The method of fundamental solutions adapted for a non homogeneous equation

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Summary

We will present the basis of a method of fundamental solutions (MFS) adapted for the non homogeneous Laplace equation. Using the approximation of the second member by fundamental solutions of the eigenvalue equation, a meshless method approaching the solution of the non homogeneous partial differential equation can be derived. Some numerical experiments showing the approximation of a function by these point sources are presented.

Introduction

Consider the Poisson equation on a bounded connected domain $\Omega \subset \mathbf{R}^d$ with Dirichlet boundary conditions

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

To find the solution u, we can take the usual strategy:

(i) First find a particular solution $u_P : \Delta u_P = f$, in an open set that contains Ω.

(ii) Then we solve the homogeneous problem $\Delta u_H = 0$ in Ω, with $u_H = g - u_P$ on $\partial\Omega$.

(iii) $u = u_H + u_P$ verifies the problem.

To produce an approximation of u_H, we can use the method of fundamental solutions, using Laplace's fundamental solution, $\Phi_0(x) = \frac{-1}{2\pi}\log(|x|)$.This is done by taking an approximation with point sources functions (e.g. [2]), that is

$$\tilde{u}_h(x) = \sum_{k=1}^{N} \alpha_k \Phi_0(x - y_k),$$

where y_k are chosen points outside $\bar{\Omega}$. However, it is clear that this technique no longer applies to the approximation of a non homogeneous problem. Several well known methods can be used to derive an approximation of the particular solution u_P. A direct approach, gives a u_P calculated by the domain integration of the Newtonian potential, $u_P(x) = \int_\Omega f(y)\Phi_0(x - y)\, dy$. However, this means that a mesh must be made in the domain Ω with the integration of the singular integral. One way to deal with this problem is to consider only integrations on the boundary $\partial\Omega$, by using the dual reciprocity method, or by using radial basis functions (cf. [2]).

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Eigenvalue equation point sources

We now consider the fundamental solutions of the eigenvalue equation $\Delta u = \lambda u$, which are known to be: • in the 3D case $\Phi_\lambda(r) = \frac{e^{r\sqrt{\lambda}}}{4\pi r}$, • in the 2D case, $\Phi_\lambda(x) = \frac{1}{2\pi}K_0(\sqrt{\lambda}r)$, if $\lambda > 0$, and $\Phi_\lambda(x) = \frac{i}{4}H_0^{(1)}(-\sqrt{\lambda}|x|)$, if $\lambda > 0$. Here K_0 is the Bessel function of the third kind, and $H_0^{(1)}$ is the first Hänkel function. Therefore, one has $(\Delta - \lambda)\Phi_\lambda = -\delta$,(where δ is the Dirac delta) and in particular, for $x \in \mathbf{R}^d \backslash \{0\}$

$$\Delta\Phi_\lambda(x) = \lambda\Phi_\lambda(x).$$

Now, let us take v to be a linear combination of point sources $\Phi_\lambda(x-y)$,

$$\tilde{u}_p(x) = \sum_{i=1}^{p}\sum_{j=1}^{n} \alpha_{ij}\Phi_{\lambda_i}(x-y_j),$$

where y_j are points placed in some admissible source set $\hat{\Gamma}$ (cf. [1]) such that $\hat{\Gamma} \subset \bar{\Omega}^c$.

If we can approximate f with

$$\tilde{f}(x) = \sum_{i=1}^{p}\sum_{j=1}^{n} \beta_{ij}\Phi_{\lambda_i}(x-y_j),$$

then it is clear that $\tilde{u}_p$ with $\alpha_{ij} = \frac{1}{\lambda_i}\beta_{ij}$ will approximate the particular solution. To justify this approximation, we will prove some theorems.

Theorem 1: Let $y_1, ..., y_n \notin \bar{\Omega}$. The functions

$$\Phi_{\lambda_1}(x-y_1), ..., \Phi_{\lambda_1}(x-y_n); ...; \Phi_{\lambda_p}(x-y_1), ..., \Phi_{\lambda_p}(x-y_n)$$

restricted to any open set Ω are linear independent.

Proof. Take

$$v(x) = \sum_{i=1}^{p}\sum_{j=1}^{n} \alpha_{ij}\Phi_{\lambda_i}(x-y_j),$$

and assume that $v(x) = 0$ on Ω. Since the fundamental solutions are analytic outside the origin, it is clear that v is an analytic function outside $\{y_1, ..., y_n\}$. Therefore we have v null on the open set Ω and by analytic continuation $v \equiv 0$ in $\mathbf{R}^d \backslash \{y_1, ..., y_n\}$. We can also write in the distribution sense,

$$v = \sum_{i=1}^{p}\sum_{j=1}^{n} \alpha_{ij}\delta_j * \Phi_{\lambda_i},$$

where $\delta_j(x) = \delta(x - y_j)$. Reordering the terms on the finite sum, we have

$$v = \sum_{j=1}^{n} \delta_j * A_j, \text{ with } A_j = \sum_{i=1}^{p} \alpha_{ij}\Phi_{\lambda_i}.$$

Since we have proved that $v \equiv 0$ in $\mathbf{R}^d \backslash \{y_1, ..., y_n\}$, we must have $A_1 = ... = A_n = 0$. Now, the functions A_j are linear combinations of the functions Φ_{λ_i} which are clearly linear independent. Thus, for every j, $\alpha_{1j} = ... = \alpha_{pj} = 0$. $\square$

By proving the linear independence, we conclude that the set of functions $\{\phi_{11}, ..., \phi_{pn}\}$,with $\phi_{ij}(x) = \Phi_{\lambda_i}(x - y_j)$, restricted to Ω, form a basis of a discrete subspace of analytic functions $Q = \langle \phi_{11}, ..., \phi_{pn} \rangle$. We will search for an approximation of f in this finite subspace Q. Considering a Hilbert space V, such that Q is a linear subspace of V, then the best approximation in Q to a function $f \in V$ is given by the projection $P_Q f$

$$P_Q f = \sum_{i=1}^{p} \sum_{j=1}^{n} P_{ij} f \ \phi_{ij}$$

where P_{ij} is the component in ϕ_{ij} of the projection of f to the subspace Q. These projection values $P_{ij}f$ can be easily obtained by solving the least squares system

$$[< \phi_{lm}, \phi_{ij} >]_{np \times np} \, [P_{ij} f]_{np \times 1} = \; [< \phi_{lm}, f >]_{np \times 1} \, .$$

Using $V = L^2(\Omega)$,we have for a regular boundary,

$$||u_P - \tilde{u}_P||_{H^2(\Omega)} \leq C||f - \tilde{f}||_{L^2(\Omega)}.$$

This means that it is sufficient to produce a good approximation in $L^2(\Omega)$ of f to ensure that the approximation of u. The question now is, can we produce a good approximation of a $L^2(\Omega)$ function by just using spaces like Q? We can do that based on a density result following [1], by assuming that the source points y_k lie on some admissible source set $\hat{\Gamma}$.

Theorem 2. Let $\hat{\Gamma}$ be an admissible source set and I an interval in $]-\infty, 0]$. The space

$$\mathbf{S}_{\hat{\Gamma}, I, \Omega} = \text{span}\{\Phi_\lambda(x - y)|_\Omega : y \in \hat{\Gamma}, \lambda \in I\}$$

is dense in $L^2(\Omega)$.

Proof. Let

$$v_\lambda(y) = \int_\Omega \alpha(x) \Phi_\lambda(y - x) \, dx \, .$$

We must see that if $v_\lambda(y) = 0$, for all $y \in \hat{\Gamma}$ and for all $\lambda \in I$ then $\alpha \equiv 0$. Let λ be fixed. We know that $v_\lambda = (\alpha \delta_\Omega) * \Phi_\lambda$,which means that v_λ verifies the Helmholtz equation with null jumps $[v_\lambda]$ and $[\partial_n v_\lambda]$ on the boundary, that is,

$$\begin{cases} (\Delta - \lambda) v_\lambda = \alpha & \text{in } \Omega \\ [v_\lambda] = 0 & \text{on } \Gamma \\ [\partial_n v_\lambda] = 0 & \text{on } \Gamma \end{cases}$$

and v_λ verifies the radiation condition at infinity. Since we assume that $v_\lambda = 0$ on $\hat{\Gamma}$, and $\hat{\Gamma}$ is an admissible source set, using a result in [1], we know that $v_\lambda \equiv 0$ in $\bar{\Omega}^c$. Thus, the exterior traces are null, i.e. $v_\lambda^+ = 0$, $\partial_n v_\lambda^+ = 0$,on Γ, and since we have no jumps on the boundary, also the interior traces are null i.e. $v_\lambda^- = 0$, $\partial_n v_\lambda^- = 0$,on Γ. Consider now any w such that $\Delta w - \lambda w = 0$ in Ω. We have by Green's formula,

$$\int_\Gamma w \partial_n v_\lambda - \int_\Gamma v_\lambda \partial_n w = \int_\Omega w \Delta v_\lambda - \int_\Omega v_\lambda \Delta w,$$

and since $v_\lambda^- = 0$, $\partial_n v_\lambda^- = 0$ on Γ, we get

$$0 = \int_\Omega w(\alpha + \lambda v_\lambda) - \int_\Omega v_\lambda \lambda w \Longleftrightarrow 0 = \int_\Omega \alpha w,$$

which means that α is orthogonal in the L^2 norm to every function w such that $\Delta w = \lambda w$.

In particular, one can choose the functions $w(x) = e^{\sqrt{\lambda}x.d}$,with $d \in \partial B(0,1)$ (plane acoustic waves). We take in particular,

$$w(x,\xi) = e^{-ix.\xi},$$

by choosing $\lambda = -|\xi|^2$ and $d = -\xi/|\xi|$. Notice that the values of $|\xi|$ are limited to the values of $\lambda \in I$, therefore if $I = [-b^2, -a^2]$, we have $|\xi| \in [a,b]$,which means that $\xi \in \bar{\mathbf{A}}_{ab} = \bar{B}(0,b)\backslash B(0,a)$. Therefore,

$$F_\alpha(\xi) = \int_\Omega \alpha(x) e^{-ix.\xi} dx = 0, \forall \xi \in \bar{\mathbf{A}}_{ab}.$$

This function F_α is the Fourier transform of $\alpha\chi_\Omega$, thus an analytic function in the whole space. Since it vanishes in the open set $\mathbf{A}_{ab}$, by analytic continuation it vanishes in the whole space. By Plancherel's formula $||\alpha\chi_\Omega||_{L^2} = ||F_\alpha||_{L^2} = 0$, and this implies $\alpha \equiv 0$. □

We have just shown that any $L^2(\Omega)$ function can be approximated by a sequence of functions of $\mathbf{S}_{\hat{\Gamma},I,\Omega}$, which means that f can be approached by a sequence of functions

$$f_k(x) = \sum_{i=1}^{p_k} \sum_{j=1}^{n_k} \beta_{ij}^{(k)} \Phi_{\lambda_i^{(k)}}(x - y_j^{(k)}),$$

each of them is in some finite subspace Q.

• The question is now to know if we want to proceed with the approximation in $L^2(\Omega)$, using the the projection defined in this space. To calculate the projection, we must calculate the integrals $< \hat{\phi}_{ij}, f >_{L^2(\Omega)} = \int_\Omega \hat{\phi}_{ij}(x) f(x) dx$,and this can mean building up a mesh and leave the features of a meshless method. Instead of considering the continuous inner product, we will consider a discrete inner product on prescribed points of Ω,

$$< \hat{\phi}_{ij}, f >_{l^2(\Omega)} = \sum_{k=1}^{m} \hat{\phi}_{ij}(x_k) f(x_k),$$

where $x_1, ..., x_m \in \Omega$ are the collocation points. Notice that the approximation obtained with this discrete l^2 inner product can be easilly controlled by checking the difference between the given f and the calculated $\tilde{f}$.

Remarks:

(i) The theoretical density result suggests to consider the location of the source points also as an unknown to the minimization problem. However this implies to consider a non linear minimization problem, instead of considering a simple least squares method. To keep the simplicity of the method, we will only consider *steady source points.* It will be a subject of future research to investigate the other possibilities, like in [4].

(ii) We have introduced the problem for the Poisson equation, but this technique can be held easily for other equations, for instance, the Helmholtz equation.

Numerical results

• First, we test the approximation of the function $f(x, y) = e^{x-y} + e^x \cos(y)$ on the unitary disc $B(0, 1) = \Omega$, using the eigenvalue point sources. We took $m = 200$ points inside Ω disposed in spiral, as shown in Figure 1 (small dots), and $n = 20$ source points outside (in circles with radius 3 and 6), as shown in Figure1 (big dots).

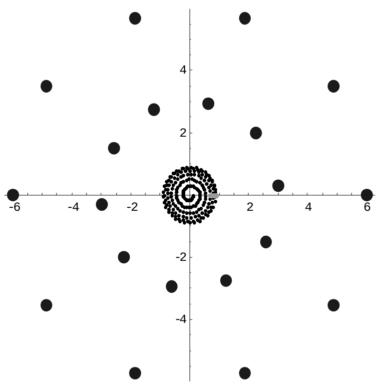

Figure1: *Collocation and source points in the unitary disc.*

In a first experiment we only consider one frequency $\lambda = -1$, and the results were not very good. In Figure 2, on the left, we plotted the true value of the function (thick line) and the approximated value (dashed line) along the points in the spiral line. We can see that the difference is usually less than 0.2, but in some cases it can be 0.8. This can be seen in the Figure 2, on the right, where the error was plotted in the all disc. The highest errors are on the boundary of the disk, suggesting that the collocation points should be placed on the border also.

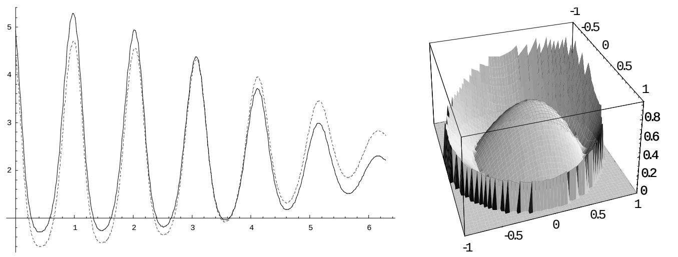

Figure 2: *Error plots with one frequency.*

In a second experiment we doubled the number of points, taking $m = 400, n = 40$. However increasing the number of points did not produce better results, the maximum errors were still to close to 1. The density result was obtained for several frequencies an we were only using one.

In a third experiment we took $m = 200, n = 20$, as in the first experiment, but we changed the values of the frequencies, allowing 4 different frequencies, $\lambda = -1, -4, -9, -16$. In fact we took $n = 5$ and $p = 4$, because we used the same 20 points as before, only changing the frequencies. The precision of the results increases, the maximum error is now 0.0476 which corresponds to a maximum 1% relative error and in almost all of the points is even less than 0.2%. In Figure 3, on the left, we plot the error on the spiral line and in Figure3, on the right, we plot the error in the whole disk.

• We now test a non smooth function $f(x, y) = \sin(3|x - y|)$ on the unitary disc $B(0, 1) = \Omega$, having a derivative discontinuity on the line $x = y$. In Figure 4, on the left, we plot the function. The approximation with $m = 200, n = 20$, and using $\lambda = -1, -4, -9, -16$ is not good, because along the

discontinuity the function is not well approximated. By taking more points, $n = 80, m = 800$, see Figure 4, on the center, we obtain a bettter result, but the error is still too high. In fact this has a perfect reasonable explanation... the approximation of non smooth functions by analytic functions is not the most appropriated one.

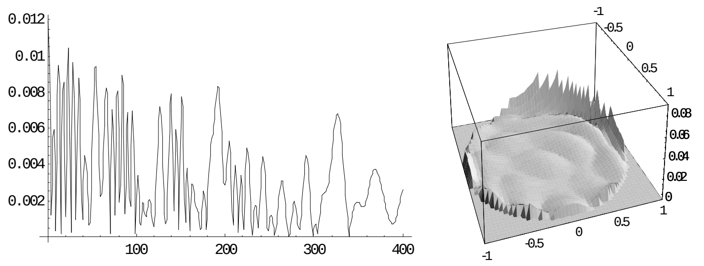

Figure 3: *Error plots with several frequencies.*

• To see the effect of the discontinuity we took the function $f(x, y) = \sin(3(x - y))$ which is only different of the previous one because it is regular (the modulus was dropped). The results are completely different and we obtain a perfectly good approximation with a smaller number of points $n = 40, m = 400$. In Figure 4, on the right, we plotted the error in the whole disk. The maximum error is less than 0.065, and in most parts of the region we have a relative error less than 1%.

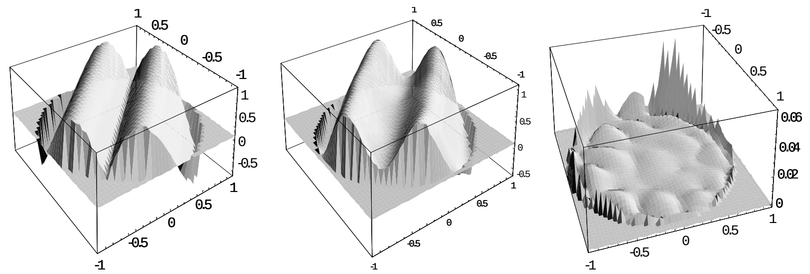

Figure 4: *Approaching a non smooth function.*

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