RECOVERY OF CRACKS USING A RECIPROCITY GAP FUNCTION
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ABSTRACT

We are interested in this paper in the recovery of cracks from boundary measurements. We will be making use of a function that we will call *point-source reciprocity gap function*, which comes as a particular case of the reciprocity gap functional, applied to point sources. This function can be calculated in each point of the outer domain, and we will show that the analytic continuation of this function to the domain may provides a possible tool to the identification of the cracks, using functions that we will call *cracklets*.

KEYWORDS

Inverse heat conduction problems, crack detection, point sources.

INTRODUCTION

Identifying the location and shape of cracks inside a material is an inverse problem with major applications in industry, related to other non destructive inverse problems. We will state the problem as an inverse heat conduction problem in the steady-state case (electric conduction inverse problems, for instance, would obviously be treated in the same way). The main idea is to consider the reciprocity gap functional, introduced in [3], in the special case of point-sources. This allows the introduction of a new function, the *point-source reciprocity gap function,* which can be used to retrieve the crack, and some numerical methods are suggested.

The inverse problem here addressed has been treated in [9], and more recently in [1], where uniqueness of identification for insulating flat cracks is proven. In the case of conductive cracks, in [2] the result was proved for any connected crack, imposing positive boundary Dirichlet data, by using a simple maximum principle argument.

It is well known that not all fluxes are *identifying*, as has been stated in [9]. A sufficient condition for a flux to be identifying is that the singular part of the solution does not vanish [7]. In that case, it has been proved in [6] that the subset of the crack where the jump vanishes is neglectible.

An identifying flux is the starting point of any recovery task. As for the little bit more, which is that the flux indeed produces singularity, it has been proved more than once to be a necessary condition for stability (c.f. [4], [7]). Actually, fluxes not producing singularities are those orthogonal to a dual singular function, which are thus highly unlikely to meet.

We will slightly address the problem of identification and suggest a numerical method to retrieve the shape and location of the crack (or cracks). Numerical experiments are being currently carried.

THE CRACK RECOVERY PROBLEM

Let Ω be an open bounded set in $\mathbf{R}^d$; $d = 2$ or 3, with boundary Γ, and let σ be a *crack* inside the body. By *crack* we will understand any piecewise C^1 curve (orientable surface, in $\mathbf{R}^3$). We are interested in recovering this unknown crack by means of boundary measurements, that is by setting some flux ϕ on the boundary and measuring the resulting temperature. It has been noticed in [3] that the presence of cracks generates a so called reciprocity gap, a suitable handling of which may give rise to fast recovery algorithms. However this has mostly been worked out so far in the case of 3D planar or 2D line-segment cracks. In such cases, the reciprocity gap provides us with explicit formulae that localise the host plane or line, and this constitutes the starting point for the numerical part of the algorithm [6, 7]. In the present paper, we are investigating an alternative use of it, extending its generality to other cracks. Let us first recall somme definitions. The steady state heat problem we are dealing with is the following:

$$\begin{cases} \Delta u_\sigma & = & 0 & \text{in } \Omega_\sigma \\ \partial_n u_\sigma & = & \phi & \text{on } \Gamma \\ \partial_n u_\sigma & = & 0 & \text{on } \sigma \end{cases} \tag{1}$$

where ϕ is a known flux prescribed on the boundary Γ, and we hold some measurements on the boundary, that is, we assume that:

$$u_\sigma = f \qquad \text{on } \Gamma, \tag{2}$$

is known. Our goal is to retrieve the crack σ from the pair (ϕ, f).

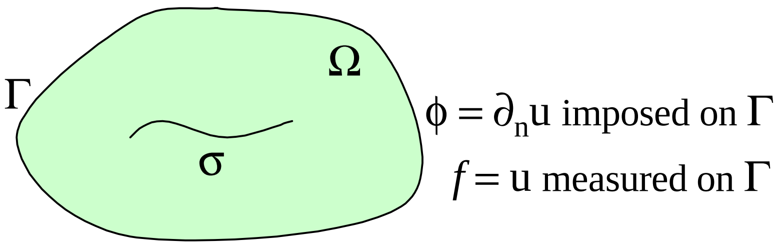

Figure 1. The unknown crack σ is to be determined from the pair (ϕ, f).

Definition 1. *Let ϕ be a flux and f_1, f_2 the two measurements produced by any two cracks, σ_1 and σ_2 (respectively). We say that the flux ϕ is an* identifying flux, *if $f_1 = f_2 \Leftrightarrow \sigma_1 = \sigma_2$.*

This definition means that no other crack, picked up in a suitable class of admissible cracks to be precised later on, than the actual one may produce the same measurements on the boundary. Since we assume that a crack σ is orientable, we can consider it as having two sides. On each one of these sides a trace of u is defined, and they will be called u^- and u^+. Using this notation, the jump of the solution on the crack σ is $[u_\sigma] = u^- - u^+$. We can easily deduce the following result on the jump.

Lemma 1. *Assume that ϕ is an identifying flux, then* $\text{supp}[u_\sigma] = \sigma$.

Proof. Suppose that $[u_\sigma]$ vanishes in some open subset ω of σ, and let τ be the crack σ deprived of ω. Since u_σ is continuous across ω, as well as its normal derivative, it is harmonic in $\Omega \setminus \tau$ and hence solves:

$$\begin{cases} \Delta u_\sigma & = & 0 & \text{in } \Omega \setminus \tau \\ \partial_n u_\sigma & = & \phi & \text{on } \Gamma \\ \partial_n u_\sigma & = & 0 & \text{on } \tau \end{cases} \tag{3}$$

as well as $u_\sigma = f$ on Γ. The flux ϕ is therefore not an identifying one since the crack τ is producing the same measurements as σ on the boundary. □

Thanks to the above lemma, the set $\overset{\circ}{\sigma} = \sigma \setminus \partial\sigma$ (i.e. the crack without its boundary) can be characterized by $\overset{\circ}{\sigma} = \{x \in \sigma\,;\, |[u_\sigma](x)| > 0\}$.

Now, given any function v harmonic in Ω, the reciprocity gap states that the scalar $\int_\Gamma(\phi v - f\,\partial_n v)$ is not vanishing as it would be if the domain were safe, and moreover its value can be related by a simple integration by parts to an integral on the crack itself, involving the jump of the solution u_σ:

$$\int_\Gamma (\phi\, v - f\,\partial_n v) = \int_\sigma [u_\sigma]\,\partial_n v, \quad \forall\, v \in \{w \in H^1(\Omega) : \Delta w = 0 \text{ in } \Omega\} \tag{4}$$

POINT-SOURCE RECIPROCITY GAP FUNCTION

Reciprocity gap algorithms are based on the choice of an appropriate set of harmonic functions v in order to derive relevant informations on the crack from the above formula. Let us go ahead along this path: given any point $x \in \bar{\Omega}^{\rm C}$, which is the outer domain, the Green function (in the 2D case, $G(x,y) = -\frac{1}{2\pi}\log|x-y|$, in the 3D case, $G(x,y) = \frac{1}{4\pi|x-y|}$), is harmonic in the inner domain Ω, and formula (4) applies.

Definition 2. *We introduce the* point-source reciprocity gap *as a function defined by*

$$g_\sigma(x) = \int_\Gamma \big(\phi(y)\,G(x,y) - f(y)\,\partial_{n_y}G(x,y)\big)\,ds_y. \tag{5}$$

The above function is harmonic in $\Omega \cup \bar{\Omega}^{\rm C}$, since it is the sum of a single layer potential and a double layer one, with sources on Γ. It should be pointed out that this part does not depend on the crack, but only on the given and measured data on the external boundary Γ. It is exactly the reciprocity gap functional, as introduced in [3], if we consider only the Green functions placed at $x \in \bar{\Omega}^{\rm C}$. Notice that in this case it is a function and not a functional. It is clear that this function g_σ verifies $\Delta g_\sigma = 0$, in $\mathbf{R}^d\backslash\Gamma$, with $[g_\sigma] = f$, $[\partial_n g_\sigma] = \phi$ on Γ, and appropriate asymptotic conditions (depending on the dimension d) when $r = |x| \to \infty$.

Now, let v_σ be the analytic function defined (for any $x \in \mathbf{R}^d\backslash\sigma$) by

$$v_\sigma(x) = \int_\sigma [u_\sigma]\,(y)\partial_{n_y}G(x,y)\,ds_y.$$

We notice that $[v_\sigma] = [u_\sigma]$.

Lemma 2. *We have $g_\sigma = v_\sigma$ in $\bar{\Omega}^{\rm C}$. Thus, the analytic extension of g_σ from $\bar{\Omega}^{\rm C}$ to $\Omega\backslash\sigma$ must be v_σ.*
Proof. If $x \in \bar{\Omega}^{\rm C}$ then $G(x,y)$ is harmonic for all $y \in \Omega$, and from (4) it follows $g_\sigma(x) = v_\sigma(x)$. □

Theorem 1. *In $\mathbf{R}^d$ we have $g_\sigma + u_\sigma\chi_\Omega = v_\sigma$.*
Proof. From Lemma 2, the equality follows in $\bar{\Omega}^{\rm C}$. Note also that

$$\begin{aligned} g_\sigma(x) &= \int_\Gamma \big(\phi(y)\,G(x,y) - f(y)\,\partial_{n_y}G(x,y)\big)\,ds_y \\ &= \int_{\Omega\backslash\sigma} (\Delta u_\sigma(y)\,G(x,y) - u_\sigma(y)\,\Delta_y G(x,y))\,ds_y + \int_\sigma [u_\sigma]\,(y)\partial_{n_y}G(x,y)\,ds_y \\ &= -\int_{\Omega\backslash\sigma} u_\sigma(y)\,\delta(x,y)\,ds_y + \int_\sigma [u_\sigma]\,(y)\partial_{n_y}G(x,y)\,ds_y \end{aligned}$$

Thus, if $x \in \Omega\backslash\sigma$, we have $g_\sigma(x) = -u_\sigma(x) + v_\sigma(x)$.
Therefore, in Γ, $g_\sigma^+ = v_\sigma$ and $g_\sigma^- = -u_\sigma + v_\sigma$, meaning $[g_\sigma]_\Gamma = g_\sigma^+ - g_\sigma^- = u_\sigma = f$, as mentioned.
Also, $[\partial_n g_\sigma]_\Gamma = (\partial_n g_\sigma)^+ - (\partial_n g_\sigma)^- = \partial_n u_\sigma = \phi$.
Finally, in σ, we note that $[g_\sigma]_\sigma = [v_\sigma]_\sigma - [u_\sigma]_\sigma = 0$. □

We use the notion of *analytic singular support* (eg. [8]), $\text{sing}_{\rm A}\,\text{supp}(g_\sigma)$, as the complement of the largest open set in $\mathbf{R}^d$ where g_σ is an analytic function. It is clear that $\text{sing}_{\rm A}\,\text{supp}(v_\sigma) \subseteq \sigma$ and that $[v_\sigma] = [u_\sigma]$, therefore

$$\text{sing}_{\rm A}\,\text{supp}(v_\sigma) = \text{supp}\,[u_\sigma].$$

From Lemma 2, g_σ^* the unique analytic extension of g_σ from $\bar{\Omega}^{\mathrm{C}}$ to σ^{C} is v_σ, and we conclude the following result.

Corollary 1. *If we impose an identifying flux ϕ, then the crack σ is perfectly determined by*

$$\sigma = \mathrm{sing}_{\mathrm{A}}\,\mathrm{supp}(g_\sigma^*)$$

Proof: Immediate, by Lemma 1, because $\mathrm{sing}_{\mathrm{A}}\,\mathrm{supp}(g_\sigma^*) = \mathrm{sing}_{\mathrm{A}}\,\mathrm{supp}(v_\sigma)$ =supp $[u_\sigma] = \sigma$. □

Remarks.

(i) Notice that even if the flux is not identifying we can only conclude that $\mathrm{sing}_{\mathrm{A}}\,\mathrm{supp}(g_\sigma^*) = \varsigma \subseteq \sigma$. What this means is that on $\sigma \backslash \varsigma$ we have null jump, $[u_\sigma] = 0$, and therefore the function is analytic there. A way to overcome this problem is to impose that the jumps belong to the space

$$\mathcal{C}(\sigma) := \left\{ q \in H_{00}^{1/2}(\sigma) : q(x) \neq 0 \text{ æ.on } \sigma \right\}. \tag{6}$$

Notice that is not a major restriction. Fluxes not producing singularities are those orthogonal to a dual singular function (cf. [10]), which are thus highly unlikely to meet. In addition, if missing, a component in the dual singular function will anyway arise from computational errors.

Therefore, if the chosen flux ϕ only generates jumps $[u_\sigma] \in C(\sigma)$, then the crack σ is perfectly determined by σ =$\mathrm{sing}_{\mathrm{A}}\,\mathrm{supp}(g_\sigma^*)$.

(ii) From Theorem 1, we also conclude that the solution u is the sum of a function $-g_\sigma$, harmonic in Ω, with a function v_σ. Thus, when doing the extension of g_σ from $\bar{\Omega}^{\mathrm{C}}$, the function obtained g_σ^* coincides with u_σ only if $u_\sigma = v_\sigma$. In the following, when we consider solutions u_σ of the type v_σ, then we are really recovering u_σ if we do the analytic extension of g_σ.

Example 1. To show the effect of this technique, we consider $\Omega =]-1,1[^2$ and

$$\sigma = \{\frac{1}{5}(-4+7t, 4-7t) : t \in [0,1]\}.$$

Given the pairs (ϕ_1, f_1) and (ϕ_2, f_2) generated by two different solutions,

$$u_1(x) = \int_\sigma \partial_n G(x,y) ds_y, \;\text{ and }\; u_2(x) = u_1(x) + \frac{x_1^2 - x_2^2}{20}.$$

In Figure 2.a) we plotted the solution u_2 in $]-3,3[^2$. It becomes clear that the information that we will retrieve in the pair (ϕ_2, f_2) will be added to the undesirable effect of the harmonic function $\frac{x_1^2 - x_2^2}{20}$. This effect could be much worse, if we added a more significant harmonic function. However, this effect will disappear if we calculate g_σ. The reciprocity gap function will act like a *filter*, keeping only the relevant information. In Figure 2.b) we plot g_2 associated to u_2, and in Figure 2.c) we plot g_1 associated to u_1. The only difference between the two plots lies inside Ω, as predicted in Theorem 1. Since u_1 will be of the form v_σ, we will have a null g_1 in Ω. This reduction to v_σ situations will help the reconstruction.

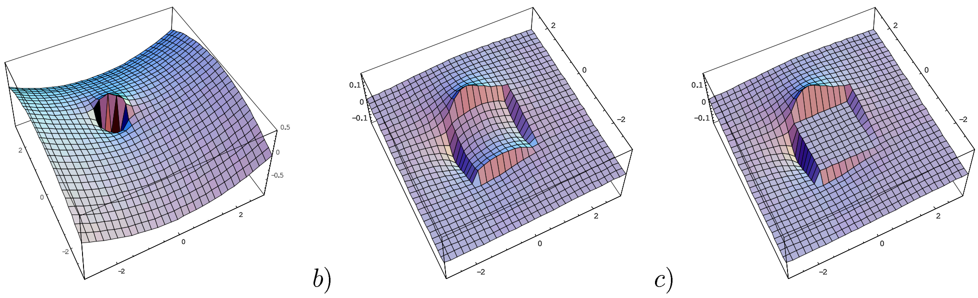
a) *b)* *c)*

Figure 2. a) Plot of u_1.b) Plot of g_1. c) Plot of g_2.

FLAT CRACKS

In the case σ is a plane crack (or a line crack in 2D), $\sigma \subset \Pi$, where Π is the plane of the crack, then we can establish the following criteria

Theorem 2.

$$\sigma \text{ is a plane crack in } \Pi \Leftrightarrow g_{\sigma(\phi)}(x) = 0, \forall x \in \Pi \backslash \sigma, \forall \phi$$

Proof.

($\Rightarrow$) For instance, in the 2D case, suppose ν is the normal to $\Pi \supset \sigma$, then

$$g_\sigma(x) = \int_\sigma [u_\sigma](y) \partial_{n_y} G(x,y)\, ds_y = \int_\sigma [u_\sigma](y) \frac{\nu \cdot (x-y)}{2\pi\, |x-y|^2}\, ds_y.$$

Therefore, if $x \in \Pi \backslash \sigma$, for all $y \in \sigma \subset \Pi$.we get $\nu \cdot (x-y) = 0$.

($\Leftarrow$) If σ is not a plane crack, we can always take a flux ϕ that produces a jump $[u_\sigma]$ not orthogonal to $\partial_{n_y} G(x,\cdot)$ in $L^2(\sigma)$. □

Remark: Likewise, if σ is almost a plane crack, with $|\nu \cdot \frac{x-y}{|x-y|}| \leq \varepsilon$, for all $y \in \sigma, x \in \Pi \backslash \bar{\Omega}$, we have

$$|g_\sigma(x)|^2 \leq \int_\sigma |\,[u_\sigma](y)|^2 ds_y \int_\sigma \frac{|\nu \cdot (x-y)|^2}{2\pi\, |x-y|^4}\, ds_y \leq \frac{||u||^2_{L^2(\sigma)} |\sigma|}{2\pi \text{dist}(\sigma, \Pi \backslash \bar{\Omega})^2}\, \varepsilon^2$$

Thus, one expects $|g_\sigma(x)|$ to be small when $x \in \Pi \backslash \sigma$.

Example 2.

Consider a domain $\Omega =]-1,1[^2$ and a non flat crack defined by

$$\sigma = \{\frac{1}{5}(-4 + 6t + 3\cos(4t), 4 - 6t) : t \in [0,1]\}.$$

We take measurements in $\Gamma = \partial\Omega$ given by the traces and normal traces of $u(x) = \int_\sigma \partial_{n_y} G(x,y) ds_y$. In Figure 2.a) we plotted the solution u and its extension to $[-3,3]^2$. In that plot one clearly sees the jump of the solution field in the crack. Since σ is an almost flat crack, and by the previous remark, one expects $|g|$ to be almost null along some line Π that crosses the crack σ. This fact can be seen in Figure 2.b), where the $|g|$ is plotted in $[-3,3]^2$. Inside the domain $\Omega =]-1,1[^2$ the field g is null, as predicted in Theorem 1.

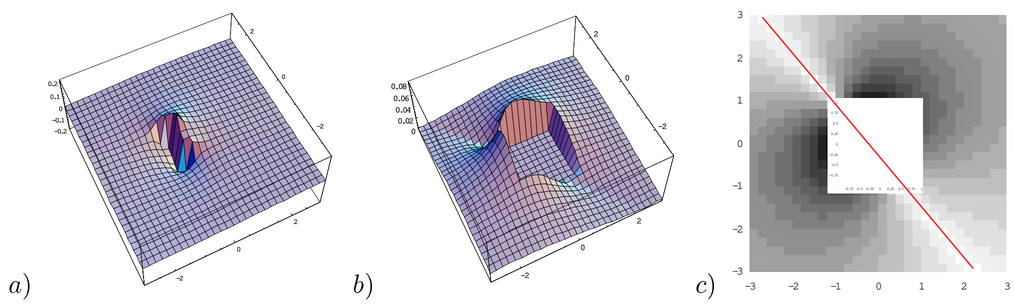

Figure 3: a) Plot of the solution u, with the jump in the crack σ, and the analytic extension outside Ω.
b) 3D-plot of $|g_\sigma|$. Note that g is null inside Ω and also along the predicted line Π.
c) Density plot of $|g_\sigma|$. We also plotted the predicted line Π, to point out that it crosses the crack σ.

LOW SENSIBILITY TO NOISY DATA

The data given by the reciprocity gap function g_σ smooths the possible noise that arises in the measurement of f or even the noise in the input field ϕ. In fact, suppose that the values with noise are $\tilde{\phi} = \phi + \varepsilon_\phi$ and $\tilde{f} = f + \varepsilon_f$, then

$$\tilde{g}_\sigma(x) = g_\sigma(x) + \int_\Gamma \left(\varepsilon_\phi(y)\, G(x,y) \;-\; \varepsilon_f(y)\, \partial_{n_y} G(x,y)\right) ds_y.$$

Since we assume the noise to be random, we consider $\int_\Gamma \varepsilon_\phi$ and $\int_\Gamma \varepsilon_f$ to be almost null. Thus, if $\mathrm{dist}(x,\Gamma)$ is not too small, we can avoid the singularity of the integral, bounding $|G(x,y)|$ and $|\partial_{n_y} G(x,y)|$. Thus,

$$|\tilde{g}_\sigma(x) - g_\sigma(x)| \leq \max |G(x,y)| \left| \int_\Gamma \varepsilon_\phi(y)\, ds_y \right| + \max |\, \partial_{n_y} G(x,y)| \left| \int_\Gamma \varepsilon_f(y)\, ds_y \right|$$

may be quite small. In the next example we present a case in which the result is not too perturbed even adding 40% random noise.

Example 3.

Consider the same domain as before, and a planar crack defined by

$$\sigma = \{\frac{1}{5}(-4+7t, 4-7t) : t \in [0,1]\},$$

like in Example 1. We have added up to 40% noise in the measurements of f and in the input data ϕ, given by the solution $u(x) = \int_\sigma \partial_{n_y} G(x,y) ds_y$. Since the solution is of the form v_σ we notice that we are in a *favorable situation*. Any significant harmonic perturbation would lead to worst results, since the noise would be added to the proportions of the 'non filtered data'. In Figure 4.a) we plotted the solution in $[-3,3]^2$, and in Figure 4.b) we plotted $|\tilde{g}_\sigma|$. One can see already an almost null field on the direction of the line crack, which becomes more clear in Figure 4.c). In that figure we also plotted two lines corresponding to small values of $|\tilde{g}_\sigma|$, and the only significant difference appears on the line more distant to the crack.

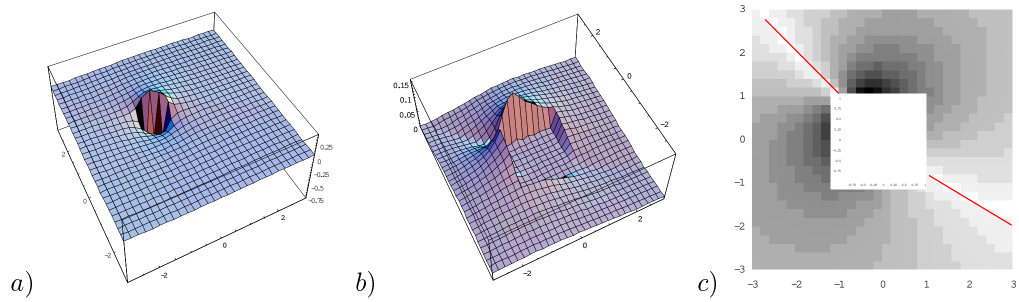

Figure 4 : a) Plot of the solution u. b) 3D-plot of $|\tilde{g}_\sigma|$. c) Density plot of $|\tilde{g}_\sigma|$.

ANALYTIC EXTENSION

One possible idea to retrieve the crack would be to consider the analytic extension of the point-source reciprocity gap function inside the domain Ω, since this would give the function v_σ, and therefore we would be able to get an approximate location of the crack by determining the discontinuous jump. However, one must be aware that this analytic extension could only be possible until the crack was reached, and the main point is that we do not know the location of the crack, and therefore if we consider the extension to $\Omega\backslash\omega$, with $\omega \subset \Omega$, we can not ensure that σ does not intersect $\Omega\backslash\omega$, and therefore the analytic extension was not possible to be made.

On the other hand, one must be aware that even considering functions with a behavior similar to v_σ, if one tries to fit this function to g_σ along some direction, fitting the data outside Ω, we may get functions that are almost identical to g_σ but their analytic extension inside Ω is completely different, even if we take a large number of points outside Ω. This feature is completely evident in Figure 5.a), where we plotted the correct solution (grey line) and the approximate solution (black line). Even if these two functions almost coincide in the points on the line outside Ω, when their behavior inside the domain Ω is completely different, compromising the location of the jump. In Figure 5.b) we plotted the result of the reconstruction of a crack

$$\sigma = \{(-\frac{3}{10} + \frac{t}{2} + \frac{3t}{8}\sin(t), -\frac{3}{10} + \frac{t}{4}) : t \in [0,1]\},$$

using a simple procedure based on the approximation of g_σ by dipole functions restricted along vertical lines. The result allows, in this case, to identify the zone of the crack, but the reconstruction is quite poor and could be even misleading.

Despite this fact, crossing this rough information with the knowledge of the approximate direction of the crack, dealt in the previous paragraph, we are able to define an approximate configuration of the crack. It suffices to remark that if we doing the reconstruction along vertical lines, the jumps will appear in the vertical lines that cross the crack (even if these jumps are not correctly placed). This defines a segment that gives an approximate location of the almost flat crack.

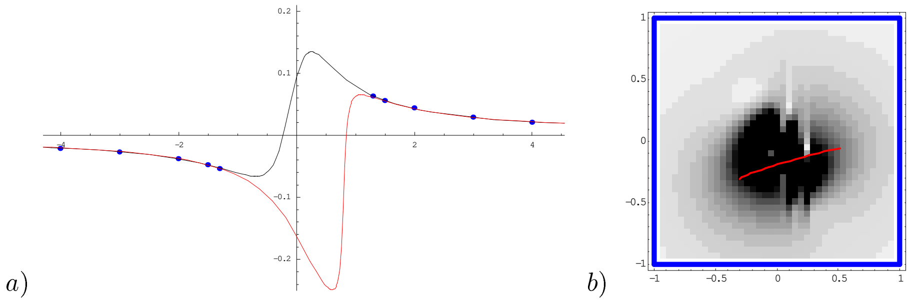

Figure 5. Difficulties with the analytic extension.

USING CRACKLETS

If one uses only piecewise constant densities, we get v_σ approximated by

$$v_h(x) \;=\; \sum_{i,j=1}^{n} q_{ij} \int_{\sigma_{ij}} \partial_{n_y} G(x,y)\,dy \;=\; \sum_{i,j=1}^{n} q_{ij}\xi_{ij}(x),$$

where to the functions ξ_{ij} we call *cracklets.*

For instance, in the 2D case, one can explicitly calculate the functions with an elementary crack $\tau = [(0,0),(1,0)]$.

In fact, given the *elementary cracklet* function

$$\Xi(x) = \int_0^1 \partial_{y_2} G(x,y)dy_1 = \arctan(\frac{1-x_1}{x_2}) + \arctan(\frac{x_1}{x_2}),$$

we can define any cracklet $\xi_{[a,b]}$ on the segment $[(a_1,a_2);(b_1,b_2)]$ by

$$\begin{aligned}\xi_{[a,b]}(x) \;&=\; \arctan(\frac{(b_1-a_1)^2+(b_2-a_2)^2-(x-a_1)(b_1-a_1)-(y-a_2)(b_2-a_2)}{(y-a_2)(b_1-a_1)-(x-a_1)(b_2-a_2)}) \\ &\quad + \arctan(\frac{(x-a_1)(b_1-a_1)-(y-a_2)(b_2-a_2)}{(y-a_2)(b_1-a_1)-(x-a_1)(b_2-a_2)}).\end{aligned}$$

Minimization algorithm to find one single crack

Given an unknown crack σ inside a domain Ω, we take a discrete set of data points given by the point-source reciprocity gap calculation,

$$D = \{(x_i, g_\sigma(x_i)) : x_i \in \bar{\Omega}^{\mathrm{C}}\}.$$

If we assume that the crack is almost flat, then it makes some sense to approach this data by nonlinear least squares using the cracklets $\xi_{[a,b]}$.

The simplest approach in this context is to minimize functions of the form

$$\xi_c(x) = \arctan(\frac{c_0 - c_1 - c_2 x_1 - c_3 x_2}{c_4 + c_5 x_1 + c_6 x_2}) + \arctan(\frac{c_1 + c_2 x_1 + c_3 x_2}{c_4 + c_5 x_1 + c_6 x_2})$$

such that $c = (c_0, ..., c_6)$ minimizes

$$Q_\sigma(c) = \sum_i |g_\sigma(x_i) - \xi_c(x_i)|^2.$$

The parameters $c_0, ..., c_6$ will be given by a standard nonlinear minimization algorithm (we used the routine NonlinearFit from *Mathematica-Wolfram Research*)..

Example 4.

We checked the results testing for three different cracks in $\Omega =]-1,1[^2$:

$$\begin{aligned} \sigma_1 &= \{(-\frac{3}{10} + \frac{t}{2} + \frac{3t}{8}\sin(t), -\frac{3}{10} + \frac{t}{4}) : t \in [0,1]\}, \\ \sigma_2 &= \{(-\frac{1}{2} + \frac{t}{4}\sin(4t), -\frac{1}{2} + \frac{t}{2}) : t \in [0,1]\}, \\ \sigma_3 &= \{\frac{1}{5}(-4 + 6t + 3\cos(4t), 4 - 6t) : t \in [0,1]\}. \end{aligned}$$

In Figure 6 we plotted the fields ξ_c that minimizes $Q_{\sigma_k}(c)$, using 75 random points plotted in $[-3,3]^2 \backslash \bar{\Omega}$. Although the reconstruction are not good (we obtained a bigger jump instead of a larger crack), this gives not only the idea of the location of the crack, but also the orientation, which also confirm the results obtained in the previous paragraph while taking the minimum of $|g_\sigma|$ (see Figure 6.c). In the last picture we also took a 5% random noise.

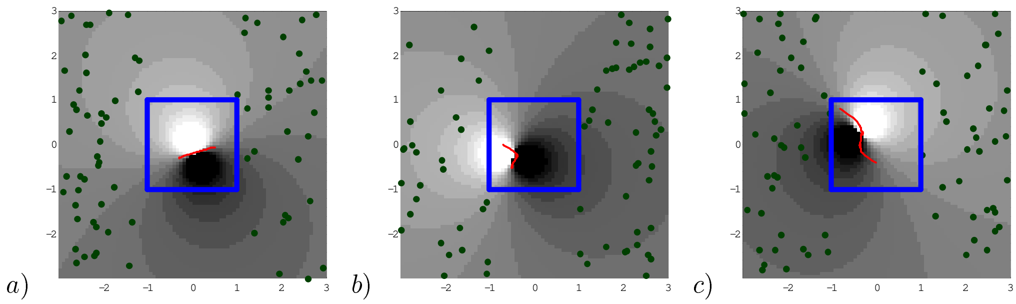

Figure 6. Density plots of the fitting for the cracks $\sigma_1, \sigma_2, \sigma_3$ (respectively).

In this example we only used a single standard flux. It is clear that other fluxes may iluminate more one side of the crack than the other. This simple reconstruction will search for the most 'iluminated part', where the jump is bigger. By considering other fluxes we expect that several portions of the crack may be revealed. The sum of all those informations may reveal a better reconstruction of the shape of the crack.

CONCLUSIONS

i) The point-source reciprocity gap function allows to filter relevant information, not only by vanishing noise but also because it filters the harmonic contribution that it is not relevant for crack detection.

ii) In the case of almost flat cracks, the line for which $|g_\sigma|$ is minimal may allow the identification of the main direction of the crack, provided the flux (or several fluxes) is well chosen.

iii) The use of a cracklet allows the identification of the zone where the crack lies and its main direction.

Extensions, using several cracklets may lead to the detection of several cracks.

We note again that all these reconstructions were made using a single flux

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