An echelon inventory based single stage cost function for a two station tandem system

José A. A. Moreira Carlos F. G. Bispo
Instituto de Sistemas e Robótica – Instituto Superior Técnico,
Technical University of Lisbon, Portugal

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Abstract

We propose an alternative cost accounting function for inventory control problems on a make to stock setting. Our proposal is based on observing that the traditional holding and backlog parameters introduce some odd short term distortions on the inventory state space. Our single stage cost function accounts for echelon inventories and possesses a pair of cost parameters for each echelon inventory variable, depending on whether it is positive or negative.

With the modified cost accounting function we study a two station tandem system producing a single product and investigate how it compares with the performances obtained with the usual single stage cost function. The results available so far show that the optimal policies approach a multi-echelon base stock structure for each machine. Also, the service levels achieved are better under the modified function without increasing the levels of finished goods inventory.

1 Introduction

The traditional cost accounting functions for inventory control assign holding costs for positive inventories along the production line and penalty costs for negative end product inventories only. It is believed that costs incurred during production are directly related to the amount of inventory circulating in a production system more than to the impact inventories have on the ability to recover from demand occurrences.

The inventory control theory has its origins in very simple systems where this principle of holding costs for positive inventory and backlog costs for negative inventory has no ambiguous interpretation nor assignment. At the bottom of the scale of simplicity is the early model of [9], known these days as the classic *Economic Order Quantity (EOQ)* model, which was made popular by [16]. This model considers the single stage production of a single product for a production system facing a deterministic and constant demand over a long time interval. More complex models have been developed since, mainly after the advent of Dynamic Programming [2]. In many of the early models the key concept of *base stock* policies has created such a stronghold that almost all the literature on inventory control addresses the issue of showing its optimality or, when this is impossible, proposes heuristic policies that rely on similar principles.

Another classic example is the famous *newsboy* problem for a single period, single product, single production stage, and random demand. Here there is a unit holding cost rate charged for holding stock at the end of the period, and a unit shortage penalty cost rate for any unsatisfied demand. Also, there exists a unit ordering cost rate for the amount acquired at the beginning of the period. The optimal policy was shown by [1] to be of the base stock type, that is, there is an amount to order up to, given that the initial inventory is under this amount. Although very simple, this system and the previous one constitute the basic building blocks of inventory control.

For single machine and single product systems, base stock policies have been shown to be optimal in a variety of settings. Single period problems, multiple period finite horizon problems, and multiple period infinite horizon problems all exhibit the same structure for the optimal control policy in situations where there is no bound on the machine capacity, [10, 12]; for deterministic capacity bounds, [7, 8]; or even for stochastic capacity bounds, [4].

For multiple machines in series and a single product, [5] show that base stock policies are optimal in terms of *multi-echelon inventory*. That is, they define multi-echelon inventory of a given machine as the sum of inventory from that machine all the way down the production system to the end product inventory. For each of these echelon inventories there is a critical number to order up to for finite horizon problems. Their machines have no capacity bounds. This result was extended to the infinite horizon case by [6].

There are other models for which a base stock policy has been shown to be optimal. However, for cases of multiple capacitated machines in series, producing single or multiple products, except for the work reported in [13] little is known about the optimal policies.

As we said earlier, much of the effort has been put into characterizing the structure of the optimal policies but little has been done regarding questioning whether the usual cost accounting system is adequate. There are some alternative cost functions, but all of them rely explicitly on the concept that the bigger intermediate inventories grow the more penalized they are and that only negative finished goods inventory incurs some penalty cost.

In this paper we start by characterizing the essential issues of the classic single stage cost function and the optimal policies derived under this accounting scheme in Section 2. Next, we observe some distortions introduced by the classic cost and, based on those observations, we propose a *modified* cost function in Section 3. Then, in Section 4 we study the structure of optimal policies obtained with this modified cost function, comparing them with the *classic* cost function policies for a production system consisting of two machines in series, having two buffers, and subject to a single product random demand for finished goods. Finally, in Section 5 we present some conclusions.

2 Classic single stage cost function

Take a two station tandem production system with an internal buffer and an output buffer producing discrete parts to stock. Machine 2 processes raw material and places parts in the intermediate buffer that feeds machine 1. Machine 1 takes parts from the intermediate buffer and places the finished parts in the output buffer. The external random demand depletes this output buffer. Demand that cannot be immediately satisfied is backlogged for replenishment at a later period. We will assume a Markov model for our setting. That is, the processing times of both machines follow exponential distributions and the demand interarrival intervals are also exponentially distributed. Each machine processes only one part at a time and there are no simultaneous occurrences of demand.

Our model is no different from that of [15], except for our numbering of machines and buffers. We will use much of their model as the basis for the classic and modified cost function comparative study presented in this paper.

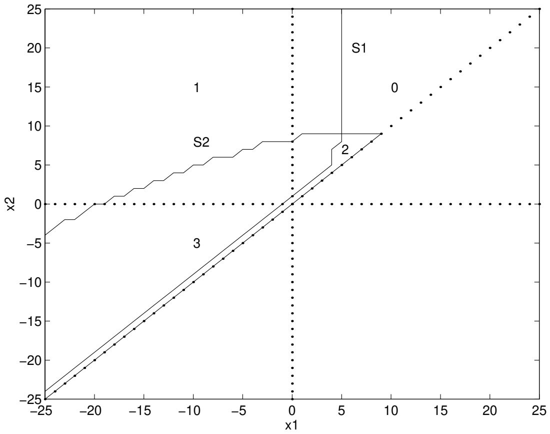

Figure 1: Optimal switching curves for the classic cost function.

In [15] the single stage cost function is defined as

$$g(I_1, I_2) = h_2 I_2 + h_1 I_1^+ + p_1 I_1^-, \tag{1}$$

where I_1 denotes the finished goods inventory, I_2 denotes the intermediate inventory, I^+ denotes $\max\{0, I\}$, I^- denotes $\max\{0, -I\}$, h_i is the unit holding cost for stage i, with $i = 1, 2$, and p_1 is the unit backlog cost. This is what we call the *classic single stage cost function* where the explicit dependence on time is omitted to avoid a heavy notation. Representing the state of the system in terms of the echelon inventories, (1) becomes

$$\hat{g}(x_1, x_2) = h_2(x_2 - x_1) + h_1 x_1^+ + p_1 x_1^-, \tag{2}$$

where $x_1 = I_1$ and $x_2 = I_1 + I_2$ are the echelon inventory variables for the same system.

Given this single stage cost, the overall infinite horizon discounted cost can be defined as

$$J(t_0) = \int_{t_0}^{\infty} e^{-\beta t} \hat{g}(x_1(t), x_2(t)) dt, \tag{3}$$

where $\beta > 0$ is the discount factor. By transforming the continuous time problem into a discrete time one, it is possible to write a Dynamic Programming recursion in order to compute the optimal policy for this type of system by means of the value iteration algorithm [3, 15]. The optimal policy for this system is characterized by two switching curves, one for each machine. Typical curves are displayed in Fig. 1, for a 90% load and non-identical machines, where we use the echelon inventories as variables of the problem. These curves are also similar to those presented in [14], which were determined for a discrete time and deterministic demand rate inventory problem with failure prone machines. In [14] the authors call "surplus variables" what we denote by echelon inventory variables.

From Fig. 1, whenever the system state lies in region 0, no machine should work; in region 1, only machine 1 should work; in region 2, only machine 2 works; and both machines work in region 3. In [15] the authors characterize the structure of the curves S_1 and S_2 based on the fact that this system is *transition monotone.*

By observing the structure of the switching curves, a number of observations can be made. Clearly, the optimal policy for each machine is not a base stock policy in terms of its local inventory. Only machine 1 comes very close to it. Also, the optimal policy for each machine is not a base stock policy in terms of echelon inventories. Again, only machine 1 comes close to it for obvious reasons.

The optimal policy for machine 2 is such that as the backlog increases in absolute value, there appears to exist a base stock level in terms of local inventory, given that the slope of S_2 approaches one. Once the backlog drops in absolute value and x_2 becomes positive, there appears to exist an optimal base stock in terms of echelon inventory, given that the slope of S_2 approaches zero. So, one possible sub-optimal policy would be to approximate the two switching curves by two curves like those displayed in Fig. 2. This has been done by other authors. A good example is the so-called *two-boundary control policy* of [14], but this is not the central issue of this paper.

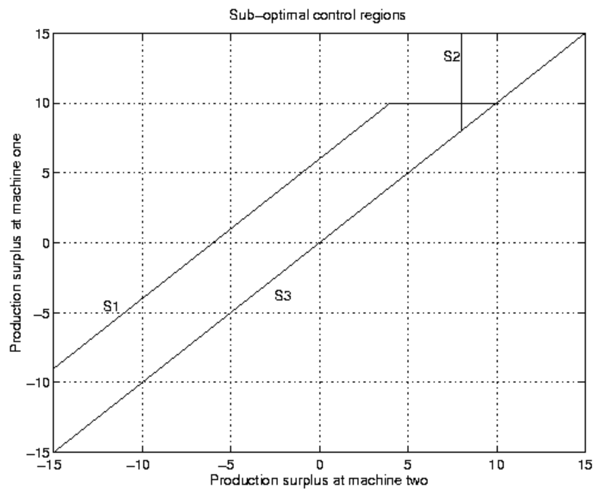

Figure 2: Sub-optimal switching curves for the classic cost function.

3 Modified single stage cost function

The motivation of our approach has to do with the quest for reasons why we lose the nice base stock structure once we put two capacitated machines in line as in [15] but keep it if the machines have no capacity bounds, as in [5].

One first observation concerns the fact that the classic single stage cost does not seem to have the appropriate monotone behavior in some regions of the state space. Take, for instance, a production system that described above with $I_1^1 = -10$ and $I_2^1 = 0$. Take the same system with $I_1^2 = -10$ and $I_2^2 = 1$. We use the superscript to differentiate between the two situations. According to the classic single stage cost function the first state costs less, given that only $10p_1$ will be incurred, whereas the second state incurs an additional h_2 due to the fact that there is inventory sitting in the intermediate buffer.

However it does not seem appropriate to consider the second state as being worse than the first one, given that this second state is in an immediate better position to recover from its backlog than the first state. It looks like we should account for costs in such a way that the first state is worse than the second state. Therefore, if we incur costs for echelon inventories rather than for local inventories, we may be able to distinguish from, say, a units of inventory between machine 2 and 1 depending on whether it occurs in a deep backlog situation or not.

So, we propose the following single stage cost function

$$\tilde{g}(x_1, x_2) = h_2^{'} x_2^+ + p_2^{'} x_2^- + h_1^{'} x_1^+ + p_1^{'} x_1^-, \tag{4}$$

where we introduce a new parameter, $p_2^{'} \geq 0$, that accounts for the fact that the overall echelon inventory seen from machine 2 may be negative and therefore a penalty cost should be incurred.

With this type of cost function the single stage cost of the previous two states would have a more convenient monotone behavior. The first state would incur a cost of $10p_2^{'} + 10p_1^{'}$ and the second state would incur a cost of $9p_2^{'} + 10p_1^{'}$, which is lower, reflecting the fact that system two is in a better position to recover from the present backlog.

Naturally, one could argue that as we look for the optimal policy by means of a dynamic programming algorithm, with the classic single stage cost function the effect of having no inventory on hand when the system is in deep backlog will eventually show up. Then the switching curves will be such that a compensation will eventually appear. In fact this is the case, and after a few iterations the cost of the first state will be higher than the cost of the second.

Note that any real valued function f can be expressed as $f^+ - f^-$. We can use this property to express (2) in terms of the modified function variables

$$\begin{aligned} \hat{g}(x_1, x_2) &= h_2(x_2 - x_1) + h_1 x_1^+ + p_1 x_1^- \\ &= h_2(x_2^+ - x_2^- - (x_1^+ - x_1^-)) + h_1 x_1^+ + p_1 x_1^- \\ &= h_2 x_2^+ - h_2 x_2^- + (h_1 - h_2) x_1^+ + (p_1 + h_2) x_1^-. \end{aligned} \tag{5}$$

Given that usually $h_2 \leq h_1$ and all coefficients are non-negative, it turns out that the classic single stage cost function deals with negative echelon inventories on the second machine by compensating for their existence with a negative cost parameter.

Furthermore, the above equation shows that in the classic single stage cost function we already make use of the modified single stage cost with a particular choice of those parameters. That is,

$$\begin{aligned} h_2^{'} &= h_2 \\ p_2^{'} &= -h_2 \\ h_1^{'} &= h_1 - h_2 \\ p_1^{'} &= p_1 + h_2. \end{aligned} \tag{6}$$

So the question here is, among others, whether it makes sense to have a negative coefficient for negative values of echelon inventory seen from machine 2, as in the classic function, or if it makes more sense to have a set of four non-negative cost coefficients that account not only for immediate, or short term, impact of the existing inventory levels, but also for the medium term impact of those variables.

What remains to be seen is whether there is a significant advantage in using one model as opposed to another. In the next section we present some preliminary data that can be used as a term of comparison between the two alternative functions.

4 Modified versus Classic

Once we define the new format of the single stage cost function, the first thing that needs to be addressed is how to establish means of comparison between the modified cost function and the classic. Observing (6) it is not clear what should be the value of the various parameters of (4) that can fairly compare with the parameters of (2).

The only thing we know for certain is that $p_2^{'}$ should be a positive parameter. Therefore, we generated the optimal policies for different systems moving from the classic single stage cost function to the modified function solely by means of increasing the value of $p_2^{'}$ from $-h_2$ up to some positive value.

In Fig. 3 we present the switching curves for the classic (left plot) and the modified function (right plot), both obtained for 80% load and machines with the same production rate. As can be clearly seen, there is a large qualitative difference between the switching curves of machine 2, while there is no visible difference for S_1. Under the modified function, curve S_2 becomes closer to a multi-echelon base stock policy curve. The change between the plot on the left and the plot on the right occurs smoothly as a function of $p_2^{'}$, but the qualitative change on the S_2 curve occurs very rapidly as we start increasing its value from $-h_2$ (see Fig. 5 for an illustration of this transition).

To better illustrate the impact on performance for each function, we compute the probability distribution function for the echelon variables under each of the cost functions. Figure 4 displays the probability distribution of x_1 (on the left) and x_2 (on the right) for both cost functions.

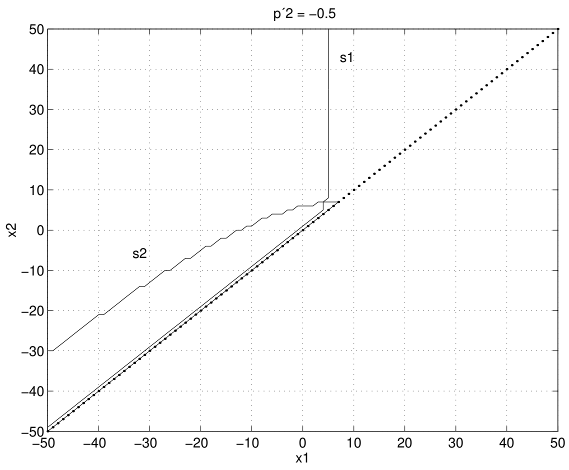

Figure 3: Optimal switching curves for the classic and the modified cost functions.

The first observation to be made is that the maximum finished goods inventory in both cases is invariant, given that the curves on the left plot reach one for the same value of inventory. What changes is the distribution of the inventories for intermediate values. We see that as $p_2^{'}$ increases, all other parameters being constant, the service level increases without changing the switching curve of machine 1. This means that the modified cost function, while not changing the overall maximum amount of finished goods inventory, is able to improve the service level by improving the ability to recover from backlog.

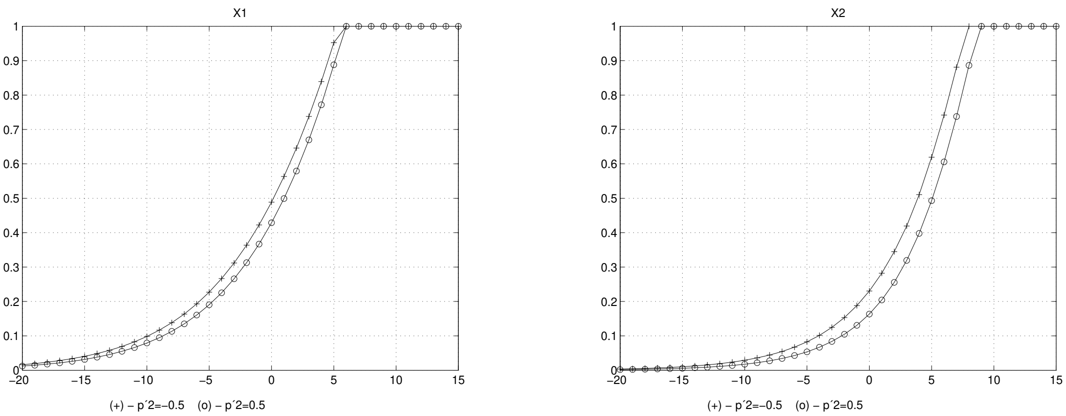

Figure 4: Probability distributions for x_1 and x_2 as functions of $p_2^{'}$.

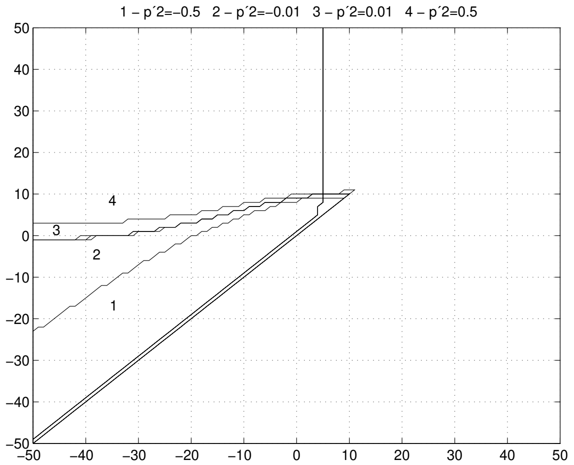

Figure 5: Evolution of the switching curves as a function of p_2'.

5 Conclusions

We have presented a different single stage cost function for inventory control and showed how it compares with the classic single stage cost function in terms of the qualitative behavior of the control policies and also in terms of quantitative differences regarding the distribution of inventories under each control policy. Our single stage cost function accounts for all echelon inventories with a couple of parameters for each variable, depending on whether it is negative or positive.

The optimal control policies approach the multi-echelon base stock policies for the modified single stage cost function. The approach to multi-echelon base stock policies is naturally done at the expense of the fact that during operation the inventory between the two stages may take very high values, as the switching curves show. High values of intermediate inventory are usually considered to be a bad characteristic of a production system, given the relationship between inventories and waiting times, as expressed by Little's law [11].

We show, by means of the probability distributions of echelon inventories, that systems being controlled under policies derived with the modified function recover faster from backlog than the classic control policies without the additional price of increasing end product maximum inventory levels, and improving the service level. So, although there is a possible increase in intermediate inventories if a given system enters a deep backlog, that increase occurs in a controllable way.

Our current research is aimed at extending this approach to the problem of a tandem two machine system with unreliable machines and batch production and exploring other ways of comparing these two cost functions. Furthermore, we want to extend our analysis to systems with more than two-machines.

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