SCREEN DETECTION WITH NEAR FIELD MEASUREMENTS

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ABSTRACT

We consider the inverse obstacle problem of identifying sound-soft plane screens. For spherical incident waves and for quite general scatterers we use the integral representation of the scattered field and a theoretical result is established in order to distinguish plane sound-soft screens from other obstacles. The scattered field is the solution of the Helmholtz equation satisfying Dirichlet boundary conditions and the Sommerfeld radiation condition. A consequence of this result is the presentation of a criterium for the location and identification of such screens. Numerical simulations are presented to illustrate these results.

KEYWORDS

Acoustic scattering, screens, inverse problems.

INTRODUCTION

Identifying the shape of an object by means of the field pattern of acoustic waves, scattered by the object, has been studied from different points of view. The book of Colton and Kress [5] gives a good overview of the subject and applications in industry as well.

In this text we derive a theoretical result for distinguishing plane sound soft screens from other obstacles, a counterpart of the results obtained in [4] (for sound hard cracks) and in [2] (for plane incident waves and far field measurements). The main purpose is the inverse obstacle problem of identifying sound-soft plane screens and to present criteria for their location, allowing several possibilities of sets of local measurements and of incident point-sources.

In the usual framework, this problem is solved considering the measurement of the far field amplitude generated by incident plane waves (e.g. [2], [5], [7]).

ACOUSTIC SCATTERING IN THE NEAR FIELD

We consider acoustic scattering by an obstacle Γ with regular boundary in a three dimensional space. Γ can be the boundary of a round shape Ω or an admissible screen. We define *admissible screens* to be open subsets of the Lipschitz boundary of an open set in $\mathbf{R}^3$.

The problem consists of determining the solution u of the Helmholtz equation (with wavenumber $k > 0$) satisfying Dirichlet boundary conditions and the Sommerfeld radiation condition, i.e.

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in } \mathbf{R}^3 \backslash \bar{\Omega}, \\ u = -u^{inc} & \text{on } \Gamma, \\ \partial_r u - iku = o(\frac{1}{r}), & \text{when } r = |x| \to \infty, \end{cases} \quad (1)$$

(when the obstacle is a screen, $\bar{\Omega}$ should be replaced by Γ). The well posedness of this problem has been established for round shapes Ω (e.g.[5]) and for admissible screens Γ (cf. [8]).

The asymptotic expansion of the scattered wave u is given by

$$u(x) = \frac{e^{ik|x|}}{|x|} u_\infty(\hat{x}) + O(\frac{1}{|x|^2})$$

where $\hat{x} = \frac{x}{|x|}$ and u_∞ is an analytic function, called *far field amplitude*, defined on the unit sphere. The usual framework of this problem is to consider incident plane waves with direction d, $u^{inc}(x) = e^{ikx.d}$ and to measure u_∞. In [2], Alves and Ha-Duong present an overview of the developments of this framework and establish criteria to distinguish the far-field of plane screens from others.

Here we will be interested in incident point sources, given by the fundamental solution of the Helmholtz equation, $\Phi(x) = \frac{e^{ik|x|}}{4\pi|x|}$,i.e. we take $u^{inc}(x) = \Phi(x - y)$, which we will also call spherical incident wave centered on y, and, at the same time, we will consider local measurements of the scattered field.

We will consider point-sources located (centered) on an *admissible set* Σ (also called source set) defined by one of the following conditions (cf. [4]):

i) Σ is the border of an open set S that contains Γ. (see Fig. 1).

ii) Σ is the border of an open set S in $\mathbf{R}^3 \backslash \Gamma$ (k^2 is not an eigenvalue of the interior Dirichlet problem). (see Fig. 1).

iii) Σ is an open set of ∂S, with S as defined in i) or ii), assuming that ∂S is an analytic surface. (see Fig. 1).

iv) Σ is an open set of ∂S, where ∂S is an analytic surface that cuts $\mathbf{R}^3$ in two unbounded domains, S and $\bar{S}^c \supset \Gamma$. We will be mainly interested in the case when ∂S is a plane. (see Fig. 2).

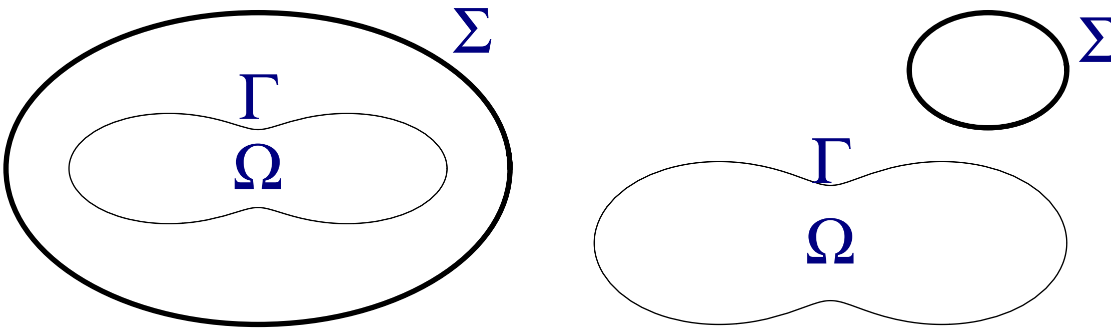

Fig. 1: The source set and the scatterer.

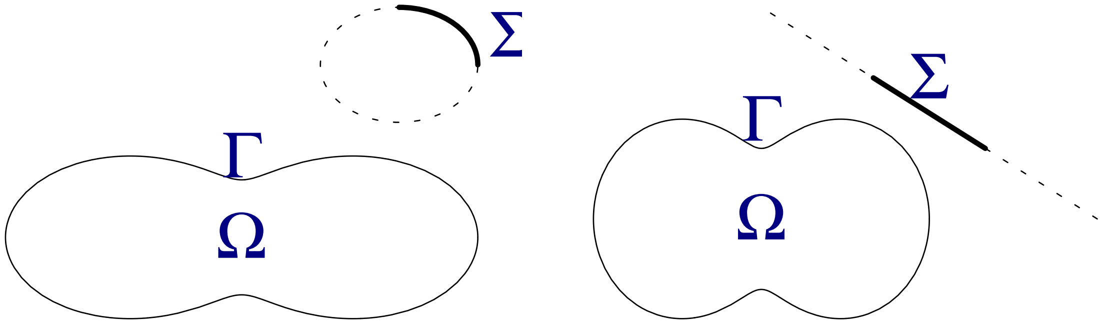

Fig. 2: Plane source set .

If we consider a source set Σ around the obstacle, for instance a sphere with sufficiently large radius, each spherical incident wave centered in y can be viewed asymptotically as an incident plane wave with direction $-\hat{y}$. Therefore, the asymptotic approach (e.g. [2]) assumes that both the source set and the observation set (the set of points where the measurements take place) are located far away from the obstacle. As a consequence other possibilities we want to consider here are excluded (see also [4]).

SOFT SCREEN DETECTION

In the following, $u(x, y)$ will denote the amplitude (measured at a point x) of the wave scattered by the object (with boundary Γ) after the incidence of a spherical wave centered on y. The measurements will be taken on an admissible set Θ which may coincide with Σ.

In the case of sound-hard obstacles we can identify a plane crack from other shapes, using spherical incident waves and near field measurements, since the scattered field vanishes on the plane of the crack (see [4]). This is the case since the scattered field can be expressed by the double layer potential

$$
\begin{aligned}
u(x,y) &= \int_\Gamma \psi(z) \frac{\partial \Phi}{\partial \mathbf{n}}(x-z)\, ds_z \\
&= \int_\Gamma \psi(z)\, \mathbf{n} \cdot \frac{x-z}{|x-z|}\left(\frac{1}{|x-z|} - ik\right)\Phi(x-z) ds_z
\end{aligned}
$$

with $\psi = [u]$, the jump of the field across Γ. For x on the plane of the crack we have $\mathbf{n} \cdot (x-z) = 0$.

In [2], by considering incident plane waves and far field measurements, a condition on the tangential derivative of the far field was obtained which was used to distinguish plane screens from other obstacles.

In the case of sound-soft obstacles the scattered field is given by the single layer potential,

$$u(x,y) = \int_\Gamma \psi(z)\Phi(x-z)\, ds_z, \tag{2}$$

with $\psi = [\partial_n u] \in H^{-1/2}(\Gamma)$, the jump of the normal derivative across Γ.

In this work, using spherical incident waves, we establish a condition on the derivatives of the near field from which we derive criteria for identification and location of sound-soft plane screens. In fact, using (2) we obtain, for some fixed vector $\mathbf{n}$,

$$\mathbf{n}\cdot\nabla_1 u(x,y) = \int_\Gamma \psi(z)\, \mathbf{n}.\nabla_x\Phi(x-z)\, ds_z, \tag{3}$$

where ∇_1 stands for the gradient with respect to the first three variables. Note that

$$\mathbf{n}\cdot\nabla_x\Phi(x-z) = \mathbf{n}\cdot\frac{x-z}{|x-z|}\left(\frac{1}{|x-z|} - ik\right)\Phi(x-z). \tag{4}$$

Thus if $\mathbf{n}\perp(x-z)$ we get $\mathbf{n}\cdot\nabla_1 u(x,y) = 0$. This is the case of plane screens if we assume that the observation point x is taken on the plane of the crack.

We will now prove that this situation occurs only for plane screens. We begin by stating a density lemma which can be proved in a similar way as for density lemmas presented in [1], [4] where such results were used to prove uniqueness for the inverse obstacle problem for spherical incident waves.

Lemma. *Let B be an open set with Lipschitz boundary such that k^2 is not an eigenvalue of the interior Dirichlet problem in B. Consider any $\gamma\subset\partial B$, and take Σ an admissible source-set. Then*

$$\mathbf{F}(\gamma) = \mathrm{span}\{\Phi(x-y)_{|\gamma} : y\in\Sigma\}$$

is dense in $H^{-1/2}(\gamma)$.

Theorem. *Consider Σ an admissible source set and Γ an admissible boundary (of a round object or a screen), such that $\Gamma\cap S = \emptyset$. There exists x^* and a vector $\mathbf{n}$ such that $\mathbf{n}\cdot\nabla_1 u(x^*,y) = 0$ for all $y\in\Sigma$, if and only if Γ is a screen on Π, the orthogonal plane to $\mathbf{n}$ defined by x^*.*

Proof. The necessary condition is clear from (3) and (4). Suppose now there is a point $w\in\Gamma, w\notin\Pi$, then there exists a neighborhood V_w of w that does not intersect Π. By the previous lemma $\mathbf{F}(\Gamma)$ is dense in $H^{-1/2}(\Gamma)$ and by the well-posedness of the Dirichlet problem (if k^2 is not an eigenvalue of the interior Dirichlet problem in Ω), we can produce an incident field u^{inc} on Γ such that there is a function ψ with support equal to V_w, and then

$$\mathbf{n}\cdot\nabla_x u(x) = \int_\Gamma \psi(z)\,\mathbf{n}\cdot\nabla_x\Phi(x-z)\, ds_z = \int_{V_w}\psi(z)\,\mathbf{n}\cdot\nabla_x\Phi(x-z)\, ds_z \neq 0 \quad (x\in\Pi\backslash\Gamma) \tag{5}$$

choosing a small V_w, because $\psi\neq 0$ in V_w and $x\in\Pi$. When k^2 is an eigenfunction of the interior Dirichlet problem in Ω, we can follow the same steps of [2] using a mixed potential representation.

Corollary. *If* Γ *is a sound-soft screen on the plane* Π *then there exist three points* y_1, y_2, y_3 *in* Σ *such that*

$$\det \mathbf{M}(x) = \det(\nabla_1 u(x, y_1), \nabla_1 u(x, y_2), \nabla_1 u(x, y_3)) = 0, \ \ (x \in \Pi \backslash \Gamma).$$

Moreover, for a such $x^* \in \Pi\backslash\Gamma$ *we have* $\dim(\ker(\mathbf{M}(x^*))) = 1$ *and therefore the plane* Π *can be identified by* x^* *and by the normal vector* $\mathbf{n}$ *solution of* $\mathbf{M}(x^*)\mathbf{n} = 0$.

Proof. The first part is a direct consequence of the necessary condition of the theorem, because we have $\mathbf{n}\cdot\nabla_1 u(x^*, y) = 0$ for all $y \in \Sigma$ which implies that the vectors $\nabla_1 u(x^*, y_1), \nabla_1 u(x^*, y_2), \nabla_1 u(x^*, y_3)$ lie on the plane Π and therefore they can not be linear independent. Suppose now that the dimension of the kernel is never equal to one. Then for arbitrary $y_1, y_2, y_3 \in \Sigma$ there exist two linear independent vectors $\mathbf{n}$, $\mathbf{m}$ such that $\mathbf{n}\cdot\nabla_1 u(x, y_i) = 0, \mathbf{m}\cdot\nabla_1 u(x, y_i) = 0$, and this implies by the sufficient condition of the theorem that Γ should be a screen on a plane Π orthogonal to $\mathbf{n}$ and to $\mathbf{m}$ which is impossible.

Remark. The reciprocity relation $u(x, y) = u(y, x)$ for all $x, y \in \mathbf{R}^3\backslash\Gamma$ implies that $\nabla_1 u(x, y) = \nabla_2 u(y, x)$, and this allows to establish a criterium where the roles in the previous results are interchanged. For instance, in the corollary, y_1, y_2, y_3 will be measurement points and x^* will be a point-source.

NUMERICAL SIMULATIONS

To derive a numerical approximation of the wave scattered by a plane screen, we use a boundary finite element method (as in [3] for the Neumann case), applying the variational formulation of the single layer potential on $H^{-1/2}(\Gamma)$ (see [6]),

$$\int_\Gamma \int_\Gamma \Phi(x - y)\psi(x)\bar{\phi}(y) ds_x ds_y = \int_\Gamma -u^{inc}(x)\bar{\phi}(x)\, ds_x. \tag{6}$$

The density ψ is approximated using a finite element method on the boundary $\Gamma \subset \mathbf{R}^2\times\{0\}$. The scattered field is then given by the single layer potential (2), using the approximation $\tilde{\psi}$.

In Fig. 3 we present the scattered field generated by a plane screen

$$\Gamma = \{(x, y, 0) : x^2 + y^2 < 1\}.$$

We remark that this is an axisymmetric problem and therefore we neglect the y axis. An incident spherical wave centered on $(3, 0, 3)$ and wavenumber $k = 6$ is considered.

To check the necessary criterium established for finding the location of the plane of the screen, we considered several situations.

Only two source points are considered (instead of three as stated in the corollary) because of the axisymmetric feature of Γ. We measure $\det(M(x))$ for $x \in \Theta$, whereΘ is a line that connects the two points x_1 and x_2, and $M(x) = (\nabla u(x, y_1), \nabla u(x, y_2))$. We notice that if no axisymmetric feature occurs we should consider Θ to be a part of a plane.

The following experiments were considered:

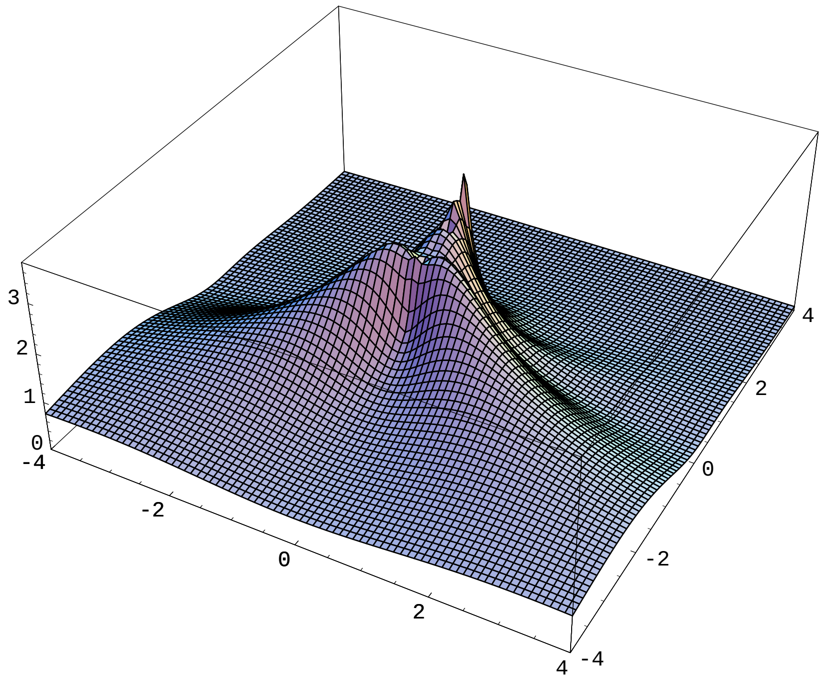

Fig. 3: Plot of $|u(x,y)|$ for fixed $y = (3,0,3), k = 6$.

i) Fig. 4
Source points: $y_1 = (-3,0,3)$ and $y_2 = (3,0,3)$.
Measurement line: $x_1 = (5,0,-5)$ and $x_2 = (3,0,3)$

ii) Fig. 6
Source points: $y_1 = (-3,0,3)$ and $y_2 = (3,0,3)$.
Measurement line: $x_1 = (-6,0,2)$ and $x_2 = (4,0,2)$

iii) Fig. 8
Source points: $y_1 = (3,0,3)$ and $y_2 = (4,0,4)$.
Measurement line: $x_1 = (2,0,4)$ and $x_2 = (2,0,-4)$

iv) Fig. 10
Source points: $y_1 = (3,0,4)$ and $y_2 = (4,0,4)$.
Measurement line: $x_1 = (2,0,4)$ and $x_2 = (2,0,-4)$

In Fig. 5, on the left, we plot the graph of $(t, |\det M(x_1 + t(x_2 - x_1))|)$ for case (i) and we confirm that $\det M(x) = 0$ for the only x on the plane of the screen. Moreover, in Fig.5, on the right, we plot the angles made by the vectors $\nabla u(x, y_1), \nabla u(x, y_2)$ and we confirm that they coincide for an angle of zero degrees which corresponds to the orientation of the crack.

The same experiment was made in the other cases. We now notice that in case (ii) (see Fig. 6) the measurement line does not intersect the line of the crack, and therefore the determinant is not null (Fig. 7, on the left), as predicted. However, the determinant can have small values near the edges of the line, since the amplitude decreases fast. When we look at the angle variation (Fig. 7, on the right), we see that the graphs are quite different and never give the angle orientation of the crack.

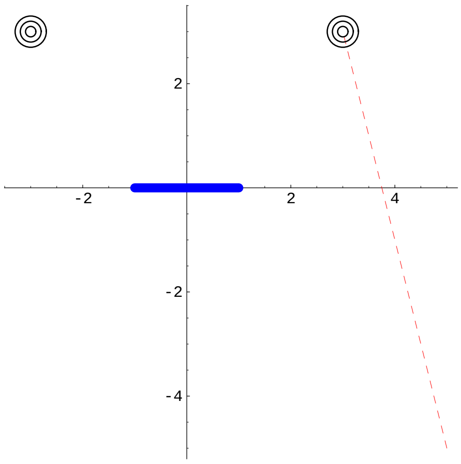

Fig. 4: Case (i). Location of the screen and of the measurement set.

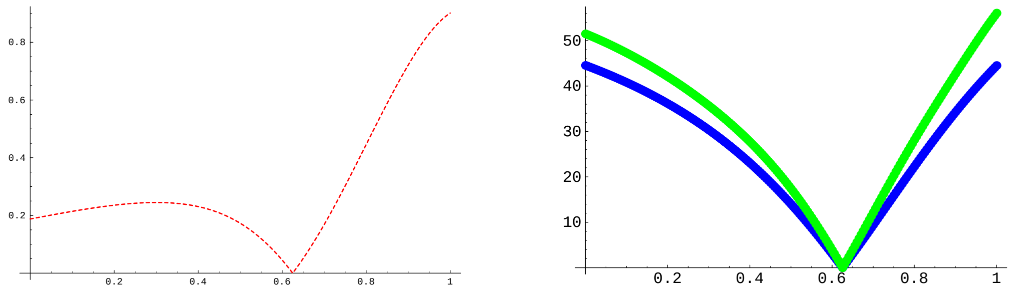

Fig. 5: Case (i). Plot of the determinant and angle variation.

In figures Fig. 8 and Fig. 10 we consider two similar experiments (cases (iii) and (iv), respectively), where the only change was in the position of a source point. In case (iii) the extension of the line that connects the two source points crosses the crack and the values of the determinant (see Fig. 9, on the left) are ten times lower than in case (iv) (see Fig. 11, on the left), and therefore the two lines of the angle variation plot almost coincide (see Fig. 9, on the right), if we compare it with the angle variation on case (iv) (see Fig. 11, on the right). This high difference can be partially justified, since in case (iv) this line extension would verify the criteria in the definition of an admissible source set location, which does not happens in case (iii). Of course, this two points do not mean much of the all source set Σ, and if Σ is not a line, other points could be chosen. However, this gives a good example on the choice of the source points.

All this experiments were made without noisy data. There is a difficulty with noisy data,

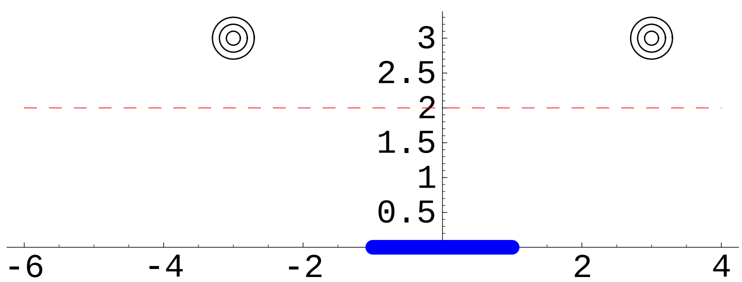

Fig. 6: Case (ii). Location of the screen and of the measurement set.

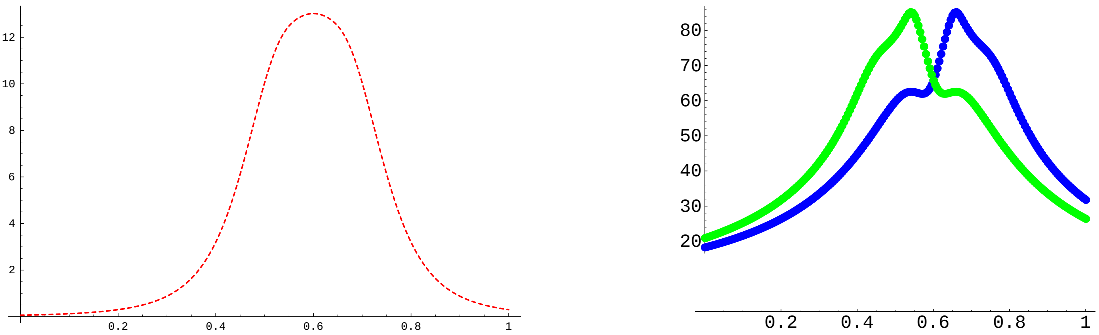

Fig. 7: Case (ii). Plot of the determinant and angle variation.

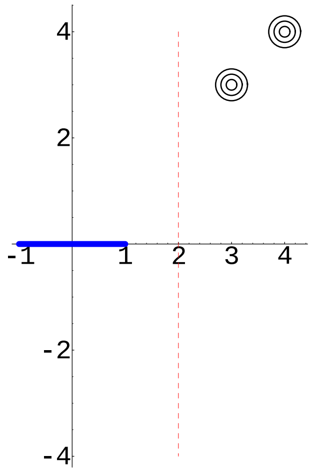

Fig. 8: Case (iii). Location of the screen and of the measurement set.

on the calculation of the derivatives, since no regularity should be expected, that can be dealt with filter techniques. In fact, a very simple approach it to use the relation

$$\nabla u(x) = < \delta_x, \nabla u >$$

since we can approach the Dirac delta on x by a mollifier, or simply by a hat function, μ_x, such that

$$\nabla u(x) \sim - < \nabla \mu_x, u > .$$

However, extra problems are expected with small values of the determinant. This analysis will be the subject of a future work.

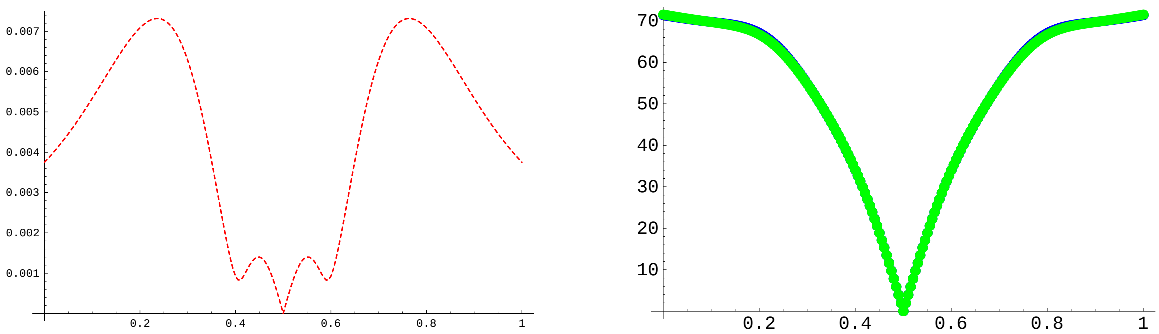

Fig. 9: Case (iii). Plot of the determinant and angle variation.

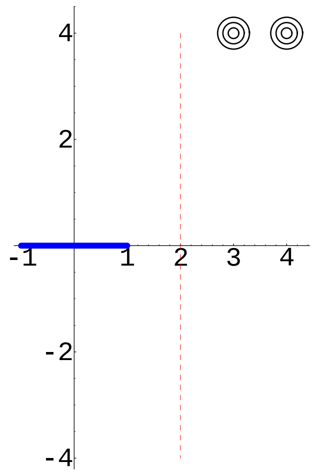

Fig. 10: Case (iv). Location of the screen and of the measurement set.

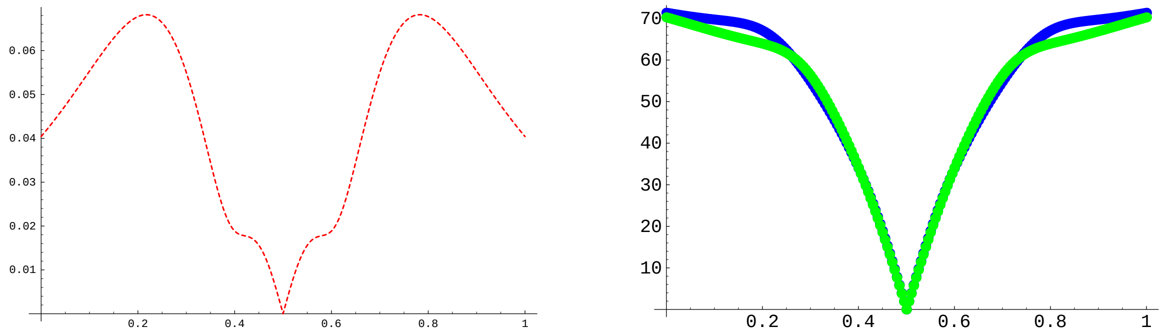

Fig. 11: Case (iv). Plot of the determinant and angle variation.

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