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COUPLING MFS WITH BEM APPROXIMATION

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Abstract. The method of fundamental solutions (MFS) is known to produce remarkable approximations for some types of boundary data, usually analytic functions and simple boundaries. It is known that in some non favorable situations, the approximation gets worse, even with a better choice of the fictitious sources. On the other hand, the boundary element method (BEM) does not suffer from this limitation, but it leads to large dense matrices to obtain comparable results. In this work, we consider a technique that couples both methods to improve the MFS results, using a less expensive BEM approximation. We present some numerical results to illustrate this.

1. INTRODUCTION

Some numerical methods use fundamental solutions of the PDEs as a main tool for reconstruction, this is the case of the method of fundamental solutions (MFS) and of the boundary element method (BEM). The MFS is a global numerical method, included in the broad class of methods using particular solutions, also known as Trefftz' methods. It may also be included in the class of spectral methods, as it presents exponential rate of convergence, under some regularity assumptions. The MFS has been introduced in the 1960's by Kupradze and Alekside in [8] and readdressed several times in the 1970's and 80's, for instance by Mathon and Johnston [9] and by Bogomolny [6] (see the review paper by Fairweather and Karageorghis [7]). In the context of meshfree methods, it regained some attention in the last years, also in the engineering community. The method offers high accuracy with the cost of ill conditioning, and some limitations in applications, such as the case of discontinuous boundary data, that we will address here. In Section 2 we begin by recalling the general MFS setting and in Section 3 we present an enrichment technique, with BEM functions, appropriated to deal with discontinuities in the boundary data. Finally in Section 4 we present numerical results that show the performance of this enhanced MFS for discontinuous data.

This follows the work by Alves and Leitão [4], and also recent results by Antunes and Valtchev [5]. The coupling with BEM is not exactly in the sense of obtaining two different solutions by implementing different methods (see for instance Tadeu et al. [10]), it is used in the sense of adding a small number of BEM basis functions to enrich the MFS and obtain a much higher accuracy.

2. GENERAL MFS SETTING

The method of fundamental solutions can be related to boundary integral equation methods like BEM, considering the discretization of the single layer potential

$$u(x) = L_\gamma \alpha(x) = \int_\gamma \alpha(y)\Phi(x-y)dy \approx \tilde{u}(x) = \sum_k \alpha_k \Phi(x-y_k),$$

where α_k are unknown coefficients, y_k prescribed source points, and Φ is the fundamental solution of the PDE, $\mathcal{D}\Phi = \delta$.

Both methods relate to layer potentials, but there are important differences:

(a) In BEM we consider $\gamma = \partial\Omega$, and the source set coincides with the boundary. In the MFS, the source set $\gamma \subset \mathbb{R}^d \backslash \bar{\Omega}$ is artificial, and we avoid singular integrations, at the cost of having higher ill conditioning problems.

(b) In BEM we can use double layer representation in Dirichlet problems (or single layer in Neumann problems) to avoid ill conditioning. In the MFS this strategy does not avoid ill conditioning problems. Moreover, for regular data, the results get better for artificial boundaries far from the true boundaries, where the conditioning is worse.

(c) In the MFS, while taking $u = L_\gamma \alpha = g$ on $\partial\Omega$, with external $\gamma \subset \mathbb{R}^d \backslash \bar{\Omega}$, this can only have a solution for analytic g. In fact, any convolution with the analytic fundamental solution Φ is analytic except at γ and therefore the MFS only provides a sequence of analytic approximations to g. There is no hope in solving the BIE $L_\gamma \alpha = g$ in a single stage, for non analytic g.

(d) In the MFS the choice for the source set is quite arbitrary. These lead (almost) always to linearly independent particular solutions, and in the sense of a Trefftz method we may add as many source points as we which. This can not be done in a random fashion since the approximation will not improve much, and it can even get worse, because of the ill conditioning problems.

The classical MFS using a source set γ may be resumed in the following steps:

(i) - *setting nodes and sources.* Define some collocation points on the boundary $x_1, \cdots, x_n \in \partial\Omega$, and the external source points $y_1, \cdots, y_m \in \gamma$ (usually with $n \geq m$) and the corresponding point-source approximations

$$v_m(x) = \sum_{j=1}^{m} \Phi_{y_j}(x)\alpha_j \tag{1}$$

which are particular solutions to $\mathcal{D}v_m = 0$.

(ii) - *solving the system.* The boundary conditions $\mathcal{B}v = g$, at the given collocation points, lead to the system

$$\sum_{j=1}^{m} \mathcal{B}\Phi_{y_j}(x_i)\alpha_j = g(x_i). \qquad (i = 1, \ldots, n), \tag{2}$$

the coefficients α_j are found such that (2) is verified. In the interpolation sense we consider $n = m$, in the least-squares sense $n \geq m$, with or without regularization.

A matrix $\mathbf{M}$ is defined by the entries $M_{ij} = \mathcal{B}\Phi_{y_j}(x_i)$, and considering $\mathbf{a}$ the vector defined by the coefficients α_j, and $\mathbf{g}$ the vector defined by $g(x_i)$, then:

– in the interpolation sense, we may obtain the coefficients by solving a linear system $\mathbf{Ma} = \mathbf{g}$,

– and in the least-squares sense, by solving $\mathbf{M}^*\mathbf{Ma} = \mathbf{M}^*\mathbf{g}$.

Since these systems in general are ill-conditioned, a simple way to circumvent machine precision difficulties is to consider a Tikhonov regularization procedure with a very small parameter $\tau > 0$, and solve

$$(\tau\mathbf{I} + \mathbf{M}^*\mathbf{M})\mathbf{a} = \mathbf{M}^*\mathbf{g}. \tag{3}$$

(iii) - *checking th approximation.* Let $\alpha_j^{(n)}$ be the coefficients calculated in **(ii)**. The approximating solution is

$$v_{m,n}(x) = \sum_{j=1}^{m} \Phi_{y_j}(x)\alpha_j^{(n)}, \tag{4}$$

satisfying the differential equation $\mathcal{D}v_{m,n} = 0$ on Ω and approximating the boundary condition

As, in general, there are density results (see [1], [3], [6]), but not exactly convergence results. The MFS is only an *a posteriori* method, and $v_{m,n}$ can only be regarded as an approximation if we check the boundary data.

The well-posedness of the boundary value problem implies that the error

$$E_{m,n} = g - \mathcal{B}v_{m,n}, \quad \text{on } \partial\Omega,$$

defines the quality of the approximation (for instance, by the maximum principle in Laplace problems, we know that the internal error will be smaller than the error on the boundary approximation).

In general we may rely on the **a posteriori error estimate**:

$$||u - v_{m,n}||_{\Omega} \leq C\, ||g - \mathcal{B}v_{m,n}||_{\partial\Omega},$$

for some theoretical constant $C > 0$ in the appropriate Sobolev (or Hölder) norms, using the well posedness of the direct problem (for the Laplace equation we may use the maximum norm). Note that, in discrete terms, it is not enough to check the boundary approximation at the collocation points - different boundary points, must also be used.

Therefore, the MFS is one of the simplest PDE numerical methods to implement and it provides remarkable results for some standard data (regular boundaries and boundary conditions). Unlike other classical methods, we can obtain almost machine precision accuracy with small matrices and no meshing procedure.
The MFS is reliable as it allows to control the error by direct inspection of the boundary approximation – this fact, depicted in step **(iii)**, is usually misregarded.
However, there is a counterpart. As the boundary element method, the MFS is restricted to some PDEs, while needing a fundamental solution, and it was also restricted to homogeneous problems ($f = 0$). On the other hand, unlike classic meshing methods, MFS and BEM are particularly suited for exterior problems, as they do not require the construction of an artificial boundary to bound the numerical domain. Also, there are techniques to both BEM and MFS that allow to solve other PDE problems, linear or non linear, for instance by iterative procedures. Both methods reduce the dimension of the linear system, but while the BEM presents singular integration difficulties, the MFS presents ill-conditioning difficulties. An advantage of the MFS (over BEM) is that it also avoids a boundary mesh, especially important for 3D problems.
The ill-conditioning feature of the MFS strongly depends on the choice of source points. The singularity of the fundamental solution means that choosing source points very close to the boundary collocation points leads to big diagonal entries, and in the limit, to a diagonally dominant matrix. However, this means that we are almost in a BEM choice, without singular integration. In that case the ill conditioning problems disappear but, while ignoring the singular integration, MFS performs poorly.
This is specially the case when we have discontinuous boundary data, since the analytical feature of the fundamental solutions is poorly adapted to these kind of data.
In this situation, the approximation shows a similar feature to Fourier approximation and a Gibbs-like phenomena occurs. In this paper we show how this problem can be avoided using an enrichment with basis functions arising from the BEM formulation.

3. MFS AND BEM ENRICHMENT

To approximate boundary data with a strong discontinuity, we may enrich the MFS with functions used in BEM, as these integrations have been already performed, and are well known.
For instance considering the Laplace equation, we may use the calculation of the functions

$$\Psi_b(x) = \int_{\Gamma_b} \partial_{n_y} \Phi(x - y) ds$$

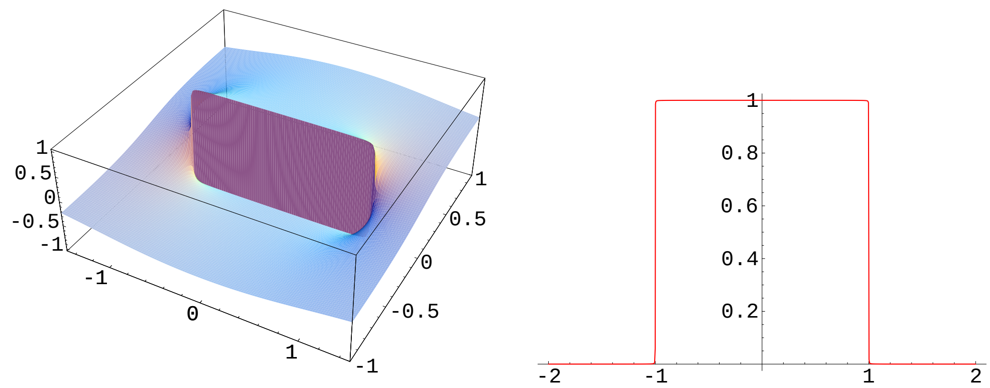

Figure 1: Plot of the function Ψ with $a = 1, b = 0$, on the left. Plot of the upper trace $b^+ = 0^+$, on the right.

where Γ_b is some simple boundary. In $\mathbb{R}^2$, take for instance the reference segment $\Gamma_b = [-a, a] \times \{b\}$. Then we know that

$$\Psi(x) = \frac{1}{\pi}\left(\arctan\frac{a - x_1}{x_2 - b} + \arctan\frac{a + x_1}{x_2 - b}\right), \tag{5}$$

and this function has the property of verifying the Laplace equation outside Γ_b. We plot this function in Figure 1.

The function Ψ has two traces, one when $x_2 \to b^+$ and an opposite when $x_2 \to b^-$, but it may be regarded as a characteristic function of the interval $[-1, 1]$ along the direction $x_2 = b^+$, as shown in Figure 1.

Therefore this function is a good candidate to approximate some discontinuous data, at the same time it verifies the PDE.

Instead of having an approximation only based on external point sources, we may consider to add *line sources*, or *cracklets* (as they were introduced in [2]), to enrich the approximation basis and deal with strong discontinuties on the boundary data.

These cracklets are available in BEM calculations, as they are the base of a piecewise $\mathbb{P}_0$ approximation.

We propose now to change the MFS matrix $\mathbf{M}$, adding collumns for these new basis functions, such that the approximation will be given by

$$v_m(x) = \sum_{j=1}^{m} \Phi_{y_j}(x)\alpha_j + \sum_{j=m+1}^{m+p} \Psi_j(x)\alpha_j.$$

The system will be solved in the least squares sense, and we can check the approximation quality exactly in the same way.

4. NUMERICAL SIMULATIONS

To illustrate the performance of the MFS with this enrichment, we consider a Dirichlet problem for the Laplace equation with a non trivial domain,

$$\Omega = (-4, 4)^2 \backslash \bar{B}_1(-1, 0) \backslash \bar{B}_1(2, 1),$$

as plotted in Figure 2.

In the the external square boundary we imposed $u(x) = -\frac{1}{4}x_2^2$ everywhere, except at $[-1, 1] \times$

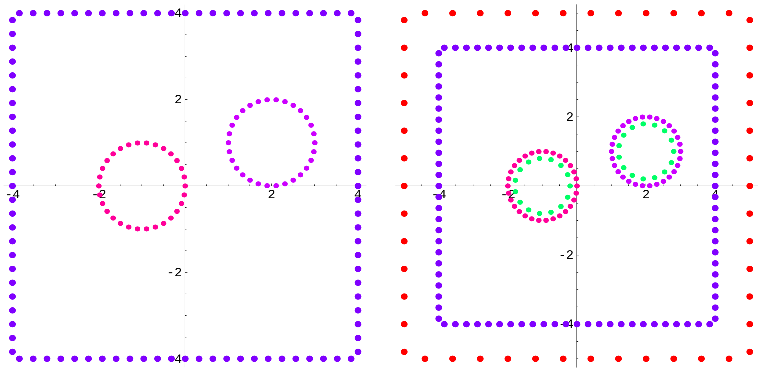

Figure 2: The domain Ω consists in a square except two disks. Collocation points on the boundary (left picture), together with external source points (right picture).

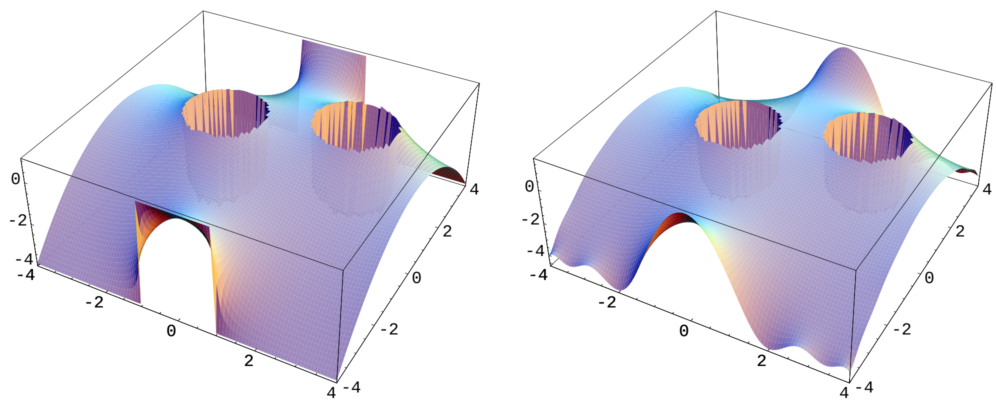

Figure 3: Numerical results for the solution u. Using the MFS with an enrichment with two extra functions Ψ_1 and Ψ_2 (picture on the left) the solution fits the discontinuous data, while this does not occur with the standard MFS (on the right).

$\{\pm 4\}$ where we considered $u(x) = 1$. This means a jump in the tips of these two boundary segments from -4 to 1, and will cause a difficulty that we will deal by introducing two appropriate basis functions Ψ_1 and Ψ_2 using $a = 1$ and $b = \pm 4$. In both inner circles boundary we considered $u(x) = 1$ as the Dirichlet boundary condition. The results are presented in Figure 3.

We do not plot the errors, as it is plain to see the difference on the boundary, that determines the internal precision (by the maximum principle). The MFS enriched with the two appropriate BEM functions Ψ_1 and Ψ_2 perfectly fits the boundary conditions even with the strong discontinuities, while the standard MFS does a rough smooth oscillating approximation, not being able to reproduce the discontinuities. Several other similar tests were made, even changing the feature of the discontinuity, i.e. not being a step discontinuity, like the one used in the basis functions. In those other cases, these same basis functions were able to fit the data, and produced similar very good results.

The main point is that these functions are able to deal with the discontinuity problem. The other features in the approximation are dealt by the standard point sources, that show an advantage with respect to standard BEM. An approximation with BEM for this domain would consider a rough approximation of the inner circles boundary, and of the boundary data, to achieve similar results with small matrices.

5. CONCLUSIONS

The introduction of appropriate basis functions in the MFS, available in BEM literature for

different PDEs, allows to obtain an accurate approximation to boundary value problems with strong discontinuities in the data, where the standard MFS usually fails. Therefore this simple procedure here presented can be extended to other PDE boundary value problems and eigenproblems, as introduced in [4] and already applied in [5].

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