Density results and the method of fundamental solutions for Cauchy data reconstruction.

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Keywords: Cauchy data reconstruction, Laplace and Helmholtz equations, method of fundamental solutions.

Abstract. In this work we address the problem that consists in fitting Cauchy boundary data using the method of fundamental solutions (MFS). We study some density properties in order to justify the MFS fitting for full and partial Cauchy data. We focus on fitting partial data and in particular we will present some theoretical and numerical results concerning the reconstruction of data in an inaccessible part of the boundary.

Overview

The reconstruction of Cauchy boundary data is an inverse ill-posed problem with many applications in non intrusive evaluation problems. The problem here addressed consists in, given a pair of data (g,g_n) solve the Cauchy problem

$$\begin{cases} \left(\Delta + k^2\right) u = 0 & \text{in } \mathbf{\Omega} \\ \left.\begin{aligned} u &= g \\ \partial_n u &= g_n \end{aligned}\right\} & \text{at } \mathbf{\Sigma} \subseteq \mathbf{\Gamma} = \partial\mathbf{\Omega} \end{cases}$$

where $\partial_n u$ denotes the normal derivative of u and the normal vector points outwards with respect to the regular (2D or 3D) domain Ω (that, for simplicity will be assumed to be simply connected). The part of the boundary Σ is assumed to be a relatively open set on the whole boundary Γ. In the following we shall consider the wave number $k \geq 0$ to be constant. Notice that, for $k=0$ we have a Cauchy problem for the Laplacian whereas for $k>0$ we are considering a Helmholtz problem.
One of the most direct approaches for the numerical solution of the above Cauchy problem is the MFS. This meshfree boundary method has been mostly considered as a numerical method for direct boundary value problems (see the first papers by Krupradze [1] and Bogomolny [2]). Concerning inverse problems, the method was firstly applied in a decomposition method for acoustic scattering (see [3]) and has been mainly used for Cauchy data fitting (eg. [4] and [5]). For a survey on the subject, see [6].

Recall that a fundamental solution, Φ^k, is a response to a Dirac delta centered at the origin, that is,

$$-\left(\Delta + k^2\right)\phi^k = \delta$$

and that the point source function $\mathbf{\Phi}_y^k$ is defined by $\mathbf{\Phi}^k(\bullet - y)$. For the Helmholtz equation ($k > 0$) we consider

$$\Phi_y^k(x)=\begin{cases}\frac{i}{4}H_0^{(1)}\left(k\left|x-y\right|\right) & \text{in 2D}\\ \frac{e^{ik|x-y|}}{4\pi\left|x-y\right|} & \text{in 3D}\end{cases}$$

and for *k=0* (Laplace equation),

$$\Phi_y^k(x)=\begin{cases}-\frac{1}{2\pi}\log\left(\left|x-y\right|\right) & \text{in 2D}\\ \frac{1}{4\pi\left|x-y\right|} & \text{in 3D}\end{cases}.$$

The method of fundamental solutions for the Cauchy problem consists in taking the approximation

$$u\left(x\right)=\sum_j \alpha_j\Phi^k\left(x-y_j\right)$$

where the source points are placed in the exterior of Ω. Thus, the above function satisfies the PDE $\left(\Delta+k^2\right)u=0$ in Ω. In order to solve numerically the Cauchy problem using u, we compute the coefficients α_j by imposing the two boundary conditions,

$$\begin{cases}\sum_j \alpha_j\Phi^k(x-y_j)=g\left(x\right)\\ \sum_j \alpha_j\partial_n\Phi^k(x-y_j)=g_n\left(x\right)\end{cases}. \qquad (1)$$

Notice that the Cauchy problem may not have a solution. Even if such solution exists, the above linear system may not be solvable. Therefore, in order to deal with the ill-posed nature of Cauchy problem we usually compute the coefficients by considering some sort of regularization method. In turn, we obtain a function that provides a fitting for the given Cauchy data.
An important theoretical question is, given a pair of Cauchy data (on appropriate functional spaces), can we obtain a good fitting using fundamental solutions basis functions ?
It turns out that the question is related to the location where the sources are placed. Usually, the source points are placed at fictitious boundaries located outside the domain of interest Ω. The number of such boundaries is related to the number of boundary conditions. In this paper we study this density problem, for two situations:

1. The case where $\Sigma = \Gamma$ (full boundary data) and
2. For partial boundary data, meaning that the Cauchy is available at part of the boundary.

Density results for full boundary data. Consider the subspace of $H^1(\Omega)$ defined by

$$H_{\Delta+k^2}=\left\{u\in H^1(\Omega):\left(\Delta+k^2\right)u=0\right\}$$

and notice that the trace and normal trace of $u\in H_{\Delta+k^2}$ belongs to $H^{1/2}\left(\Gamma\right)$ and $H^{-1/2}\left(\Gamma\right)$ respectively. Let $H_\Gamma=H^{1/2}\left(\Gamma\right)\times H^{-1/2}\left(\Gamma\right)$ be the data space and consider the linear map $\Lambda_\Gamma:H_{\Delta+k^2}\to H_\Gamma$ defined by

$$\Lambda_\Gamma\left(u\right)=\left(u_{|\Gamma},\partial_n u_{|\Gamma}\right).$$

Notice that, due to Holmgren's lemma, the above map is injective. The Cauchy problem can now be recast as,

given (g,g_n) on H_Γ solve the linear problem $\Lambda_\Gamma(u)=(g,g_n)$.

The range of Λ_Γ is the so called *set of compatible Cauchy data* and is a proper subspace of H_Γ, that is, $R(\Lambda_\Gamma)\neq H_\Gamma$. Consider now the boundary set of fundamental solutions

$$S_\Gamma=\left\{\left(\Phi_y^k|_\Gamma,\partial_n\Phi_y^k|_\Gamma\right):y\in\Gamma\right\}$$

where Γ is the boundary of a regular domain containing Ω (notice that there are other possible choices for artificial curves). Clearly, $span\, S_\Gamma\subset R(\Lambda_\Gamma)$ and system *(1)* is solvable if and only if $(g,g_n)\in span\, S_\Gamma$.
We have:

Proposition. If - k^2 is not an eigenvalue for the Laplace-Dirichlet problem in Ω, the set of compatible Cauchy data is closed in H_Γ

and

Proposition. Under the same above no resonance assumptions, the subspace $span\, S_\Gamma$ is dense in $R(\Lambda_\Gamma)$.

The first result means that for some pair of H_Γ data there is no nearby data coming from $H_{\Delta+k^2}$, In particular, it makes no sense to consider the MFS fitting for general data in H_Γ taking only source points in the exterior of Ω. However, for compatible data, the second proposition states that the MFS provides a good fitting by taking sources located only in the exterior of Ω. One way to deal with the lack of density in the data space is to consider also interior source points, located at an extra interior artificial curve. This approach leads to a data fitting taking fundamental basis functions satisfying the PDE only at a subset of the whole domain Ω.

Density results for partial boundary data. Consider a decomposition of the boundary

$$\Gamma=\Sigma\dot{\cup}\Pi\dot{\cup}\left(\Gamma\setminus\bar{\Sigma}\right)$$

where Σ and $\Gamma\setminus\bar{\Sigma}$ are relatively open (non empty) subsets of Γ with common boundary Π. As above, we define the space of partial data by $H_\Sigma=H^{1/2}(\Sigma)\times H^{-1/2}(\Sigma)$ (see [7] for the definition of these trace spaces) and the map $\Lambda_\Sigma:H_{\Delta+k^2}\to H_\Sigma$. Again from Holmgren's lemma, the map Λ_Σ is injective hence, given $\left(g^\Sigma,g^\Sigma{}_n\right)\in R(\Lambda_\Sigma)$ there exists an unique $u\in H_{\Delta+k^2}$ and $\left(g^{\Gamma\setminus\bar{\Sigma}},g^{\Gamma\setminus\bar{\Sigma}}{}_n\right)\in R\left(\Lambda_{\Gamma\setminus\bar{\Sigma}}\right)$ such that

$$\begin{cases}\Lambda_\Sigma(u)=\left(g^\Sigma,g^\Sigma{}_n\right)\\ \Lambda_{\Gamma\setminus\bar{\Sigma}}(u)=\left(g^{\Gamma\setminus\bar{\Sigma}},g^{\Gamma\setminus\bar{\Sigma}}{}_n\right)\end{cases}$$

The pair $\left(g^{\Gamma\setminus\bar{\Sigma}},g^{\Gamma\setminus\bar{\Sigma}}{}_n\right)$ will be referred as the missing data.

Proposition. Under the no resonance assumptions, the set of compatible data, $R(\Lambda_\Sigma)$, is dense in H_Σ.

Notice that the set of full compatible data $R(\Lambda_\Gamma)$ is closed in the data space H_Γ but is not dense (otherwise, all the data would be compatible). On the other hand, the set of partial compatible data $R(\Lambda_\Sigma)$ is dense but is not closed in the data space H_Σ.

Moreover, we have the following density result concerning fundamental solutions basis functions.

Proposition. Assuming the no resonance assumption, the subspace $span\left\{\left(\Phi_y^k|_\Sigma, \partial_n \Phi_y^k|_\Sigma\right) : y \in \Gamma\right\}$ is dense in H_Σ.

Therefore, we can obtain a good fitting of partial data taking fundamental solutions with source points located only in the exterior of Ω.

An iterative method for Cauchy data reconstruction. In the classical MFS approximation for the Cauchy problem described at the beginning of the paper, the H_Σ pairs of data are fitted simultaneously. However, in practice, one of the data is imposed (usually assumed with negligible error) and the other measured (usually contaminated with noise). In such situations, a simultaneous fitting of both data may lead to poor reconstruction results.

Assume, for instance, that the Dirichlet datum g^Σ is imposed and the Neumann datum g_n^Σ is measured. Assume further that $\left(g^\Sigma, g_n^\Sigma\right) \in R(\Lambda_\Sigma)$ and define the linear map

$$\psi_{g^\Sigma} : H^{1/2}\left(\Gamma \setminus \bar{\Sigma}\right) \to H^{-1/2}(\Sigma),\ h \mapsto \partial_n u|_\Sigma$$

where $u \in H_{\Delta+k^2}$ is determined by the boundary conditions $u|_\Sigma = g^\Sigma$ and $u|_{\Gamma\setminus\bar{\Sigma}} = h$. It follows that there exists an unique $u \in H_{\Delta+k^2}$ satisfying $\psi_{g^\Sigma}\left(u|_{\Gamma\setminus\bar{\Sigma}}\right) = g_n^\Sigma$ and in particular, the missing boundary data is $\Lambda_{\Gamma\setminus\bar{\Sigma}}(u)$. Such solution can be obtained by solving the equation

$$\psi_{g^\Sigma}(h) = g_n^\Sigma.$$

Linearizing, we get the equation on the update h_u

$$\psi'_{g^\Sigma}(h) h_u = g_n^\Sigma - \psi_{g^\Sigma}(h).$$

Numerical examples

In order to illustrate the proposed methods we present two numerical simulations. First case, concerns Cauchy data reconstruction for the domain presented in Fig. 1 using a direct MFS approach. The partial data was obtained at 60 observation points (green dots) and the source points are represented by the red dots. In Fig. 2 we present several reconstruction results, taking observations at different parts of the boundary (the corresponding location is represented by the green dots on the x axis).

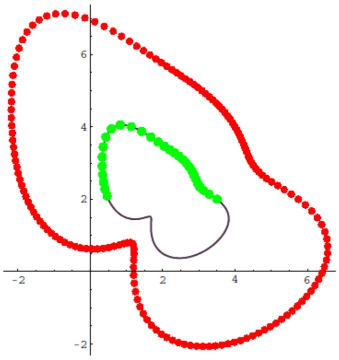

Fig. 1 Domain geometry and location of source and observations points.

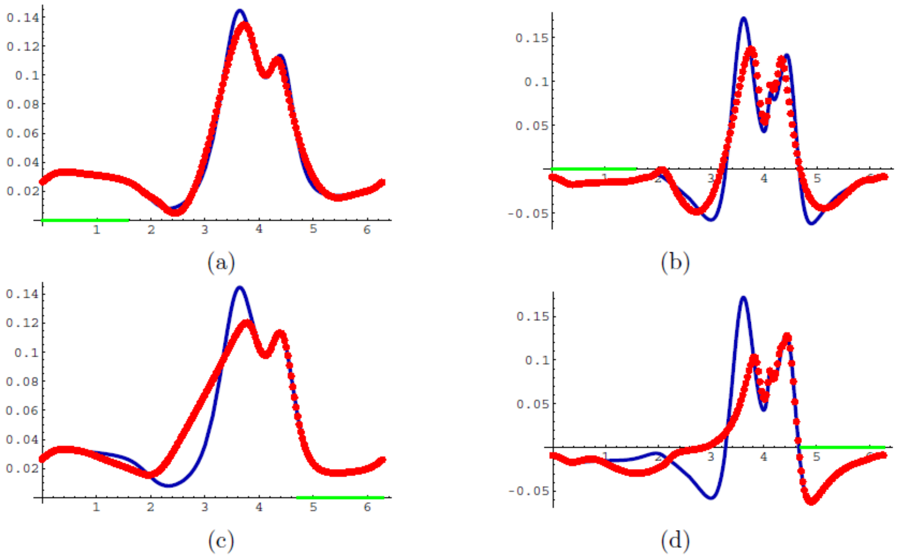

Fig. 2 Cauchy data reconstruction. Plots (a) and (c) concerns
Dirichlet datum and plots (b) and (d) Neumann datum.

Second simulation concerns a comparison between a direct MFS data reconstruction (Fig. 3) and the proposed iterative method (Fig. 4).

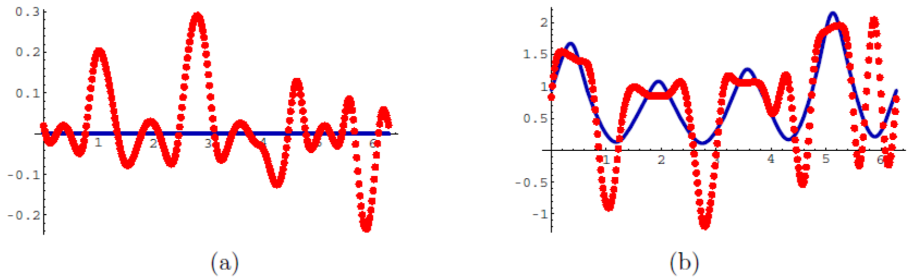

Fig. 3 Data reconstruction using a direct MFS approach

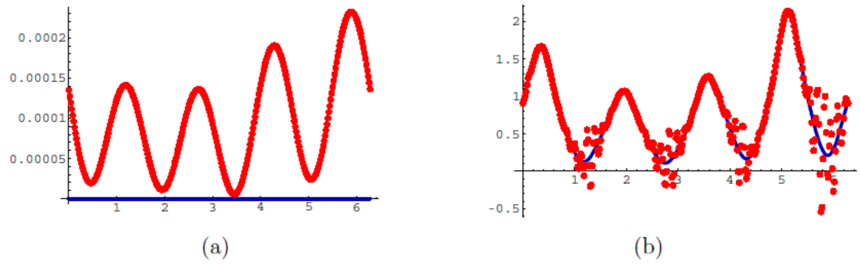

Fig. 4 Data reconstruction using an iterative approach

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