On the determination of elastic point sources

Carlos J. S. Alves * Abdellatif El Badia† Tuong Ha Duong‡

Abstract

In this work we show that the algebraic method used in [4] can be applied to retrieve the positions and intensities of elastic point sources inside a bounded domain Ω using one boundary observation. The algebraic method allows to identify completely the number, location and intensity of all sources.

1 Introduction

In this work we address the problem of retrieving the location and intensities of point sources, inside an homogeneous elastic body, using a single boundary measurement. As an application we can refer the detection of the magnitudes and hypocenters of several small internal sources inside the elastic body.

Several authors have also addressed this type of source problems in the static and transient cases (e.g. [1], [3], [4], [5], [6]).

Here we propose to extend to the elastic static case an algebraic method that was developed in [4] and applied to scalar potential problems. The extension to the transient case, following [5], is in progress.

2 Detection of point sources in an elastic medium

We will use the notation $\Delta^* = \mathrm{div}(\nabla^*)$ and $\partial_n^* \mathbf{u} = \nabla^* \mathbf{u} \cdot \mathbf{n}$, where ∇^* stands for the stress tensor,

$$[\nabla^* \mathbf{u}]_{ij} = \lambda \, \mathrm{div}(\mathbf{u}) \delta_{ij} + \mu(\partial_i u_j + \partial_j u_i).$$

noticing that $\Delta^* \mathbf{u} = \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u}$.

Consider $\Omega \subset \mathbf{R}^d$ (with $d = 2$ or $d = 3$), to be a bounded open domain with regular border $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 is a part of the boundary where we can measure the data. The aim is to find the intensity and location of dipole sources, a set $S = \{s_1, ..., s_N\}$, which is known to be inside Ω. We consider two situations:

(i) *Dirichlet boundary condition* on Γ.
Measuring $\mathbf{f} = \partial_n^* \mathbf{u}$ on Γ_0, the elastic displacement $\mathbf{u}$ verifies

$$\begin{cases} -\Delta^* \mathbf{u} = \sum_{k=1}^N \mathbf{a}_k \delta_{s_k} & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \Gamma \\ \partial_n^* \mathbf{u} = \mathbf{f} & \text{on } \Gamma_0 \end{cases} \tag{1}$$

*Centro de Matemática Aplicada, Instituto Superior Técnico, 1049-001 Lisboa, Portugal
†
‡Université de Technologie de Compiègne, 60 205 Compiègne, France

(ii) *Neumann boundary condition* on Γ.

Measuring $\mathbf{f} = \mathbf{u}$ on Γ_0, the elastic displacement $\mathbf{u}$ verifies

$$\begin{cases} -\Delta^* \mathbf{u} = \sum_{k=1}^{N} \mathbf{a}_k \delta_{s_k} & \text{in } \Omega \\ \partial_n^* \mathbf{u} = 0 & \text{on } \Gamma \\ \mathbf{u} = \mathbf{f} & \text{on } \Gamma_0 \end{cases} \tag{2}$$

In both situations, our aim is to find the locations s_k and the intensities $\mathbf{a}_k$. The constant vectors $\mathbf{a}_k$ stand for the unknown intensities associated to each point source location s_k. We will also write

$$\mathbf{F} = \sum_{k=1}^{N} \mathbf{a}_k \delta_{s_k} = \mathbf{a} \delta_S. \tag{3}$$

It is clear that the associated direct problems have a well defined solution. Take, for instance, the Dirichlet case. We can decompose $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ verifies

$$-\Delta^* \mathbf{v} = \mathbf{a}\delta_S, \text{ in the whole } \mathbf{R}^d.$$

Therefore, using $\mathbf{\Phi}$, the fundamental tensor of elasticity equation (Kelvin's solution),

$$\Phi_{ij}(x) = \begin{cases} \frac{1}{16\pi\mu\,|x|} \left(\frac{\lambda+3\mu}{\lambda+2\mu}\, \delta_{ij} + \frac{\lambda+\mu}{\lambda+2\mu} \frac{x_i x_j}{|x|^2} \right), & \text{in } \mathbf{R}^3 \\ \frac{1}{8\pi\mu} \left((\frac{\lambda+3\mu}{\lambda+\mu}) \log(\frac{1}{|x|}) \delta_{ij} + \frac{x_i x_j}{|x|^2} \right), & \text{in } \mathbf{R}^2 \end{cases} \tag{4}$$

we explicitly get the solution

$$v(x) = \mathbf{\Phi} * (\mathbf{a}\delta_S) = \sum_{k=1}^{N} \mathbf{\Phi}(x - s_k)\mathbf{a}_k.$$

Now, taking $\mathbf{w}$ verifying

$$\begin{cases} -\Delta^* \mathbf{w} = 0 & \text{in } \Omega \\ -\mathbf{w} = \mathbf{v} & \text{on } \Gamma \end{cases} \tag{5}$$

which is a well posed problem (the data $\mathbf{v}$ on the boundary has the same regularity as the boundary), we get $\mathbf{w} \in H^1(\Omega)^d$.

It is now clear that $\mathbf{u} = \mathbf{v} + \mathbf{w}$ verifies

$$\begin{cases} -\Delta^* \mathbf{u} = \mathbf{a}\delta_S & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \Gamma \end{cases} \tag{6}$$

and the solution of the Dirichlet problem (i) is well defined.

The same argument can be used for the Neumann case (ii).

3 Identifiability

We will now prove that the locations and intensities of point sources are uniquely determined by a single measurement taken in the boundary part Γ_0.

Theorem 1 . *A source* $\mathbf{F}$ *(3) is uniquely determined by a single measurement* $\mathbf{f}$.

Proof. Assume that $\mathbf{u}_1$ and $\mathbf{u}_2$ are solutions of the problem (P) that produce the same displacement $\mathbf{f}$ on Γ_0, for different values of non null intensities $\mathbf{a}_k^{(m)}$ and locations $s_k^{(m)}$, with

$m = 1, 2$ (respectively). The number of sources can also be different, we will denote them by N_1 and N_2, respectively.

It follows that $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ is null on Γ_0 and unknown in Ω. We will prove that $\mathbf{v} \equiv 0$ in Ω.

For that purpose we consider $\mathbf{v}$ extended by zero outside $\bar{\Omega}$, and $\mathbf{v}$ it is known to be the unique solution of the problem

$$\begin{cases} -\Delta^* \mathbf{v} = \sum_{k=1}^{N} \mathbf{a}_k \delta_{s_k} & \text{in } \Omega, \\ \partial_n^* \mathbf{v} = 0 & \text{on } \Gamma, \\ \mathbf{v} = 0 & \text{on } \Gamma_0, \\ \mathbf{v}(x) = \begin{cases} \mathbf{O}(\log(|x|)), & \text{if } d = 2 \\ \mathbf{O}(|x|^{-1}), & \text{if } d = 3 \end{cases} & \text{when } |x| \to \infty. \end{cases} \tag{7}$$

where $\mathbf{a}_k$ and s_k are defined in a way such that

$$\sum_{k=1}^{N} \mathbf{a}_k \delta_{s_k} = \sum_{k=1}^{N_1} \mathbf{a}_k^{(1)} \delta_{s_k^{(1)}} - \sum_{k=1}^{N_2} \mathbf{a}_k^{(2)} \delta_{s_k^{(2)}} \tag{8}$$

holds, ie. we reorder the terms in the difference, and we now must prove that the remaining $\mathbf{a}_k$ are null.

It is well known that the solution of the problem (7) is an analytic function in $X = \mathbf{R}^d \backslash (\Gamma_1 \cup S_1 \cup S_2)$, given by

$$\mathbf{v}(x) = \sum_{k=1}^{N} \mathbf{\Phi}(x - s_k)\mathbf{a}_k + \int_{\Gamma_1} \partial_n^* \mathbf{\Phi}(x - y)[\mathbf{v}](y) ds_y,$$

where $\mathbf{\Phi}$ is the fundamental tensor of the elasticity equation (note that $\partial_n^* \mathbf{\Phi}$ is a matrix), and $[\mathbf{v}]$ is the jump of $\mathbf{v}$ on Γ_1.

Since $\mathbf{v} \equiv 0$ in $(\bar{\Omega})^{\complement}$, we must have $\mathbf{v} \equiv 0$ in X. Now, defining

$$\mathbf{w}(x) = \mathbf{v}(x) - \int_{\Gamma_1} \partial_n^* \Phi(x - y)[\mathbf{v}](y) ds_y = \sum_{k=1}^{N} \Phi(x - s_k)\mathbf{a}_k, \tag{9}$$

$\mathbf{w}$ is analytic in $\mathbf{R}^d \backslash (S_1 \cup S_2)$. If we take $z \to s_k$, in one hand we have

$$\mathbf{w}(z) = -\int_{\Gamma_1} \partial_n^* \Phi(z - y)[\mathbf{v}](y) ds_y,$$

which converges to a finite limit, and on the other hand

$$\mathbf{w}(z) = \sum_{k=1}^{N} \Phi(z - s_k)\mathbf{a}_k$$

explodes, unless $\mathbf{a}_k = 0$. This means that the sources and the intensities coincide. $\diamond$

Remark: The same analysis can be made for the Dirichlet case (i).

4 Recovery of the point sources

We assume that $\Gamma_0 = \Gamma$, and take for instance the Neumann case (ii), introducing the reciprocity gap functional,

$$\mathcal{R}(\mathbf{v}) = \int_{\Gamma} \partial_n^* \mathbf{u} \cdot \mathbf{v}\, ds - \int_{\Gamma} \partial_n^* \mathbf{v} \cdot \mathbf{u}\, ds = -\int_{\Gamma} \partial_n^* \mathbf{v} \cdot \mathbf{f}\, ds \tag{10}$$

(the dot in these formulas stands for the scalar product in $\mathbf{R}^d$). Using Betti's formula,

$$\mathcal{R}(\mathbf{v}) = \int_\Omega \Delta^* \mathbf{u} \cdot \mathbf{v}\, dx - \int_\Omega \mathbf{u} \cdot \Delta^* \mathbf{v}\, dx.$$

and we now choose $\mathbf{v}$ such that $\Delta^* \mathbf{v} = 0$ in Ω. Thus,

$$\mathcal{R}(\mathbf{v}) = \int_\Omega (\sum_{k=1}^N \mathbf{a}_k \delta_{s_k}) \cdot \mathbf{v}\, dx = \sum_{k=1}^N \mathbf{a}_k \cdot \mathbf{v}(s_k), \forall \mathbf{v} \in \mathcal{H}, \tag{11}$$

where $\mathcal{H} = \{\mathbf{v} \in (H^1(\Omega))^d : \Delta^* \mathbf{v} = 0\}$.

The question is now to obtain simple functions in $\mathcal{H}$ such that a simple algebraic calculation allows to retrieve the locations and the intensities of the point sources. For this purpose, we must introduce appropriated test functions in the 2-dimensional and in the 3-dimensional cases.

It is important to note that these test functions have no physical meaning, they are used only for auxiliary calculus purposes, and this allows us to consider functions with complex values. In the following, if $\mathbf{v} = \mathbf{v}_1 + \mathrm{i}\mathbf{v}_2$ then $\mathbf{a} \cdot \mathbf{v} = \mathbf{a} \cdot \mathbf{v}_1 + \mathrm{i}(\mathbf{a} \cdot \mathbf{v}_2)$.

4.1 Test functions

Consider the following function $\mathbf{w}$ defined in $\mathbf{R}^3$ with complex values (in $\mathbf{C}^3$),

$$\mathbf{w}_p(x, y, z) = (x + \mathrm{i}\, y)^p \, [1 \quad \mathrm{i} \quad 0]^\top$$

one can see that it verifies the elasticity equation. We will use this type of functions $\mathbf{w}_p$ as test functions, having

$$\mathcal{R}(\mathbf{w}_p) = \sum_{k=1}^N \mathbf{a}_k \cdot \mathbf{w}_p(s_k) = \sum_{k=1}^N a'_k (s'_k)^p$$

where $a'_k = \mathrm{a}_{k,1} + \mathrm{a}_{k,2}\, \mathrm{i}$, and $s'_k = s_{k,1} + s_{k,2}\, \mathrm{i}$.

Take $M \geq N$, a bound for the maximum number of sources inside Ω. We now define the complex matrices

$$\mathbf{A}_m = \begin{bmatrix} (s'_1)^m & \cdots & (s'_N)^m \\ \vdots & \ddots & \vdots \\ (s'_1)^{m+M-1} & \cdots & (s'_N)^{m+M-1} \end{bmatrix}_{M \times N},$$

which are Vandermonde-like matrices if all s'_k are assumed to be different (see Remark 1).

Consider $\mathbf{b}_m$ to be the complex vector defined by

$$b_j = \sum_{k=1}^N a'_k (s'_k)^j, \text{ with } j = m, ..., m + M - 1$$

and let $\mathbf{a}'$ be the complex vector $\mathbf{a}' = (a'_1, ..., a'_N)$.

• We first notice that

$$\mathbf{b}_m = \mathbf{A}_m \mathbf{a}', \text{ and } \mathbf{A}_{m+1} = \mathbf{A}_m \mathbf{S},$$

where $\mathbf{S}$ is the diagonal matrix defined by $\mathbf{S} = \mathrm{diag}(s'_1, ..., s'_N)$.

Thus, we conclude that $\mathbf{A}_{m+1} = \mathbf{A}_1 \mathbf{S}^m$ and therefore $\mathbf{b}_m = \mathbf{A}_0 \mathbf{S}^m \mathbf{a}$.

Lemma 1 . *Let* $\mathbf{B} = [\mathbf{b}_0, ..., \mathbf{b}_{M-1}]$ *be a matrix* $M \times M$*. We have* $rank(\mathbf{B}) = N$*.*

Proof. We have $\mathbf{b}_0 = \mathbf{A}_0\mathbf{a}', ..., \mathbf{b}_{M-1} = \mathbf{A}_0\mathbf{S}^{M-1}\mathbf{a}'$, which means that they are all images of $\mathbf{A}_0$, an $M \times N$ matrix, and therefore $rank(\mathbf{B}) \leq N$.

We now prove that $\mathbf{b}_0, ..., \mathbf{b}_{N-1}$ are linear independent. Suppose that

$$0 = \sum_{m=0}^{N-1} c_m \mathbf{b}_m = \sum_{m=0}^{N-1} c_m \mathbf{A}_0 \mathbf{S}^m \mathbf{a}' = \mathbf{A}_0 \sum_{m=1}^{N-1} c_m \mathbf{S}^m \mathbf{a}'.$$

Since the $N \times N$ diagonal submatrix of $\mathbf{A}_0$ is a Vandermonde matrix, it is clear that it is invertible and this implies

$$(\sum_{m=0}^{N-1} c_m \mathbf{S}^m)\mathbf{a}' = 0 \quad \text{and thus} \quad \sum_{m=0}^{N-1} c_m \mathbf{S}^m = 0,$$

because we assume $a'_j \neq 0$, for all j. The diagonal matrices

$$\mathbf{S}^m = \mathbf{diag}((s'_1)^m, ..., (s'_N)^m)$$

can be understood as the lines of a Vandermonde matrix and therefore this implies $c_m = 0$, for all $m = 0, ..., N-1$. ◇

4.2 Determination of source points and their intensities.

We can easily derive the following relation

$$\mathbf{b}_{m+1} = \mathbf{A}_0 \mathbf{S}^{m+1} \mathbf{a}' = \mathbf{A}_0 \mathbf{S} \mathbf{A}_0^{-1} \mathbf{A}_0 \mathbf{S}^m \mathbf{a}' = \mathbf{A}_0 \mathbf{S} \mathbf{A}_0^{-1} \mathbf{b}_m.$$

It is clear that the matrix $\mathbf{T} = \mathbf{A}_0 \mathbf{S} \mathbf{A}_0^{-1}$ is similar to $\mathbf{S}$ and therefore the eigenvalues of $\mathbf{T}$ are exactly $s'_1, ..., s'_N$. Therefore it suffices to construct $\mathbf{T}$ to obtain the location of the source points s'_k.

The values of $\mathbf{a}'$ are obtained by solving the linear system, $\mathbf{b}_0 = \mathbf{A}_0\mathbf{a}'$.

Remarks: (i) We may assume all s'_k to be different. Notice that the distinct points $(s_k)_{1 \leq k \leq N}$ may have non distinct projections $(s'_k)_{1 \leq k \leq N}$ on the xOy-plane. We only get a part of the information – the projection points that are distinct, and the projections of the intensities, that are not null.

Actually, it is finite the number of planes that violate that distinction condition for projection of (a_k, s_k). Thus it is not a very restricted assumption to suppose that we have chosen a coordinate system such that the projections (s'_k) of (s_k) on xOy are distinct, and such that the vectors a'_k are not null.

In fact, we may assume that the number N of sources is known. We then choose a coordinate system such that the number N' of (s'_k) coincides with N.

(ii) Since there is no problem with projections, the 2D case is obviously simpler, and the same analysis can be made, with full recovery.

Recovering the third components

We now assume that (a'_k, s'_k) have been determined, and we look for the missing information $(a_{k,3}, s_{k,3})$.

Note that some $a_{k,3}$ may be null or some $s_{k,3}$ may coincide.

If we suppose that for the other coordinate planes xOz and yOz the distinction condition for projection is also verified then the recovery is completed.

However we will use that assumption. We take a function

$$\mathbf{w}(x, y, z) = z\mathbf{w}_p(x, y, z) = z(x + \mathrm{i}\ y)^p\ [1 \quad \mathrm{i} \quad 0]^\top,$$

and then

$$\mathcal{R}(\mathbf{w}) = \sum_{k=1}^{N} \mathbf{a}_k \cdot \mathbf{w}(s_k) = \sum_{k=1}^{N} a'_k s_{k,3} (s'_k)^p.$$

The matrix $\mathbf{A}_0 = [(s'_k)^{p-1}]_{k,p=1,\dots N}$ is invertible and $\mathcal{R}(\mathbf{w}) = \mathbf{A}_0 \mathbf{z}$ where $\mathbf{z}$ is the vector $[a'_k s_{k,3}]_{k=1,..,N}$. Thus, since $a'_k \neq 0$, the $s_{k,3}$ are determined. To obtain the values $a_{k,3}$ it suffices to consider

$$\mathbf{w}(x, y, z) = (x + \mathrm{i}\ y)^p\ \ [0\ 0\ 1]^\top,$$

since then

$$\mathcal{R}(\mathbf{w}) = \sum_{k=1}^{N} a_{k,3} (s'_k)^p$$

and the $(a_{k,3})$ are obtained by $(\mathbf{A}_0)^{-1}\mathcal{R}(\mathbf{w})$.

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