ON THE IDENTIFICATION OF CONDUCTIVE CRACKS

C. J. S. ALVES (1), T. HA DUONG (2) and F. PENZEL (3)

(1) *Centro de Matemática Aplicada, Instituto Superior Técnico*
Av.Rovisco Pais 1, 1049-001 Lisboa, Portugal
Email: calves@math.ist.utl.pt

(2) *Université de Technologie de Compiègne,*
BP 20529, 60205 Compiègne, France
Email: e-mail: thaduong@dma.utc.fr

(3) *Technische Universität Darmstadt, Schlossgartenstrasse 7,*
Fachbereich Mathematik, Arbeitsgruppe 12, 64289 Darmstadt, Germany,
Email: e-mail: penzel@mathematik.tu-darmstadt.de

ABSTRACT

We consider the problem of identification of a connected crack in a bounded domain. Conditions on the boundary data are presented such that the crack can be identified by the corresponding measurement. An admissible crack (or screen) is considered to be a part of a boundary of an open set with Lipschitz regularity. We show that in the case of admissible connected shapes, a single measurement is enough to determine the position and the shape of a conductive crack, or an acoustic screen.

KEYWORDS

Inverse problems, uniqueness, cracks, screens.

INTRODUCTION

Recently, results have been obtained on the identification of cracks in unbounded and bounded domains (an account of the state of the art can be found in [1]). The case of cracks in an infinite domain was first considered in [2] and in [3]. In [3] (Remark 5, Proposition 4) it was proved that the knowledge of three far field patterns generated by incident plane waves are necessary and sufficient to determine the location of a plane crack in $\mathbf{R}^3$. In this paper we shall study the problem of identification of a single crack in a bounded domain (see the references [4], [5] and [6] for recent developments). If one prescribes an infinite set of boundary data, the crack can be uniquely determined, see the citations no. 6-11 in [7]. In the paper by Friedman and Vogelius ([7]) it was proved that two measurements are necessary and sufficient to determine an insulating crack in a two-dimensional domain. More recently, new results were obtained by Alessandrini and

Diaz Valenzuela [8], and Alessandrini and DiBenedetto [9], proving that in the case of conductive cracks two measurements are also sufficient to establish uniqueness in the three dimensional case. However like in [7], the authors obtain these results using input data which are differences of two Dirac masses on the boundary, therefore excluding more regular data such as continuous or integrable functions (see also [10] where other measurements are proposed).

In this text we prove that for the case of a connected conductive crack (or acoustic screen) one measurement is enough to determine the crack if we prescribe any data that is continuous and non vanishing on the boundary.

CONDUCTIVE CRACK PROBLEMS

Throughout the paper we denote by Ω an open, bounded, simply connected set in $\mathbf{R}^{\mathbf{d}}$, (usually $d = 2$, or $d = 3$) and by Γ its boundary, which we assume to be Lipschitz. For the definition of an open set with Lipschitz boundary we refer to [11].
Let $\omega^+ \subset \Omega$ be a connected, open set with a Lipschitz boundary and $\omega^- = \Omega \backslash \overline{\omega^+}$. Let $\mathbf{n}$ be the outer normal vector-field defined a.e. on Γ or on $\partial\omega^+$ with direction into $\mathbf{R}^d \backslash \overline{\Omega}$ on Γ and with direction into ω^- on $\partial\omega^+$. We define an open submanifold of $\partial\omega^+$ to be an *admissible crack* γ. In the two dimensional case this means that any piecewise C^1 curve inside Ω is an admissible crack, and in the three dimensional case, the main restriction is on the surface orientation, excluding, for instance, the case of a Möbius strip.
For smooth manifolds γ the traces of smooth functions u for $x_0 \in \gamma$ are defined as usual by

$$u^{\pm}(x_0) = \lim_{x \in \omega^{\pm}, x \to x_0} u(x),\ \partial_n^{\pm} u(x_0) = \lim_{x \in \omega^{\pm}, x \to x_0} n(x_0) \cdot \nabla u(x).$$

If $\partial_n^+ u(x_0) = \partial_n^- u(x_0)$, we define $\partial_n u(x_0) = \partial_n^{\pm} u(x_0)$. Then each function $u \in W_2^1(\Omega \backslash \gamma)$ which satisfies $\Delta u \in L^2(\Omega \backslash \gamma)$ has traces $u^{\pm} \in H^{1/2}(\gamma)$ and $\partial_n^{\pm} u \in H^{-1/2}(\gamma)$ obtained by continuous extension of the trace operator (cf. [11]). Note that the spaces $H^{\pm 1/2}(\gamma)$ can be identified with the spaces defined by taking the restrictions of distributions from $H^{\pm 1/2}(\partial\omega^+)$ and that if $u \in W_2^1(\Omega \backslash \gamma)$ and if $\Delta u \in L^2(\Omega \backslash \gamma)$ then the jumps $[u] = u^+ - u^-$ and $[\partial_n u] = \partial_n^+ u - \partial_n^- u$ can be extended by zero to distributions in $H^{\pm 1/2}(\partial\omega^+)$.

We consider the following Dirichlet problem

$$(D) \begin{cases} -\Delta u = 0 & \text{in } \Omega \backslash \gamma, \\ u^{\pm} = 0 & \text{on } \gamma, \\ u = f & \text{on } \Gamma, \end{cases}$$

The problem (D) is well posed in $W_2^1(\Omega \backslash \gamma)$ for any given $f \in H^{1/2}(\Gamma)$, and one can define the jump $[\partial_n u] \in H^{-1/2}(\gamma)$ across the admissible crack γ, as well as the normal derivative $g = \partial_n u \in H^{-1/2}(\Gamma)$. The function f will be called the *input data* and g the *output data.*

The *inverse problem* that we consider is to determine a crack γ from the couple (f, g) of input and output data. We emphasize that here we will consider *connected* cracks, and we will prove that, using positive (or negative) input data, *one* measurement is enough to determine a conductive crack.

Difficulties with identification. First, we notice that it is rather simple to consider an example where a single measurement is not sufficient to locate a crack. In fact, just consider the input data $f(x_1, x_2) = x_1 x_2$, and any crack located on the x_1 or on the x_2

axis. It is clear that $u(x_1, x_2) = x_1 x_2$ is a solution of problem (D), no matter what set Ω is considered, as long as it contains the cracks. Since the solution is the same, this input data does not allow to distinguish between two different conductive cracks if they are located in the axes.
Moreover, supose that the two different cracks are both located in the x_1 axis, if the experiments are made using input data $f(x_1, x_2) = (x_1 - c)x_2$, where c is any constant, the solution is $u(x_1, x_2) = (x_1 - c)x_2$, no matter where the cracks are, since we have $u(x_1, 0) = 0$, for all x_1. This shows that even an infinite number of measurements may be not sufficient to identify the crack, if the the input data is not well chosen.

In the previous examples the input data changes sign, and we will now prove that if we provide input data that does not change sign, only one measurement will be enough to identify any conductive crack. To do this we begin by proving a crucial lemma that connects the support of the jump $[\partial_n u]$ with the crack itself. Notice that in the previous examples this support was void, because the solution was in fact analytical inside the all domain Ω.

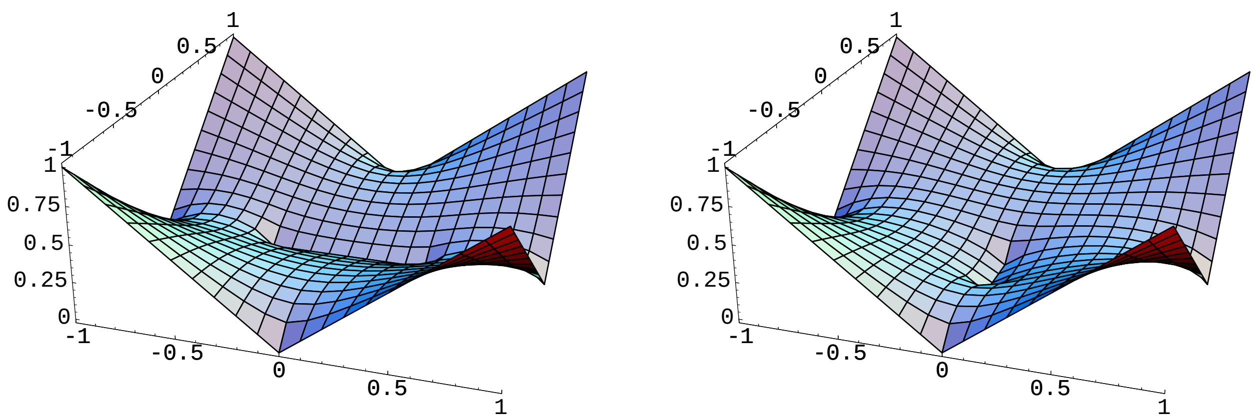

Figure 1: 3D profiles of the solution in two different cracks. One can see the discontinuities of the normal derivative on the crack.

Under the hypothesis of positiveness on Γ it is clear, by the maximum principle, that no analytical solution would be possible. It is sufficient to change the sign of f, considering $f(x_1, x_2) = |x_1 x_2|$ to see that the support of the $[\partial_n u]$ coincides with the crack. Two simple experiments with a finite difference method are presented in Fig.1. In both cases we considered $\Omega =]-1, 1[^2$. The plot on the left shows the results for a crack $\gamma_1 = [-\frac{1}{2}, \frac{1}{2}] \times \{0\}$, and the one on the right was made for a smaller crack $\gamma_2 = \{0\} \times [-\frac{1}{2}, 0]$, orthogonal to γ_1. In both cases it is clear the discontinuity of the normal derivative along the crack, which shows the relation between the support of $[\partial_n u]$ and the crack itself. In Fig.2, on the left, we plot out the difference between the output data $\partial_n u$ on the edges of the square for γ_1. It is clear that a 90 degrees rotation would produce a similar result, concerning a crack $\gamma_3 = \{0\} \times [-\frac{1}{2}, \frac{1}{2}]$, therefore it is this difference in Γ, the edges of the square $]-1, 1[^2$, that allows the distinction between an horizontal or a vertical crack. It is worth noting that this difference is relatively small, in this case it is about 8% of the absolute values measured, and this is probably due to the fact that we impose $f = 0$ on the crack lines.
More clear is the distinction between γ_2 and γ_3, as one can see in Fig.2, on the right. The dashed line shows the information on the edges which are parallel to the crack (notice that only one line is needed since the results turn to be the same either on $\{-1\} \times [-1, 1]$ or on $\{1\} \times [-1, 1]$). One can see that the difference is more significant on the right, which

corresponds also to the difference between γ_2 and γ_3 in what concerns the x_2 axis. The thick line shows the result on the $[-1,1] \times \{1\}$ edge, where the difference is larger, and the the thiner one on the $[-1,1] \times \{-1\}$ edge, where the difference is smaller. This may still be explained to the fact that the difference between γ_2 and γ_3 can be resumed to the segment $\{0\} \times [0, \frac{1}{2}]$ which is closer to the $[-1,1] \times \{1\}$ edge.

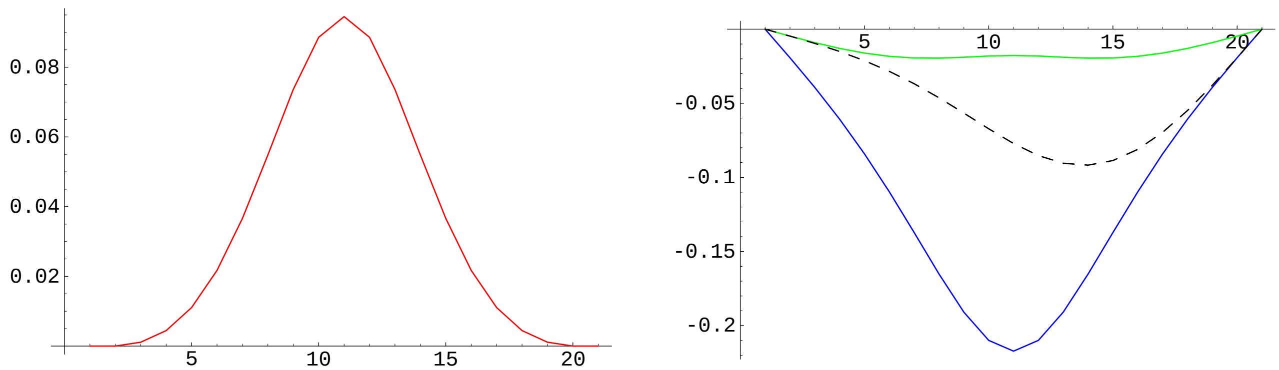

Figure 2: The difference between the boundary profiles of $\partial_n u$ of the cracks γ_1 and γ_3 (on the left), and of γ_2 and γ_3 (on the right).

Lemma 1. *Let u be the solution of problem (D). If γ is an admissible crack and we consider $f \geq 0, f \neq 0$ on Γ, then supp$([\partial_n u]) = \gamma$.*

Proof. Suppose supp$([\partial_n u]) = \gamma_1 \neq \gamma$, then $\gamma_0 = \gamma \backslash \gamma_1$ has a non void interior with respect to the topology of γ and $\gamma_0 \cap (\Omega \backslash \gamma_1) \neq \emptyset$.
Since in γ_0 we have $[\partial_n u] = 0$ and $[u] = 0$ (in fact $u = 0$), one can deduce that $\Delta u = 0$ in $\Omega \backslash \gamma_1$. Thus, the maximum principle proves that there can not be a point x_0 in $\Omega \backslash \gamma_1$ such that $u(x_0) = 0$. That contradicts the fact that u is null on γ_0. $\square$

Remark 1. This support lemma also holds for *non connected* cracks and can be stated even for more general cracks with bifurcations, however we can not avoid the orientation of the crack, since otherwise we would not be able to define $[\partial_n u]$.

Theorem 1. *Let γ_1, γ_2 be two admissible connected cracks such that, for an input data $f \geq 0, f \neq 0$, one has $\partial_n u_1 = \partial_n u_2 = g$, where u_1, u_2 are solutions of (D) for γ_1 and γ_2 (respectively). Then $\gamma_1 = \gamma_2$.*

Proof. We consider the open set $\Omega \backslash (\gamma_1 \cup \gamma_2)$. We define Ω_c to be the connected component of $\Omega \backslash (\gamma_1 \cup \gamma_2)$ with $\Gamma \subset \partial \Omega_c$ (there is only one component in this situation because $\gamma_1, \gamma_2 \subset \Omega$), and we define $\Omega^* = \Omega \backslash \bar{\Omega}_c$ (see Fig. 3).

We have $v = u_1 - u_2 = 0$ in Ω_c because of Holmgren's theorem, because $u_1 = u_2 = f$ and $\partial_n u_1 = \partial_n u_2 = g$ on Γ implies $v = 0$ in a neighborhood of Γ inside Ω_c and by analytical extension $v = 0$ on Ω_c.

i) Suppose $\Omega^* = \emptyset$. This is the case when $\Omega \backslash (\gamma_1 \cup \gamma_2) = \Omega_c$ is connected.

In this case $v = 0$ on $\Omega \backslash (\gamma_1 \cup \gamma_2)$, therefore $[\partial_n u_1] = [\partial_n u_2]$ on γ_1 and on γ_2. Outside γ_2 we know that u_2 is analytic, meaning that $0 = [\partial_n u_2] = [\partial_n u_1]$ on $\gamma_1 \backslash \gamma_2$ and therefore supp$([\partial_n u_1]) = \gamma_1 \cap \gamma_2$. Using the Lemma we know that supp$([\partial_n u_1]) = \gamma_1$ and conclude that $\gamma_1 = \gamma_1 \cap \gamma_2$. The same argument gives $\gamma_2 = \gamma_1 \cap \gamma_2$, thus $\gamma_1 = \gamma_2$.

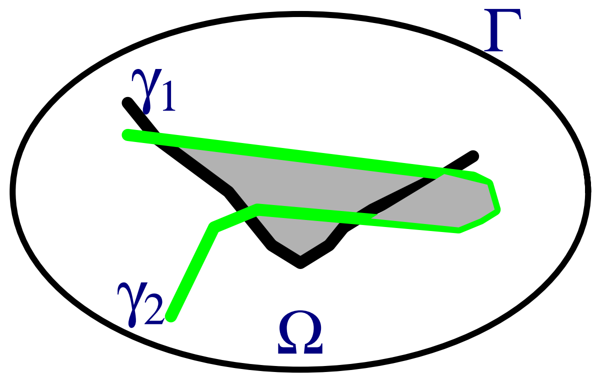

Figure 3: The filled area denotes the Ω^* set. This area can be crossed by γ_1 defining connected open components. One of this components will be the open set ω.

ii) Suppose $\Omega^* \neq \emptyset$. Let $\gamma_1^* = \overline{\Omega}^* \cap \gamma_1$, i.e. a part of γ_1 that divides Ω^* in open connected components and we take ω to be one of that components (if $\gamma_1^* = \emptyset$ this means that $\omega = \Omega^*$). Since $\partial\omega \subset \gamma_1^* \cup \partial\Omega^*$, this means that $\partial\omega = \gamma_1^* \cup (\partial\Omega^* \cap \gamma_1) \cup (\partial\Omega^* \cap \gamma_2)$ and the part $\gamma_2^* = \partial\Omega^* \cap \gamma_2$ can not be void (otherwise $\partial\omega \subset \gamma_1$).

Now, since $u_1 = u_2$ on Ω_c we have $u_1 = u_2 = 0$ on $\gamma_2^* \subset \partial\Omega_c$ and the condition $u_1 = 0$ on γ_1 imply $u_1 = 0$ on $\partial\omega$. By uniqueness of the interior Dirichlet problem $u_1 = 0$ in $\omega \subset \Omega\backslash\gamma_1$ and therefore by analyticity $u_1 = 0$ in $\Omega\backslash\gamma_1$ which implies $f = 0$. This contradicts the hypothesis and therefore $\Omega^* = \emptyset$, which brings us to case i). □

Remark 2. An extension of the proof of this theorem to non connected cracks is under current research.

SCREEN IDENTIFICATION

We now extend this results for the Helmholtz equation, concerning the identification of acoustic screens. The proof of the Theorem 1 can follow the same steps, however, the proof of the support lemma must be different, since the maximum principle is no longer available!

We now have

$$(H)\begin{cases} -(\Delta + k^2)u = 0 & \text{in } \Omega\backslash\gamma, \\ u^{\pm} = 0 & \text{on } \gamma, \\ u = f & \text{on } \Gamma, \end{cases}$$

where $k > 0$ is the wavenumber. Our proof of the support lemma will have two restrictions: (i) γ must be connected, and (ii) $0 < k < k_D$. Here k_D is the smallest eigenvalue of the interior Dirichlet problem in Ω.

Lemma 2. *Let u be the solution of problem (H). Suppose $0 < k < k_D$ and that γ is an admissible connected crack. If we consider $f > 0$ on Γ, then supp$([\partial_n u]) = \gamma$.*

Proof. Suppose again that supp$([\partial_n u]) = \gamma_1 \neq \gamma$, then $\gamma_0 = \gamma\backslash\gamma_1$ still has a non void interior with respect to the topology of γ and $\gamma_0 \cap (\Omega\backslash\gamma_1) \neq \emptyset$.
Since in γ_0 we have $[\partial_n u] = 0$ and $[u] = 0$, one can deduce that $(\Delta + k^2)u = 0$ in $\Omega\backslash\gamma_1$. Therefore γ_0 must be a part of a level line, and therefore it has an analytical extension, $\tilde{\gamma}_0$ where $u = 0$. It is clear that $\tilde{\gamma}_0$ can not cross Γ because we assume $f > 0$. Thus $\tilde{\gamma}_0$

must intersect γ, and since γ is connected this intersection defines an interior open set ω, with border $\partial\omega \subset \gamma \cup \tilde{\gamma}_0$, with Dirichlet boundary conditions $u = 0$. Since we suppose that $k < k_D$ and since $\omega \subset \Omega$, by the strong monotonicity property of the eigenvalues we conclude that k is not an eigenvalue of the Dirichlet problem in ω, and therefore $u = 0$ in ω. The result now follows immediately, since by analytic continuation, $u = 0$ in $\Omega\backslash\gamma$, because $\omega \subset \Omega\backslash\gamma$, and this contradicts $f > 0$. $\square$

Remark 3. This proof can not be extended to non connected cracks. Consider the situation where γ is defined by three components $\gamma_a = [-3,-1]\times\{1\}$, $\gamma_b = [1,3]\times\{1\}$, $\gamma_b = \partial B(0,2)\cap \mathbf{R}\times\mathbf{R}^-$. Suppose now that the support of $[\partial_n u]$ is $\gamma_1 = \gamma_a \cup \gamma_b$. Therefore $\gamma_0 = \gamma_c$, and the analytic extension of the circle intersects γ_1 in the two disjoint parts γ_a and γ_b without defining an interior open set ω, since the analytic extension can not cross γ_1 (see Fig.4).

Remark 4. The restriction on k is merely for the sake of uniqueness in ω. An extension of the proof to any k is under current research.

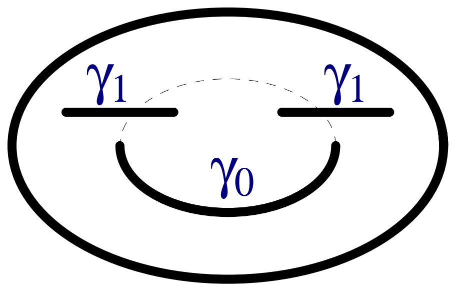

Figure 4: In the case of non connected screens, it is not possible to ensure the existence of ω, as one can see in this example, since the analytical extension of γ_0 crosses γ_1 in two non connected components.

Theorem 2. *Suppose $0 < k < k_D$ and let γ_1, γ_2 be two admissible connected cracks. If for an input data $f > 0$, one has $\partial_n u_1 = \partial_n u_2 = g$, where u_1, u_2 are solutions of (H) for γ_1 and γ_2 (respectively), then $\gamma_1 = \gamma_2$.*

Proof. Immediate consequence of the proof of Theorem 1 and of Lemma 2. $\square$

REFERENCES

[1] Isakov, V. (1998). *Inverse Problems for Partial Differential Equations.* Springer, New York.
[2] Kress, R. (1995) *Math. Meth. Appl. Sci.* 18, 267.
[3] Alves, C. J. S. and Ha-Duong, T. (1997) *Inverse Problems* 13, 1161.
[4] Bryan, K. and Vogelius, M. (1992) *SIAM J. Math. Anal.* 23, 950.
[5] Andrieux, S. and Ben Abda, A., (1996) *Inverse Problems* 12, 553.
[6] Eller, M. (1996) *Inverse Problems* 12, 395.
[7] Friedman, A. and Vogelius, M. (1989) *Ind. Univ. Math. J.* 38, 497.
[8] Alessandrini, G. and Valenzuela, A. D. (1996) *SIAM J. Contr. Opt.* 34, 913.
[9] Alessandrini, G. and DiBenedetto, E. (1997) *Indiana Univ. Math. J.* 46, 1.
[10] Kubo, S. (1991). In:*Inverse Problems in Engineering Sciences*, pp. 52-58; M. Yamaguti (Ed.), Springer, New York.
[11] Grisvard, P. (1985). *Elliptic Problems in Nonsmooth Domains.* Pitman, London.