Central results from Newton's Principia Mathematica – II. The Force of Attraction of a Spherical Body

Josef Bemelmans

Institute for Mathematics, RWTH Aachen

February 28th, 2018

Outline

Introduction

Newton in a letter to Halley, 20 June 1686:

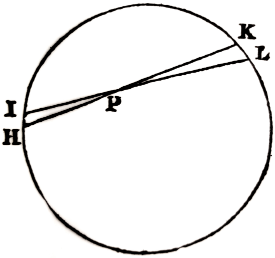
" I never extended y^e duplicate proportion lower than to y^e superficies of y^e earth & before a certain demonstration I found y^e last year have suspected it did not reach accurately enough down so low ... There is so strong an objection against y^e accurateness of this proportion, y^t without my Demonstrations ... it cannot be beleived by a judicious Philosopher to be any where accurate. "

Corr. II, pp. 435f.

Dana Densmore
Newton's Principia: The Central Argument.
Translation, Notes, and Expanded Proofs,
Green Lion Press, Santa Fe, NM, 1999.

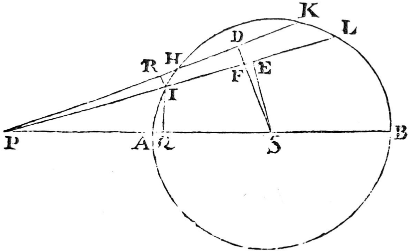
Propositio LXX. Theorema XXX

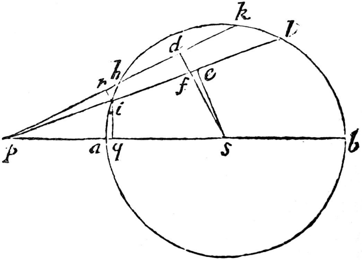
If toward each of the separate points of a spherical surface there tend equal centripetal forces as the squares of the distances from the point, I say that a corpuscle placed inside the surface will not be attracted by these forces in any direction.



Propositio LXXI. Theorema XXXI

With the same conditions being supposed as in prop.71, I say that a corpuscle placed outside the spherical surface is attracted to the center of the sphere by a force inversely proportional to the square of its distance from that same center.

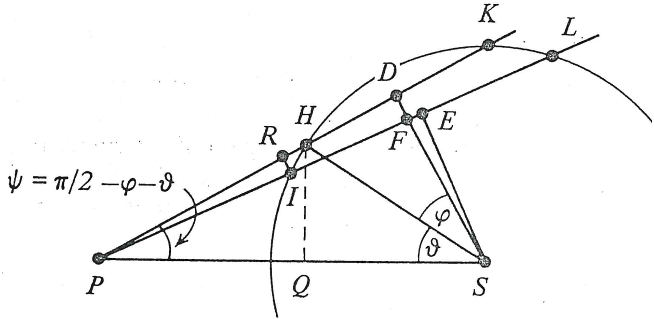




Littlewood's Proof

J.E. Littlewood, Newton and the attraction of a sphere, Mathematical Gazette 32(1948) No. 300.

Also see:
B. Bollobás (Ed.), Littlewood's miscellany, Cambridge, 1986, pp.169-174.



Define
$a = SH = SK$, $r = PS$, and Q on PS by requiring $HQ \perp PS$.

1^{st}step:

Contribution ΔF of the ring $\sigma(\widehat{HI})$ generated by rotating the arc $\widehat{HI}$ about the axis PS:

$$\Delta F = \frac{1}{PH^2} \cdot 2\pi\, HQ \cdot a\,|\Delta\vartheta| \cdot cos\psi.$$

$$r^2\,\frac{\Delta F}{\Delta\varphi} = \frac{PS^2}{PH^2} \cdot \cos\psi \cdot 2\pi a \cdot \frac{|\Delta\vartheta|}{\Delta\varphi} \cdot HQ, HQ = a\ \sin\vartheta.$$

Sine rule: $\dfrac{PS}{\sin\sphericalangle PHS} = \dfrac{PH}{\sin\vartheta}$ and $\sphericalangle PHS = \pi/2 + \varphi$,

hence $\dfrac{PS^2}{PH^2} = \dfrac{\cos^2\varphi}{\sin^2\vartheta}$.

$\cos\psi = \sin(\varphi + \vartheta)$ leads to

$$\frac{r^2}{2\pi a^2}\frac{dF}{d\varphi} = \frac{\cos^2\varphi}{\sin^2\vartheta} \cdot \sin\vartheta \cdot \sin(\vartheta + \varphi) \cdot \left|\frac{d\vartheta}{d\varphi}\right|.$$

2^{nd} step:

We show $\dfrac{d\vartheta}{d\varphi} = -\dfrac{\sin\vartheta}{\cos\varphi \cdot \sin(\vartheta+\varphi)}$.

$$\frac{a}{r} = \frac{SH}{SP} = \frac{\sin\psi}{\sin\sphericalangle PHS} = \frac{\sin(\pi/2-(\vartheta+\varphi))}{\cos\varphi}$$

$$= \frac{cos(\vartheta(\varphi)+\varphi)}{\cos\varphi} \equiv const.$$

$$\Rightarrow 0 = \frac{d}{d\varphi}\left(\frac{\cos(\vartheta+\varphi)}{\cos\varphi}\right) =$$

$$\frac{-\sin(\vartheta+\varphi)}{\cos\varphi} \cdot \left(\frac{d\vartheta}{d\varphi}+1\right) + \frac{\cos(\vartheta+\varphi)}{-\cos^2\varphi} \cdot (-\sin\varphi)$$

$$\Rightarrow \frac{d\vartheta}{d\varphi} = -1 + \frac{\sin\varphi\cos(\vartheta+\varphi)}{\cos\varphi\sin(\vartheta+\varphi)} = -\frac{\sin\vartheta}{\cos\varphi\sin(\vartheta+\varphi)}.$$

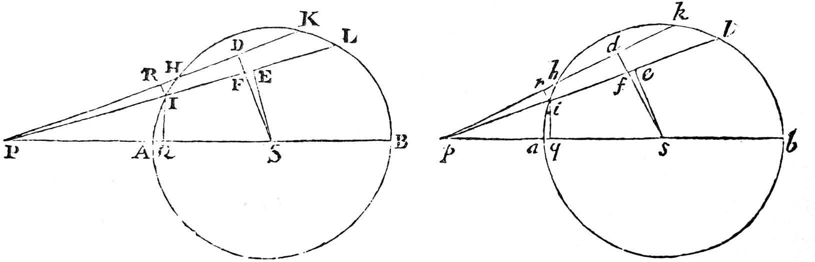
3^{rd}step:
Conclusion.

$$
\begin{aligned}
\frac{r^2}{2\pi a^2}\frac{dF}{d\varphi} &= \frac{\cos^2\varphi}{\sin^2\vartheta}\cdot\sin\vartheta\cdot\sin(\vartheta+\varphi)\cdot\frac{\sin\vartheta}{\cos\varphi\cdot\sin(\vartheta+\varphi)} \\
&= \cos\varphi.
\end{aligned}
$$

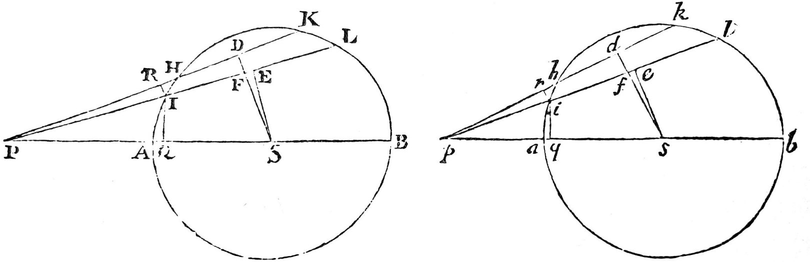
Integration gives $F = const.\cdot\dfrac{\pi a^2}{r^2}$.

Newton's Proof

Littlewood: "Now for the geometric proof (which must have left its readers in helpless wonder)." (l.c., p.172)



Result: $F(P) : F(p) = ps^2 : PS^2$,
i.e. comparison of two related quantities rather than calculation of one/both of them.



Construction of arcs $\widehat{HI}$,$\widehat{KL}$ and $\widehat{hi}$, $\widehat{kl}$ of equal length.
Main argument: $DS = ds$ and $DE = de$, construction based on theorems from Euclid's Elements, Book III.

Decisive property: contribution of ring $\sigma(\widehat{HI})$ differs only by a constant from contribution of the sphere.

Conclusive Remarks

Section 13: The attraction forces of bodies that are not spherical.
Section 14: The motion of very small bodies when agitated by centripetal forces tending to the several parts of any very great body.

Philoſophiæ Naturalis
PRINCIPIA
MATHEMATICA.

DEFINITIONES.

DEFINITIO I.

Quantitas materiæ eſt menſura ejuſdem orta ex illius denſitate et magnitudine conjunctim.

AER denſitate duplicata, in ſpatio etiam duplicato, fit quadruplus; in triplicato ſextuplus. Idem intellige de nive & pulveribus per compreſſionem vel liquefactionem condenſatis. Et par eſt ratio corporum omnium, quæ per cauſas quaſcunque diverſimode condenſantur. Medii interea, ſi quod fuerit, interſtitia partium libere pervadentis, hic nullam rationem habeo.

Hanc autem quantitatem ſub nomine corporis vel maſſæ in ſequentibus paſſim intelligo. Innoteſcit ea per corporis cujuſque pondus: Nam ponderi proportionalem eſſe reperi per experimenta pendulorum accuratiſſime inſtituta, uti poſthac docebitur.

DEFINITIO II.

Quantitas motus eſt menſura ejuſdem orta ex velocitate et quantitate materiæ conjunctim.

Motus totius eſt ſumma motuum in partibus ſingulis; ideoque in corpore duplo majore, æquali cum velocitate, duplus eſt, & dupla cum velocitate quadruplus.