Density results for the Helmholtz equation and the method of fundamental solutions.

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Summary

Density results obtained on the boundary of the obstacle using some specific sets of point-sources allow a further justification on the possible location of those points while considering the implementation of the method of fundamental solutions. Numerical examples are presented for admissible and non-admissible sets of point-sources.

Introduction

The method of fundamental solutions (MFS) is a quite straightforward method that allows to approach the solution of partial differential equations whenever a fundamental solution is known. Although there are quite old references to this method, more recently a new breath was given to this approach (e.g. [1], [2], [3], [4]). The method has two antagonic features, in one hand computational implementation is quite simple and it is a meshless method, on the other hand, there are some serious difficulties with ill conditioning.

The method consists in taking an artificial domain $\hat{\Omega}$ with boundary $\hat{\Gamma}$ outside the domain Ω where the point sources will be located. In previous references this set $\hat{\Omega}$ was usually taken to include Ω, the main goal of this work is to consider other possibilities, justifying the choices of sets based on density results.

In the following, we will consider Dirichlet problems for the Laplace and Helmholtz equations, but as shown in the references, extensions to other equations and boundaries are also possible.

A density result

We will keep ourselves to the two dimensional case, however this analysis can be carried in the three dimensional case.

Let $\Omega \subseteq \mathbf{R}^2$ be a bounded connected domain with regular boundary Γ and consider the Dirichlet problem for the Helmholtz equation (or Laplace, if $k = 0$),

$$(P) \left\{ \begin{array}{ll} (\Delta + k^2)u = 0 & \text{in } \Omega \\ u = g & \text{on } \Gamma. \end{array} \right.$$

A fundamental solution Φ of the Helmholtz equation verifies $(\Delta + k^2)\Phi = -\delta$, where δ stands for the Dirac delta distribution. There are two distinct possibilities of fundamental solutions if $k > 0$,

$$\Phi(x) = \frac{i}{4} H_0^{(1)}(k|x|) \text{ or } \bar{\Phi}(x) = -\frac{i}{4} H_0^{(2)}(k|x|),$$

(where $H_0^{(1,2)}$ are Hänkel functions) but also the convex combinations $\lambda\Phi + (1-\lambda)\bar{\Phi}$ are fundamental solutions (in the case $k = 0$ they are resumed to $\Phi(x) = -\frac{1}{2\pi}\log(|x|)$, however one usually takes Φ since it verifies the Sommerfeld radiation condition, ie. one usually consider outgoing waves. This difference between the possible fundamental solutions is only significant when we deal with the exterior problem.

Given a connected bounded domain Ω, with boundary Γ we will now introduce a class of associated artificial source sets, each one of them can be used afterwards in the numerical method.

Definition 1. If one of the following conditions is verified we will say that $\hat{\Gamma}$ is an *admissible source set* associated to Γ.

i) $\hat{\Gamma}$ is the boundary of an open set $\hat{\Omega}$ that contains Ω.

ii) $\hat{\Gamma}$ is the boundary of an open set $\hat{\Omega} \subset \mathbf{R}^3 \backslash \bar{\Omega}$.

iii) $\hat{\Gamma}$ is an open set of $\partial\hat{\Omega}$, with $\hat{\Omega}$ as defined in i) or ii), but assuming also that $\partial\hat{\Omega}$ is an analytic surface.

iv) $\hat{\Gamma}$ is an open set of $\partial\hat{\Omega}$, where $\partial\hat{\Omega}$ is an infinite analytic surface that cuts $\mathbf{R}^3$ in two unbounded domains, $\hat{\Omega}$ and $\mathbf{R}^3 \backslash \hat{\Omega} \supset \Gamma$. We must ensure also that there exists an open set Q in S^2 (the unitary sphere) of directions such that $\tau\hat{x} \in B$ for all $\hat{x} \in Q, \tau > \rho$, for some $\rho > 0$. One possibility is when $\partial\hat{\Omega}$ is a plane.

Furthermore, in the case ii) we will assume that k^2 is not an eigenvalue of the interior Dirichlet problem

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in } \hat{\Omega}, \\ u = 0 & \text{on } \hat{\Gamma}. \end{cases}$$

The following density results have been established (cf. [5], [6]).

Theorem 1. If $\hat{\Gamma}$ is an admissible source set associated to Γ then the set

$$\mathbf{S}(\Gamma, \hat{\Gamma}) = \text{span}\left\{\Phi(x-y)_{|x\in\Gamma} : y \in \hat{\Gamma}\right\}$$

is dense in $L^2(\Gamma)$, provided k^2 is not a spurious frequency for the Dirichlet interior problem in Ω.

Proof. The proof in [5] can be easily extended to the case $k = 0$, the other cases were already proven in [5] and [6]. □

We now present an example to show that in other situations this density result is no longer true.

Example 1. Consider $\Omega = B(0,1)$ and $\hat{\Gamma} = \{y \in \mathbf{R}^2 : y_1 = 0,\ y_2 \in [a,b]\}$,with $|b| > |a| > 1$.

According to definition 1, this is not an admissible artificial set associated to $\Gamma = \partial\Omega$, because the analytic extension of the line $\hat{\Gamma}$ crosses Γ.

In this case density is not possible, because $\Phi(x-y) = \Phi(x^*-y)$ for $y \in \hat{\Gamma}$, where $x^* = (-x_1, x_2)$. This is true for the fundamental solutions, depending only on $|x-y|$, since it is quite obvious that $|x-y| = |(x_1, x_2-y_2)| = |(-x_1, x_2-y_2)| = |x^*-y|$.

Therefore, in this case, any function $f \in \mathbf{S}(\Gamma, \hat{\Gamma})$ will verify $f(x) = f(x^*)$ and this excludes the possibility of approaching data on Γ that is not symmetric to the y-axis. □

Remark 1. In the case of Helmholtz equation there is also a density result (e.g. [7]) stating that

$$\text{span}\left\{(e^{ikx.y})_{|x\in\Gamma} : y \in \partial B(0,1)\right\} \text{ is dense in } L^2(\Gamma)$$

(still providing that k^2 is not a spurious frequency).

Connection with the MFS

The previous result shows that if it is possible to approach the boundary condition g with a combination of point sources $\Phi(x-y)$, with $y \in \hat{\Gamma}$, and this allows to approach the solution of problem (P), because $\Phi(x-y)$ are in fact solutions of the Helmholtz equation in Ω.

In fact the density result shows that there exists a sequence of functions u_n,

$$u_n(x) = \sum_{k=1}^{n} \alpha_{n,k}\Phi(x - y_{n,k}),$$

that converges to $u_{|\Gamma} = g$ in $L^2(\Gamma)$. These functions are clearly solutions of the Helmholtz equation, and therefore the well posedness of (P) implies $u_n|_\Gamma \to u|_\Gamma \Rightarrow u_n|_\Omega \to u|_\Omega$.

We will choose a finite set of points $y_1, ..., y_m$ in $\hat{\Gamma}$ and from the definition of $\mathbf{S}(\Gamma, \hat{\Gamma})$, we now consider a finite m dimensional function space

$$S_m(\Gamma, \hat{\Gamma}) = \text{span}\{\Phi(x - y_k)|_\Gamma : y_k \in \hat{\Gamma}, (k = 1, ..., m)\}$$

We can now use the two different approaches to the MFS that lead to an approximation of u,

$$\tilde{u}(x) = \sum_{j=1}^{m} \alpha_j \Phi(x - y_j).$$

The coefficients α_j are usually calculated by:

(i) *Collocation.* Imposition of the boundary condition using m points $x_i \in \Gamma$, requiring that

$$\sum_{j=1}^{m} \alpha_j \Phi(x_i - y_j) = g(x_i).$$

by solving the $m \times m$ system $[\Phi(x_i, y_j)][\alpha_j] = [g_i]$.

(ii) *Discrete Least-Squares.* Here we can use more points $X_M = \{x_1, ..., x_M\} \subset \Gamma$, with $M > m$, and we proceed considering the minimisation of

$$\left\| \sum_{j=1}^{m} \alpha_j \Phi(x_i - y_j) - g(x_i) \right\|_{l^2(X_M)}$$

using a standard least-squares approximation on $l^2(X_M)$.

Remark 2. One other possible way, is to consider a continuous least-squares approach, using an integration over Γ. This would have the advantage of keeping ourselves to the theoretical result on $L^2(\Gamma)$, but would mean integration over the boundary... loosing some simplicity of the meshless method.

Remark 3. Both methods are also possible for the exterior problem, since the chosen fundamental solution presents the appropriated asymptotic behaviour. In this case the choice of source sets should still obey conditions similar to (i), (iii) in Definition 1, ie. $\hat{\Gamma}$ must be the boundary (or part of an analytic boundary) of a $\hat{\Omega} \subset \Omega^c$ (assuming that Ω^c is connected).

Remark 4. The ill posedness of these methods can be understood if we see them as the result of a discretisation of a first kind Fredholm integral equation, from the single layer potential representation

$$(\mathrm{L}\alpha)(x) = \int_{\hat{\Gamma}} \alpha(y)\Phi(x-y)ds_y = g(x).$$

The integral operator L is compact in $L^2(\hat{\Gamma})$ and the inversion is therefore an ill-posed problem (cf. [4]). The MFS is in some extent more general (α is not necessarily a function, it can be a sum of Dirac distributions), otherwise we would be restricted to analytic data, since Lα is an analytic function if $\alpha \in L^2(\hat{\Gamma})$.

Numerical results

We start by showing numerical results on a peanut shape boundary, considering four different types of source sets $\hat{\Gamma}_k = \{y \in \mathbf{R}^2 : y_1 = a_k,\ y_2 \in [b_k, b_k + 2\pi]\}$, with :

(i) $a_1 = 0, b_1 = 12$, (ii) $a_2 = 3, b_2 = 12$, (iii) $a_3 = 6, b_3 = 3$, (iv) $a_3 = 12, b_4 = 3$.

The peanut shape and the four source sets are plotted in figure 1. In figures 2 and 3 we present some results for the approximation of the boundary condition function, considering a collocation method of fundamental solutions, using $m = 10, 20$ and 50 collocation points (the thickest line represents the exact function, which is $\Phi((x_1, x_2) - (5,5))|_\Gamma$).

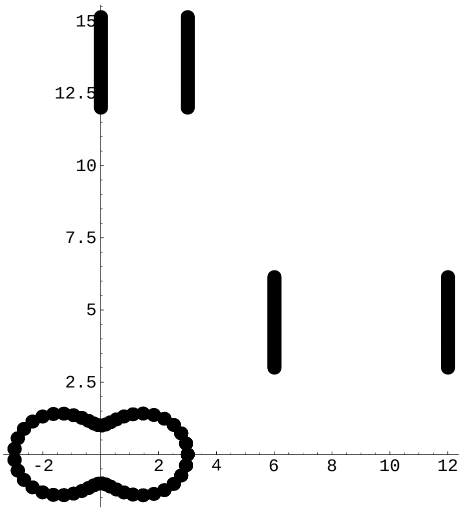

Figure 1: Peanut domain and the four different source lines.

In figure 2 one can see high instabilities for the source sets $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$, somehow expected, since the density result fails ($\hat{\Gamma}_1$ is the case presented in example 1). On the other hand, in figure 3, the high instabilities vanish for the source sets $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$, where the density result holds. This difference is not connected with the distance between the source set and the boundary.

In figures 4 and 5 we can see the same problem for source sets $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ (repectively), now using Helmholtz equation and a discrete least-squares approach. We took the wavenumber $k = 3$ and plotted the real values of the solution on the boundary (Figure 4 and 5, on the left) and on an interior line (the line on the x-axis: Figures 4 and 5, on the right). The dashed line corresponds to the approximation

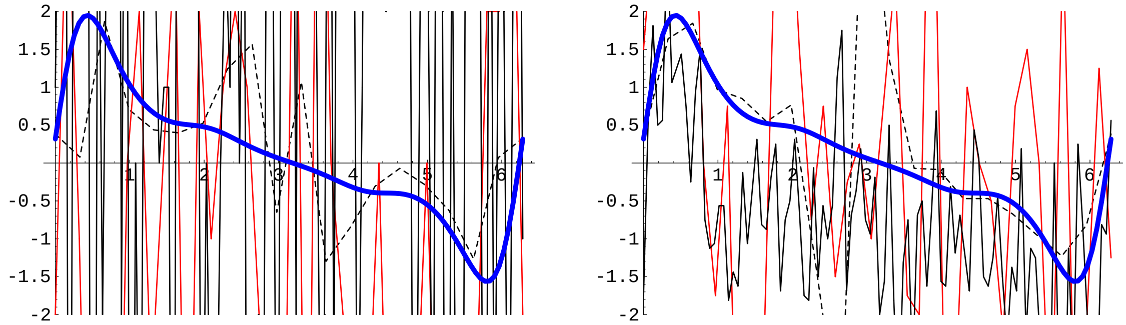

Figure 2: High instabilities for $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$.

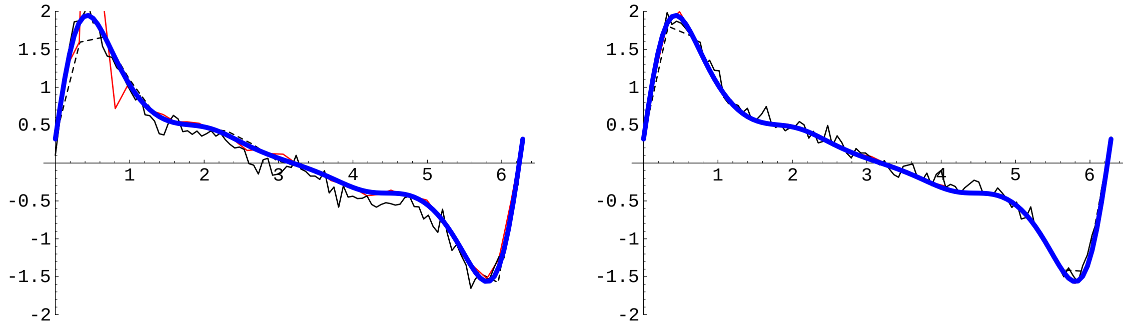

Figure 3: Approximation for $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$.

considering $m = 20$ and $M = 100$. In this case there are no high instabilities... due to the use of least-squares, but the difference is quite clear. For the source sets $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$ the results are very good, giving absolute errors inferior to 10^{-6} (...therefore we did not plot them). Finally, in figure 6, we plot the approximation in the peanut using an exterior circle $\partial B((3,3),1)$ as the location of the source points. The results are also quite good, with absolute errors inferior to 10^{-7}.

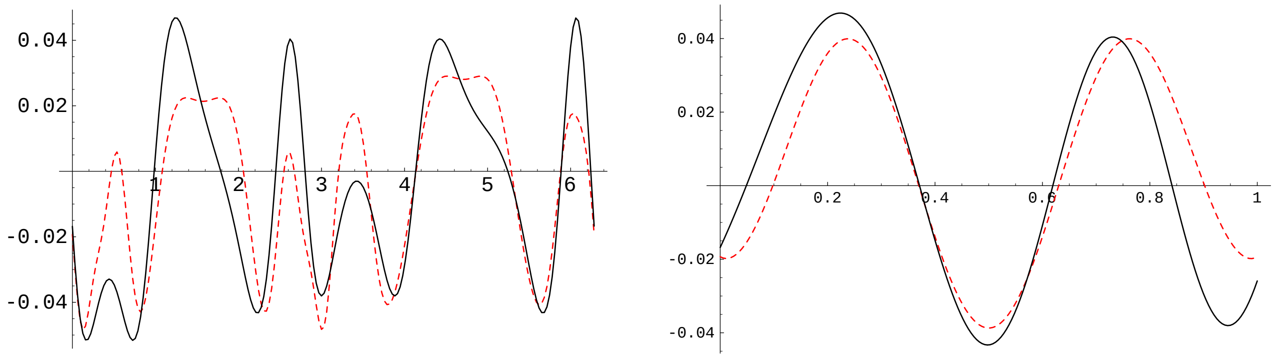

Figure 4: Helmholtz equation – MFS using least-squares on $\hat{\Gamma}_1$.

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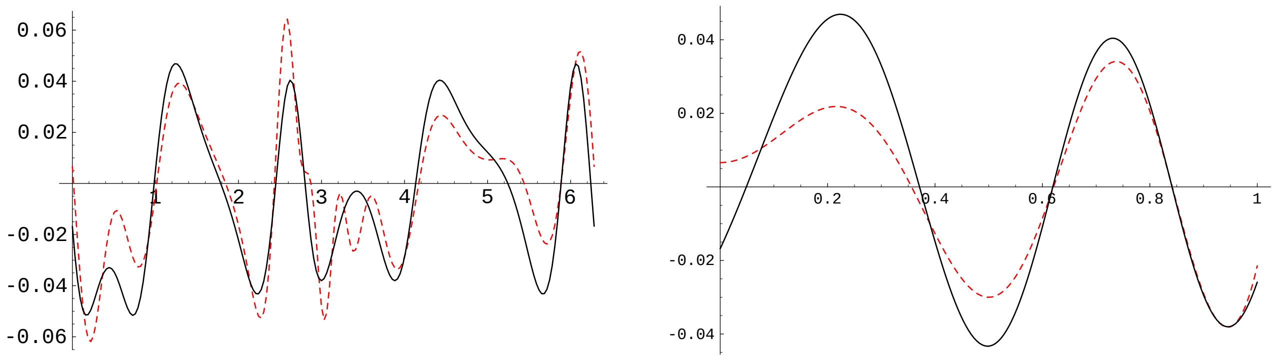

Figure 5: Helmholtz equation – MFS using least-squares on $\hat{\Gamma}_2$.

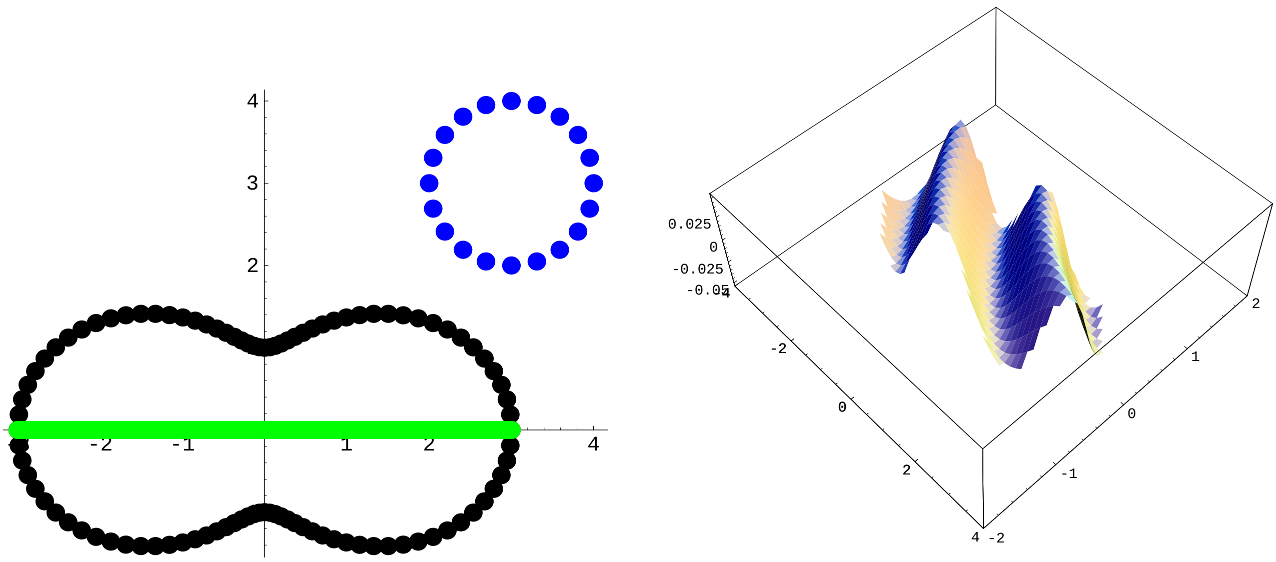

Figure 6: MFS with least-squares using an exterior circle as source set. 3D-graph of the Helmholtz solution in the peanut.

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