the pseudo-estimate of $\dot{\mathbf{r}}_{3d}$, which is given by

$$\widehat{\dot{\mathbf{r}}_{3d}} = -\frac{1}{\|\mathbf{f}_d\|^2}\left(\|\mathbf{f}_d\|\widehat{\dot{\mathbf{f}}}_d-\frac{1}{\|\mathbf{f}_d\|}\mathbf{f}_d\mathbf{f}_d^\top\widehat{\dot{\mathbf{f}}}_d\right), \tag{SM 41}$$

where

$$\widehat{\dot{\mathbf{f}}}_d = \widehat{\dot{\mathbf{f}}_d^{\parallel}}+\widehat{\dot{\mathbf{f}}_d^{\perp}}, \tag{SM 42}$$

with

$$\widehat{\dot{\mathbf{f}}_d^{\parallel}} = 2m_Q\ell\boldsymbol{\omega}^\top\widehat{\dot{\boldsymbol{\omega}}}\mathbf{q}+m_Q\ell\|\boldsymbol{\omega}\|^2\dot{\mathbf{q}}-\dot{\mathbf{q}}\mathbf{q}^\top\boldsymbol{\xi}-\mathbf{q}\dot{\mathbf{q}}^\top\boldsymbol{\xi}-\mathbf{q}\mathbf{q}^\top\widehat{\dot{\boldsymbol{\xi}}}-\dot{\mathbf{q}}\mathbf{q}^\top\widehat{\mathbf{b}}-\mathbf{q}\dot{\mathbf{q}}^\top\widehat{\mathbf{b}}-\mathbf{q}\mathbf{q}^\top\widehat{\dot{\mathbf{b}}} \tag{SM 43}$$

and

$$\widehat{\dot{\mathbf{f}}_d^{\perp}} = m_Q\ell\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q})\left(\widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{h_q}{h_\omega}\mathbf{q}_d + \frac{k_\omega}{h_\omega}\mathbf{z}_\omega + \frac{1}{m_Q\ell}\widehat{\mathbf{b}}\right) + m_Q\ell\mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})\left(\widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{h_q}{h_\omega}\mathbf{q}_d + \frac{k_\omega}{h_\omega}\mathbf{z}_\omega + \frac{1}{m_Q\ell}\widehat{\mathbf{b}}\right)$$
$$+ m_Q\ell\mathbf{S}^2(\mathbf{q})\left(\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} + \frac{h_q}{h_\omega}\widehat{\dot{\mathbf{q}}}_d + \frac{k_\omega}{h_\omega}\widehat{\dot{\mathbf{z}}}_\omega + \frac{1}{m_Q\ell}\widehat{\dot{\mathbf{b}}}\right),$$

where,

$$\widehat{\mathbf{v}}_\star = g\mathbf{e}_3 - \frac{m_Q\ell}{m_T}\|\boldsymbol{\omega}\|^2\mathbf{q} + \frac{1}{m_T}\mathbf{q}\mathbf{q}^\mathsf{T}(\mathbf{f}+\widehat{\mathbf{b}}), \tag{SM 44}$$

$$\widehat{\dot{\mathbf{z}}}_v = \widehat{\mathbf{v}}_\star - \ddot{\mathbf{p}}_d, \tag{SM 45}$$

$$\widehat{\dot{\mathbf{e}}} = \epsilon^2\mathbf{K}_\mathbf{p}\mathbf{z}_v + \epsilon\widehat{\dot{\mathbf{z}}}_v, \tag{SM 46}$$

$$\widehat{\dot{\boldsymbol{\zeta}}} = \epsilon^2(\mathbf{I}+\mathbf{K}_\mathbf{v}\mathbf{K}_\mathbf{p})\mathbf{z}_v + \epsilon(\mathbf{K}_\mathbf{p}+\mathbf{K}_\mathbf{v})\widehat{\dot{\mathbf{z}}}_v, \tag{SM 47}$$

$$\widehat{\dot{\boldsymbol{\xi}}} = m_T\left(\widehat{\dot{\boldsymbol{\zeta}}} - \dddot{\mathbf{p}}_d\right), \tag{SM 48}$$

$$\widehat{\dot{\mathbf{q}}}_d = \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q}_d)\widehat{\dot{\boldsymbol{\xi}}} \tag{SM 49}$$

$$\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = -\frac{(\dot{\mathbf{q}}\mathbf{q}^\mathsf{T} + \mathbf{q}\dot{\mathbf{q}}^\mathsf{T})\boldsymbol{\xi} + \mathbf{q}\mathbf{q}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}}{m_T} - \dddot{\mathbf{p}}_d, \tag{SM 50}$$

$$\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = \epsilon^2(\mathbf{I}+\mathbf{K}_\mathbf{v}\mathbf{K}_\mathbf{p})\widehat{\dot{\mathbf{z}}}_v + \epsilon(\mathbf{K}_\mathbf{p}+\mathbf{K}_\mathbf{v})\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}, \tag{SM 51}$$

$$\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = m_T\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \ddddot{\mathbf{p}}_d\right), \tag{SM 52}$$

$$\frac{d}{dt}\left(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = \frac{1}{\|\boldsymbol{\xi}\|^2}\left(\|\boldsymbol{\xi}\|\left(\mathbf{S}(\widehat{\dot{\mathbf{q}}}_d)\mathbf{S}(\mathbf{q}_d) + \mathbf{S}(\mathbf{q}_d)\mathbf{S}(\widehat{\dot{\mathbf{q}}}_d)\right) - \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q}_d)\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}\right)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q}_d)\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}, \tag{SM 53}$$

$$\widehat{\dot{\boldsymbol{\omega}}} = -\frac{1}{m_Q\ell}\mathbf{S}(\mathbf{q})(\mathbf{f}+\widehat{\mathbf{b}}), \tag{SM 54}$$

$$\widehat{\ddot{\mathbf{q}}} = -\mathbf{S}(\mathbf{q})\widehat{\dot{\boldsymbol{\omega}}} - \|\boldsymbol{\omega}\|^2\mathbf{q} \tag{SM 55}$$

$$\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = -\frac{1}{m_T}\left(\widehat{\ddot{\mathbf{q}}}\mathbf{q}^\mathsf{T}\boldsymbol{\xi} + \dot{\mathbf{q}}\dot{\mathbf{q}}^\mathsf{T}\boldsymbol{\xi} + \dot{\mathbf{q}}\mathbf{q}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}} + \dot{\mathbf{q}}\dot{\mathbf{q}}^\mathsf{T}\boldsymbol{\xi} + \mathbf{q}\widehat{\ddot{\mathbf{q}}}^\mathsf{T}\boldsymbol{\xi} + \mathbf{q}\dot{\mathbf{q}}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}} + \dot{\mathbf{q}}\mathbf{q}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right.$$
$$\left.+\mathbf{q}\dot{\mathbf{q}}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{q}\mathbf{q}^\mathsf{T}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\right) - \ddddot{\mathbf{p}}_d,$$

$$\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = \epsilon^2(\mathbf{I}+\mathbf{K}_\mathbf{v}\mathbf{K}_\mathbf{p})\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} + \epsilon(\mathbf{K}_\mathbf{p}+\mathbf{K}_\mathbf{v})\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_v|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}, \tag{SM 56}$$

$$\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = m_T\left(\frac{d}{dt}\left(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\zeta}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \mathbf{p}_d^{(5)}\right), \tag{SM 57}$$

and, finally,

$$
\begin{aligned}
&\frac{d}{dt}\left(\widehat{\dot{\mathbf{z}}}_\omega|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} = \mathbf{S}(\widehat{\ddot{\mathbf{q}}})\boldsymbol{\omega} + \mathbf{S}(\dot{\mathbf{q}})\widehat{\dot{\boldsymbol{\omega}}} - \frac{k_q}{h_q}\Big(\mathbf{S}(\widehat{\ddot{\mathbf{q}}})\mathbf{S}(\mathbf{q})\mathbf{q}_d + \mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\dot{\mathbf{q}})\mathbf{q}_d + \mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q})\widehat{\dot{\mathbf{q}}}_d + \mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\dot{\mathbf{q}})\mathbf{q}_d + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\ddot{\mathbf{q}}})\mathbf{q}_d \\
&+\mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})\widehat{\dot{\mathbf{q}}}_d + \mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}^2(\mathbf{q})\frac{d}{dt}\left(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\Big) - \frac{\epsilon}{h_q m_T}\Big[-\frac{1}{\|\boldsymbol{\xi}\|}[\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q})\mathbf{e}\boldsymbol{\xi}^\mathsf{T} \\
&+\mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})\mathbf{e}\boldsymbol{\xi}^\mathsf{T} + \mathbf{S}^2(\mathbf{q})\widehat{\dot{\mathbf{e}}}\boldsymbol{\xi}^\mathsf{T} + \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q})\mathbf{e}\left(\widehat{\dot{\boldsymbol{\xi}}}^\mathsf{T}\|\boldsymbol{\xi}\| - \frac{1}{\|\boldsymbol{\xi}\|}\boldsymbol{\xi}^\mathsf{T}(\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}})\right)]\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}^2(\mathbf{q})\mathbf{e}\boldsymbol{\xi}^\mathsf{T}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} \\
&-\left(\frac{1}{\|\boldsymbol{\xi}\|}\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}\left[\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q}) + \mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})\right]\mathbf{e} + \|\boldsymbol{\xi}\|\left(\mathbf{S}(\widehat{\ddot{\mathbf{q}}})\mathbf{S}(\mathbf{q}) + \mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\dot{\mathbf{q}}) + \mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\dot{\mathbf{q}}) + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\ddot{\mathbf{q}}})\right)\mathbf{e} + \|\boldsymbol{\xi}\|(\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q}) + \mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}}))\widehat{\dot{\mathbf{e}}}\right) \\
&-\left(\frac{1}{\|\boldsymbol{\xi}\|}\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}\mathbf{S}^2(\mathbf{q}) + \|\boldsymbol{\xi}\|\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q}) + \|\boldsymbol{\xi}\|\mathbf{S}(\mathbf{q})\mathbf{S}(\dot{\mathbf{q}})\right)\widehat{\dot{\mathbf{e}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \|\boldsymbol{\xi}\|\mathbf{S}^2(\boldsymbol{\xi})\frac{d}{dt}\left(\widehat{\dot{\mathbf{e}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\Big] - \Big(\frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}(\widehat{\ddot{\mathbf{q}}})\mathbf{S}(\mathbf{q}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \\
&-\frac{1}{\|\boldsymbol{\xi}\|}\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\widehat{\dot{\mathbf{q}}}_d)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{1}{\|\boldsymbol{\xi}\|^2}\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q}_d)\left(\|\boldsymbol{\xi}\|\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \frac{1}{\|\boldsymbol{\xi}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}\right) + \mathbf{S}(\dot{\mathbf{q}})\mathbf{S}\Big(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\boldsymbol{\xi}\|}\Big)\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \\
&+\mathbf{S}(\mathbf{q})\mathbf{S}\Big(\frac{\|\boldsymbol{\xi}\|\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \frac{1}{\|\boldsymbol{\xi}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}}{\|\boldsymbol{\xi}\|^2}\Big)\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} + \mathbf{S}(\mathbf{q})\mathbf{S}\Big(\frac{\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}}{\|\boldsymbol{\xi}\|}\Big)\frac{d}{dt}\left(\widehat{\dot{\mathbf{q}}}_d|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}} - \left(\mathbf{S}(\dot{\mathbf{q}})\mathbf{S}(\mathbf{q}_d) + \mathbf{S}(\mathbf{q})\mathbf{S}(\widehat{\dot{\mathbf{q}}}_d)\right) \\
&\frac{1}{\|\boldsymbol{\xi}\|^2}\left[\|\boldsymbol{\xi}\|\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{1}{\|\boldsymbol{\xi}\|}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right] - \mathbf{S}(\mathbf{q})\mathbf{S}(\mathbf{q}_d)\Big[\frac{1}{\|\boldsymbol{\xi}\|^2}\Big(\|\boldsymbol{\xi}\|\frac{d}{dt}\Big(\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\Big)_{\text{est}} \\
&-\frac{1}{\|\boldsymbol{\xi}\|}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}\Big) - \frac{1}{\|\boldsymbol{\xi}\|^3}\left(\frac{d}{dt}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} - \frac{1}{\|\boldsymbol{\xi}\|^6}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\left(\|\boldsymbol{\xi}\|^3\widehat{\dot{\boldsymbol{\xi}}}^\mathsf{T} - 3\|\boldsymbol{\xi}\|\boldsymbol{\xi}^\mathsf{T}(\boldsymbol{\xi}^\mathsf{T}\widehat{\dot{\boldsymbol{\xi}}})\right)\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}} \\
&+\frac{1}{\|\boldsymbol{\xi}\|^3}\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\boldsymbol{\xi}^\mathsf{T}\frac{d}{dt}\left(\widehat{\dot{\boldsymbol{\xi}}}|_{\mathbf{f}=\mathbf{f}_d^{\parallel}}\right)_{\text{est}}\Big]\Big). \qquad \text{(SM 58)}
\end{aligned}
$$

References

[1] Yang, W., Yu, G., Reis, J. and Silvestre, C. Robust Nonlinear 3D Control of an Inverted Pendulum Balanced on a Quadrotor. *Automatica*. pp. 1-12 (2023)