Central results from Newton’s Principia Mathematica

II. The Force of Attraction of a Spherical Body

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Outline

Introduction
Propositio LXX. Theorema XXX
Propositio LXXI. Theorema XXXI
  a) Littlewood’s Proof
  b) Newton’s Proof
Concluding Remarks
Newton in a letter to Halley, 20 June 1686:

“\ I never extended \( y^e \) duplicate proportion lower than to \( y^e \) superficies of \( y^e \) earth & before a certain demonstration I found \( y^e \) last year have suspected it did not reach accurately enough down so low ... There is so strong an objection against \( y^e \) accurateness of this proportion, \( y^e \) without my Demonstrations ... it cannot be beleived by a judicious Philosopher to be any where accurate. ”

Dana Densmore
Newton’s Principia: The Central Argument.
Translation, Notes, and Expanded Proofs,
Green Lion Press, Santa Fe, NM, 1999.
If toward each of the separate points of a spherical surface there tend equal centripetal forces as the squares of the distances from the point, I say that a corpuscle placed inside the surface will not be attracted by these forces in any direction.
Propositio LXXI. Theorema XXXI

With the same conditions being supposed as in prop.71, I say that a corpuscle placed outside the spherical surface is attracted to the center of the sphere by a force inversely proportional to the square of its distance from that same center.
Define $a = SH = SK$, $r = PS$, and $Q$ on $PS$ by requiring $HQ \perp PS$. 

\[ \psi = \frac{\pi}{2} - \varphi - \theta \]
1\textsuperscript{st} step:

Contribution $\Delta F$ of the ring $\sigma(\widehat{HI})$ generated by rotating the arc $\widehat{HI}$ about the axis $PS$:

$$\Delta F = \frac{1}{PH^2} \cdot 2\pi HQ \cdot a |\Delta \vartheta| \cdot \cos \psi.$$

$$r^2 \frac{\Delta F}{\Delta \varphi} = \frac{PS^2}{PH^2} \cdot \cos \psi \cdot 2\pi a \cdot \frac{|\Delta \vartheta|}{\Delta \varphi} \cdot HQ, \quad HQ = a \sin \vartheta.$$

Sine rule: \[
\frac{PS}{\sin \angle PHS} = \frac{PH}{\sin \vartheta} \quad \text{and} \quad \angle PHS = \pi/2 + \varphi,
\]

hence \[
\frac{PS^2}{PH^2} = \frac{\cos^2 \varphi}{\sin^2 \vartheta}.
\]

$\cos \psi = \sin(\varphi + \vartheta)$ leads to

$$\frac{r^2}{2\pi a^2} \frac{dF}{d\varphi} = \frac{\cos^2 \varphi}{\sin^2 \vartheta} \cdot \sin \vartheta \cdot \sin(\vartheta + \varphi) \cdot \left| \frac{d\vartheta}{d\varphi} \right|.$$
2\textsuperscript{nd} step:

We show \( \frac{d\vartheta}{d\varphi} = -\frac{\sin \vartheta}{\cos \varphi \cdot \sin(\vartheta + \varphi)} \).

\[ \frac{a}{r} = \frac{SH}{SP} = \frac{\sin \psi}{\sin \angle PHS} = \frac{\sin(\pi/2 - (\vartheta + \varphi))}{\cos \varphi} \]

\[ = \frac{\cos(\vartheta(\varphi) + \varphi)}{\cos \varphi} \equiv \text{const}. \]

\[ \Rightarrow 0 = \frac{d}{d\varphi} \left( \frac{\cos(\vartheta + \varphi)}{\cos \varphi} \right) = \]

\[ \frac{-\sin(\vartheta + \varphi)}{\cos \varphi} \cdot \left( \frac{d\vartheta}{d\varphi} + 1 \right) + \frac{\cos(\vartheta + \varphi)}{-\cos^2 \varphi} \cdot (-\sin \varphi) \]

\[ \Rightarrow \frac{d\vartheta}{d\varphi} = -1 + \frac{\sin \varphi \cos(\vartheta + \varphi)}{\cos \varphi \sin(\vartheta + \varphi)} = -\frac{\sin \vartheta}{\cos \varphi \sin(\vartheta + \varphi)}. \]
3\textsuperscript{rd} step:
Conclusion.

\[
\frac{r^2}{2\pi a^2} \frac{dF}{d\varphi} = \frac{\cos^2 \varphi}{\sin^2 \vartheta} \cdot \sin \vartheta \cdot \sin(\vartheta + \varphi) \cdot \frac{\sin \vartheta}{\cos \varphi \cdot \sin(\vartheta + \varphi)} = \cos \varphi.
\]

Integration gives \( F = \text{const.} \cdot \frac{\pi a^2}{r^2} \).
Newton’s Proof

Littlewood: “Now for the geometric proof (which must have left its readers in helpless wonder).” (l.c., p.172)

Result: $F(P) : F(p) = ps^2 : PS^2$,
i.e. comparison of two related quantities rather than calculation of one/both of them.
Construction of arcs $\widehat{HI}, \widehat{KL}$ and $\widehat{hi}, \widehat{kl}$ of equal length.

Main argument: $DS = ds$ and $DE = de$, construction based on theorems from Euclid’s Elements, Book III.

Decisive property: contribution of ring $\sigma(\widehat{HI})$ differs only by a constant from contribution of the sphere.
Conclusive Remarks

Section 13: The attraction forces of bodies that are not spherical.  
Section 14: The motion of very small bodies when agitated by  
centripetal forces tending to the several parts of any very great  
body.
Philosophiæ Naturalis

PRINCIPIA

MATHEMATICA.

DEFINITIONES.

DEFINITIO I.

Quantitas materiae est mensura ejusdem orta ex illius densitate et magnitudine conjunctim.

AER densitate duplicata, in spatio etiam duplicato, fit quadruplus; in triplicato sextuplus. Idem intellige de nive & pulveribus per compressionem vel liquefactionem condensatis. Et par est ratio corporum omnium, quæ per causas quascunque diversimode condensantur. Medii interea, si quod fuerit, interstitia partium libere pervadentis, hic nullam rationem habeo.
Hanc autem quantitatem sub nomine corporis vel massae in sequentibus paslim intelligo. Innotescit ea per corporis cujusque pondus: Nam ponderi proportionalem esse reperi per experimenta pendulorum accuratissime instituta, uti posthac docebitur.

DEFINITIO II.

Quantitas motus est mensura ejusdem orta ex velocitate et quantitate materiae conjunctim.

Motus totius est summa motuum in partibus singulis; ideoque in corpore duplo majore, æquali cum velocitate, duplus est, & dupla cum velocitate quadruplicis.