COMMUTATIVE ALGEBRA - PROBLEM SET 3

Problem 1. (5 points) Let R be a ring.

(1) Show that if $(R, \mathfrak{m})$ is a local ring then every element of $R \setminus \mathfrak{m}$ is invertible.
(2) Show that if $(R, \mathfrak{m})$ is a local ring then $R_{\mathfrak{m}} \simeq R$.
(3) Show that R is a local ring if and only if the following holds:

$$x + y = 1 \Rightarrow x \text{ is invertible or } y \text{ is invertible.}$$

Problem 2. (5 points) A ring R is called reduced if $r^n = 0 \Rightarrow r = 0$ (i.e., its nilradical is trivial). Show that R is reduced if and only if R_P is reduced for every prime ideal P of R.[1]

Problem 3. (5 points) Let S, S' be two rings and consider their product $R = S \times S'$.

(1) Show that the prime ideals of R are either of the form $S \times P'$ for some P' prime ideal of S' or $P \times S'$ for some P prime ideal of S.
(2) Show that

$$R_{P \times S'} \simeq S_P\,.$$

(3) Show that if S is a domain then S_P is also a domain.
(4) Conclude that if S and S' are both domains, then the localization of $S \times S'$ at every prime is a domain, but $S \times S'$ is not a domain.[2]

Problem 4. (5 points) An ideal $Q \subseteq R$ is called primary if it satisfies

$$ab \in Q \Rightarrow a \in Q \text{ or } b \in \sqrt{Q}\,.$$

(1) Show that if Q is primary then $\sqrt{Q}$ is a prime ideal (in which case we say that Q is $\sqrt{Q}$-primary).
(2) What are the primary ideals of $k[x]$ if k is algebraically closed?
(3) Give an example of an ideal on $k[x, y]$ such that $\sqrt{Q}$ is prime but Q is not primary.
(4) Show that if $\sqrt{Q}$ is maximal then Q is primary.

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[1] We say that being reduced is a local property. Note that the statement is also equivalent to $R_{\mathfrak{m}}$ being reduced for every maximal ideal $\mathfrak{m}$.

[2] The geometry is the following: if X is a disjoint union of two irreducible components (e.g. $X = \{(a, b) \colon b = 0 \text{ or } b = 1\}$) then X looks irreducible locally, but it is not. There is a converse to this problem saying that if the localization at every prime is a domain then R is a product of domains, see Proposition 2.20 in Eisenbud's book. Geometrically: X is locally irreducible if and only if every connected component of X is irreducible.