COMMUTATIVE ALGEBRA - PSET 8

Problem 1. (5 points) Let $R = k[x, y]$ and $\mathfrak{m} = (x, y)$.

(1) Show that $\mathrm{gr}_{\mathfrak{m}}(R) \simeq k[z, w]$.
(2) Show that $B_{\mathfrak{m}}(R) \simeq k[x, y, z, w]/(xw - yz)$.[1]

Problem 2. (5 points) Let M be a finitely generated R-module and

$$\mathcal{F}\colon M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$$

be an I-filtration such that all the M_i are finitely generated.

(1) Show that if $\mathcal{F}$ is I-stable then $\mathrm{gr}_{\mathcal{F}}(M)$ is a finitely generated $\mathrm{gr}_{\mathfrak{m}}(R)$ -module.
(2) Show that $\mathcal{F}$ is I-stable if and only if $B_{\mathcal{F}}(M)$ is a finitely generated $B_{\mathfrak{m}}(R)$ -module.

Problem 3. (5 points) Let k be a field of characteristic $\neq 2$.[2]

(1) Show that
$$k[\![x, y]\!]/(y^2 - x^2(x + 1)) \simeq k[\![z, w]\!]/(zw)\,.$$
(2) Show that $k[\![x, y]\!]/(y^2 - x^3)$ is not isomorphic to $k[\![z]\!]$. Argue by contradiction and start by showing that the images of x, y in $k[\![z]\!]$ would be power series $f(z), g(z)$ divisible by z and such that $g(z)^3 = f(z)^2$.

Problem 4. (5 points) Given a R-module M and a filtration

$$\mathcal{F}\colon M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$$

we can define the completion of M with respect to $\mathcal{F}$ as

$$\begin{aligned} \hat{M}_{\mathcal{F}} &= \varprojlim M/M_i \\ &= \{(m_1, m_2, \dots)\colon m_i \in M/M_i\,,\ m_i \equiv m_j \mod M_i \text{ for } i < j\}\,. \end{aligned}$$

When $\mathcal{F}$ is the $\mathfrak{m}$-adic filtration, we write $\hat{M}_{\mathcal{F}} = \hat{M}_{\mathfrak{m}}$.

(1) Let $\mathcal{F}, \mathcal{F}'$ be two filtrations of M which are intertwined, in the sense that there are increasing functions $f, g\colon \mathbb{N} \to \mathbb{N}$ such that
$$M_i \supseteq M'_{f(i)} \text{ and } M'_i \supseteq M_{g(i)}.$$
Show that the induced completions are the same, i.e.
$$\hat{M}_{\mathcal{F}} \simeq \hat{M}_{\mathcal{F}'}\,.$$
(2) Use (1) and the Artin-Rees lemma to show that if R is Noetherian and $N \subseteq M$ are finitely generated R-modules then
$$\hat{N}_{\mathfrak{m}} = \varprojlim N/\mathfrak{m}^i N \simeq \varprojlim N/(N \cap \mathfrak{m}^i M)\,.$$

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[1] The variety $\{(x, y, [z : w]) \in \mathbb{A}^2 \times \mathbb{P}^1 : xw = yz\}$ is called the blow-up of $\mathbb{A}^2$ at the origin. The projection to $\mathbb{A}^2$ is an isomorphism away from $(0, 0)$ and the fiber at $(0, 0)$ is the projective line $\mathbb{P}^1$, and it is called the exceptional divisor. The exceptional divisor is related to $\mathrm{gr}_{\mathfrak{m}}(R)$.

[2] This exercise is really about understanding singularities of algebraic curves using completions. If $a \in X$ is such that $A(X)_{\mathfrak{m}_a} \simeq k[\![z]\!]$ then X is smooth at a. If it is $\simeq k[\![z, w]\!]/(zw)$ then a is called a node, which is the case in (1); in other words, a is a node if X near a looks like $wz = 0$ near $(0, 0)$. If X near a looks like $y^2 = x^3$ then a is a cusp; (2) says that a cusp is not smooth. Singularities become even more interesting (and complicated) in higher dimensions. For example, a singularity on a surface that looks locally like $x^2 + y^3 + z^5 = 0$ is called a E_8 singularity.