Virasoro constraints for sheaf moduli spaces via wall-crossing

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History of Virasoro constraints

- In 1990, Witten proposed a conjecture saying that integrals of ψ-classes in the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ satisfy some relations which completely determine them:

$$L_k(Z) = 0 \quad \text{for } k \geqslant -1\,,$$

  where Z is the generating function of these integrals and L_k are differential operators satisfying the Virasoro bracket

$$[L_k, L_\ell] = (\ell - k)L_{k+\ell}\,.$$

- Witten's conjecture was proven in 1992 by Kontsevich. Alternative proofs by Okounkov-Pandharipande and Mirzakhani were found later.
- Eguchi-Hori-Xiong propose in 1997 a generalization to the Gromov-Witten (GW) theory of a target variety X.

History of Virasoro constraints

- In 2006, Maulik-Nekrasov-Okounkov-Pandharipande (MNOP) propose a conjecture connecting Gromov-Witten invariants on 3-folds to Donaldson-Thomas (DT) invariants, defined using the moduli space of ideal sheaves.
- An analog of Virasoro constraints should exist in DT theory! Oblomkov-Okounkov-Pandharipande make a precise conjecture by calculations in $X = \mathbb{P}^3$.
- In 2020, with Oblomkov-Okounkov-Pandharipande we prove that the MNOP correspondence intertwines the GW Virasoro and the DT Virasoro constraints (in stationary regime).
- This proves Virasoro constraints for the DT theory of toric 3-folds with stationary descendents.

History of Virasoro constraints

- In 2020 I used the previous result to prove a version of Virasoro constraints for the Hilbert scheme of points on simply-connected surfaces.
- In 2021 D. van Bree conjectures a generalization of the Hilbert scheme result to moduli spaces of stable sheaves on surfaces.
- Much more general?...

Today

I will explain joint work with A. Bojko and W. Lim containing:

- Unified formulation of Virasoro constraints for moduli spaces of sheaves and pairs.
- How the Virasoro constraints are naturally formulated using the vertex algebra that D. Joyce introduced to study wall-crossing.
- Virasoro constraints are compatible with wall-crossing.
- A proof of the Virasoro constraints for moduli spaces of stable sheaves on curves and surfaces with $h^{0,1} = h^{0,2} = 0$ by reducing everything to the rank 1 case.

Moduli spaces of sheaves

We consider moduli spaces M of stable sheaves on a smooth projective variety X (typically of small dimension) such that:

1. There are no strictly semistable sheaves in M.
2. There is an (in principle non-unique) universal sheaf $\mathbb{G}$ in $X \times M$; $\mathbb{G}$ is such that $\mathbb{G}_{|X \times \{[G]\}} \cong G$. Tensoring $\mathbb{G}$ by a line bundle pulled back from M gives another universal sheaf.
3. M admits a 2-term perfect obstruction theory with deformation theory at $[G] \in M$ given by

$$\mathrm{Tan} = \mathrm{Ext}^1(G, G)\,,\ \mathrm{Obs} = \mathrm{Ext}^2(G, G)\,,\ 0 = \mathrm{Ext}^{>2}(G, G)\,.$$

It follows that there is a virtual fundamental class $[M]^{\mathrm{vir}} \in H_\bullet(M)$.

Examples of moduli spaces

The following are the main examples that fit in the description above:

Example

- Moduli space of stable vector bundles $M_C(r,d)$ of rank r and degree d on a curve.
- Moduli space of torsion-free H-stable sheaves $M_S^H(r,\beta,\chi)$ on a surface (with $p_g > 0$).
- Moduli space of 1-dimensional H-stable sheaves $M_S^H(\beta,\chi)$ on a surface (with $p_g > 0$).
- Moduli space of ideal sheaves or stable pairs on a (Fano) 3-fold.

Moduli spaces of pairs

An important variation that plays a role are moduli spaces of pairs P, parametrizing a sheaf F together with a map $V \to F$ from a fixed sheaf V. There are two main differences:

- There is a unique universal pair $q^*V \to \mathbb{F}$ on $P \times X$
- The deformation theory at a pair $V \to F$ is governed by $\mathrm{RHom}([V \to F], F)$.

Example

- Symmetric powers of curves.
- Moduli spaces of Joyce-Song pairs or, more generally, Bradlow pairs.
- Grassmannians.
- Quot schemes on curves or on surfaces.

Other variations

Other variations that I will not talk about but for which there is also a Virasoro story with similar flavour:

Example

- Flag varieties.
- Moduli spaces of quiver representations (possibly with relations).
- Moduli spaces of parabolic bundles.

Descendents

To get numerical invariants from M we integrate certain natural cohomology classes against the virtual fundamental class.

Definition (Descendent algebra)

Let $\mathbb{D}^X$ be the free (super)commutative $\mathbb{C}$-algebra generated by symbols

$$\mathrm{ch}_i(\gamma) \quad \text{for } i \geqslant 0, \gamma \in H^\bullet(X)\,.$$

Definition (Geometric realization of descendents)

Let M be a moduli of sheaves with a universal sheaf $\mathbb{G}$ in $M \times X$. Define the geometric realization morphism $\xi_{\mathbb{G}} \colon \mathbb{D}^X \to H^\bullet(M)$ by

$$\xi_{\mathbb{G}}\left(\mathrm{ch}_i(\gamma)\right) = p_*\left(\mathrm{ch}_{i+\dim(X)-s}(\mathbb{G})q^*\gamma\right) \in H^\bullet(M)$$

for $\gamma \in H^{s,t}(X)$. p, q are the projections of the product onto M and X, respectively.

Definition

For $n \geqslant -1$ define the operators $\mathsf{L}_n \colon \mathbb{D}^X \to \mathbb{D}^X$ by $\mathsf{L}_n = \mathsf{R}_n + \mathsf{T}_n$ where:

1. The operator $\mathsf{R}_n \colon \mathbb{D}^X \to \mathbb{D}^X$ is a derivation defined on generators by
$$\mathsf{R}_n \mathrm{ch}_i(\gamma) = \left(\prod_{j=0}^{n}(i+j)\right) \mathrm{ch}_{i+n}(\gamma).$$
2. The operator $\mathsf{T}_n \colon \mathbb{D}^X \to \mathbb{D}^X$ is the multiplication by the element of $\mathbb{D}^X$ given by
$$\mathsf{T}_n = \sum_{i+j=n} i!j! \sum_s (-1)^{\dim X - p_s^L} \mathrm{ch}_i(\gamma_s^L)\mathrm{ch}_j(\gamma_s^R)\,,$$
where $\sum_s \gamma_s^L \otimes \gamma_s^R = \Delta_* \mathrm{td}(X)$.

Virasoro operators

They satisfy the Virasoro bracket:

$$[\mathsf{L}_n, \mathsf{L}_m] = (m-n)\mathsf{L}_{n+m} \,.$$

There is also a version of descendent algebra for moduli spaces of pairs, $\mathbb{D}^{X,\mathrm{pa}}$ which is generated by classes $\mathrm{ch}_i^{\mathcal{V}}(\gamma), \mathrm{ch}_i^{\mathcal{F}}(\gamma)$.
Moreover there are pair Virasoro operators $\mathsf{L}_n^{\mathrm{pa}} : \mathbb{D}^{X,\mathrm{pa}} \to \mathbb{D}^{X,\mathrm{pa}}$.
The main difference is in the linear term:

$$\mathsf{T}_n^{\mathrm{pa}} = \sum_{i+j=n} i!j! \sum_s (-1)^{\dim X - p_s^L} \mathrm{ch}_i^{\mathcal{F}-\mathcal{V}}(\gamma_s^L)\mathrm{ch}_j^{\mathcal{F}}(\gamma_s^R) \,.$$

Virasoro constraints for pairs

Conjecture (Virasoro for pairs)

Let P be a moduli space of pairs with universal pair $q^*V \to \mathbb{F}$. For any $D \in \mathbb{D}^{X,\mathrm{pa}}$ and $n \geqslant 0$ we have

$$\int_{[P]^{\mathrm{vir}}} \xi_{(q^*V,\mathbb{F})}\big(\mathsf{L}_n^{\mathrm{pa}}(D)\big) = 0\,.$$

Weight 0 descendents

The formulation of sheaf Virasoro constraints should be independent on the choice of universal sheaf. If $\mathbb{G}$ is a universal sheaf and L is a line bundle on M then $\mathbb{G}' = \mathbb{G} \otimes p^* L$ is another universal sheaf and

$$\xi_{\mathbb{G}'} = \sum_{j \geqslant 0} \frac{c_1(L)^j}{j!} \xi_{\mathbb{G}} \circ \mathrm{R}_{-1}^j \,.$$

Definition

We say that $D \in \mathbb{D}^X$ has weight 0 if $\mathrm{R}_{-1}(D) = 0$. We denote by $\mathbb{D}^X_{\mathrm{wt}_0} \subseteq \mathbb{D}^X$ the algebra of weight 0 descendents.

If $D \in \mathbb{D}^X_{\mathrm{wt}_0}$ then its geometric realization $\xi_{\mathbb{G}}(D)$ does not depend on the choice of $\mathbb{G}$, so we write

$$\int_{[M]^{\mathrm{vir}}} D = \int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(D) \,.$$

Virasoro constraints for sheaves

Let

$$\mathsf{L}_{\mathrm{wt}_0} = \sum_{n \geqslant -1} \frac{(-1)^n}{(n+1)!} \mathsf{L}_n \mathsf{R}_{-1}^{n+1} \,.$$

Fact

$$\mathsf{L}_{\mathrm{wt}_0}(D) \in \mathbb{D}^X_{\mathrm{wt}_0} \,.$$

Conjecture (Virasoro for sheaves)

Let M be a moduli space of sheaves. For any $D \in \mathbb{D}^X$ we have

$$\int_{[M]^{\mathrm{vir}}} \mathsf{L}_{\mathrm{wt}_0}(D) = 0 \,.$$

Example – rank 2 sheaves on a curve

Let $M = M_C(2, \Delta)$ be the moduli space of stable bundles on a curve C of genus g with rank 2 and fixed determinant Δ of odd degree; this is a smooth moduli space of dimension $3g - 3$.
All integrals of descendents on M can be deduced from integrals of products of certain classes

$$\eta \in H^2(M), \quad \theta \in H^4(M), \quad \zeta \in H^6(M)\,.$$

Thaddeus proved:

$$\int_M \eta^m \theta^k \zeta^p = (-1)^{g-1-p} \frac{m!g!}{(g-p)!} 2^{2g-2-p} \frac{(2^q - 2)B_q}{q!}\,,$$

where $m + 2k + 3p = 3g - 3$ and $q = m + p - g + 1$.
The Virasoro constraints for M are equivalent to

$$(g-p) \int_M \eta^m \theta^k \zeta^p = -2m \int_M \eta^{m-1} \theta^{k-1} \zeta^{p+1}\,.$$

Vertex algebras recap

1. A vertex algebra is a (graded) vector space $V_\bullet$ together with a vacuum vector $|0\rangle \in V_\bullet$, a translation operator $T : V_\bullet \to V_\bullet$ and a state-field correspondence $Y : V_\bullet \to \mathrm{End}(V_\bullet)[\![z^{\pm 1}]\!]$

$$Y(u,z)v = \sum_{n\in\mathbb{Z}} u_n v z^{-n-1} \,.$$

2. Vertex algebras often come with a conformal element ω (in which case they are called vertex operator algebras). A conformal element induces a representation of the Virasoro algebra, i.e. $\mathrm{L_n} = \omega_{\mathrm{n+1}} \in \mathrm{End}(\mathrm{V}_\bullet)$ satisfy

$$[\mathrm{L}_n, \mathrm{L}_m] = (n-m)\mathrm{L}_{m+n} + \delta_{m+n,0}\, c\, \frac{m^3 - m}{12}\, \mathrm{id}$$

where $c \in \mathbb{C}$ is a constant called the central charge of ω.

Vertex algebras recap

3. If $V_\bullet$ is a vertex algebra there is a natural Lie algebra structure on $\check{V}_\bullet = V_\bullet / T(V_\bullet)$ (called Borcherds Lie algebra). The bracket is defined by

$$[\overline{u}, \overline{v}] = \overline{u_0 v}\,.$$

Example (Lattice vertex algebras)

Let Λ be an abelian group with a symmetric bilinear pairing $Q\colon \Lambda \times \Lambda \to \mathbb{Z}$. There is a vertex algebra structure on

$$V_\bullet = \mathbb{C}[\Lambda] \otimes \mathrm{Sym}(\Lambda_{\mathbb{C}}[\![t]\!]).$$

If Q is non-degenerate then $V_\bullet$ has a natural conformal element.

Joyce's vertex algebra

Let $\mathcal{M}_X$ be the (higher) stack parametrizing perfect complexes in X, i.e. objects in $D^b(X)$.
Joyce defined a vertex algebra structure on the homology

$$V_\bullet = H_\bullet(\mathcal{M}_X)\,.$$

The translation operator T is related to the $B\mathbb{G}_m$ action $B\mathbb{G}_m \times \mathcal{M}_X \to \mathcal{M}_X$. The state-field is given explicitly by

$$Y(v,z)u = (-1)^{\chi(\alpha,\beta)} z^{\chi_{\mathrm{sym}}(\alpha,\beta)} \Sigma_* \Big((e^{zT} \boxtimes \mathrm{id}) \big(c_{z^{-1}}(\Theta) \cap v \boxtimes u \big) \Big)\,.$$

Θ is a complex on $\mathcal{M}_X \times \mathcal{M}_X$ given by

$$\Theta_{F,G} = \mathrm{RHom}(F,G)^\vee \oplus \mathrm{RHom}(G,F).$$

If M is a moduli of sheaves with universal sheaf $\mathbb{G}$, by the universal property of $\mathcal{M}_X$ there is a map $f_{\mathbb{G}}\colon M \to \mathcal{M}_X$. Define

$$[M]^{\mathrm{vir}}_{\mathbb{G}} = (f_{\mathbb{G}})_*[M]^{\mathrm{vir}} \in V_\bullet = H_\bullet(\mathcal{M}_X)\,,$$

$$[M]^{\mathrm{vir}} = \overline{[M]^{\mathrm{vir}}_{\mathbb{G}}} \in \check{V}_\bullet = V_\bullet/T(V_\bullet) \cong H_\bullet(\mathcal{M}_X^{\mathrm{rig}})\,.$$

Theorem (J. Gross)

The cohomology $H^\bullet(\mathcal{M}_X)$ is closely related to $\mathbb{D}^X$ and the cohomology $H^\bullet(\mathcal{M}_X^{\mathrm{rig}})$ is closely related to $\mathbb{D}^X_{\mathrm{wt}_0}$. Integrating descendents against the virtual fundamental class is identified with pairing between a homology and cohomology class in $\mathcal{M}_X$, i.e.

$$\int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(D) = \langle [M]^{\mathrm{vir}}_{\mathbb{G}}, D\rangle$$

$$\int_{[M]^{\mathrm{vir}}} D_{\mathrm{wt}_0} = \langle [M]^{\mathrm{vir}}, D_{\mathrm{wt}_0}\rangle\,.$$

Wall-crossing

Moduli spaces of sheaves often depend on a choice of a stability parameter μ. Denote by $M_\alpha(\mu)$ the moduli of μ-stable sheaves with Chern character equal to α. Wall-crossing is about trying to understand how $M_\alpha(\mu)$ and the corresponding numerical invariants change when μ changes.

Theorem (Joyce)

For every α there exist classes

$$[M_\alpha(\mu)]^{\mathsf{Jo}} \in \check{V}_\bullet$$

even if $M_\alpha(\mu)$ contains strictly semistable sheaves. When there are no strictly semistables, $[M_\alpha(\mu)]^{\mathsf{Jo}} = [M_\alpha(\mu)]^{\mathsf{vir}}$.

Wall-crossing

Theorem (Joyce)

Let μ, τ be two stability conditions. Then

$$[M_\alpha(\mu)]^{\mathrm{Jo}} = \sum_{\alpha_1+\ldots+\alpha_l=\alpha} U(\alpha_1,\ldots,\alpha_l;\mu,\tau)\times \Big[\ldots\big[[M_{\alpha_1}(\tau)]^{\mathrm{Jo}},[M_{\alpha_2}(\tau)]^{\mathrm{Jo}}\big],\ldots,[M_{\alpha_l}(\tau)]^{\mathrm{Jo}}\Big]$$

in $\check{V}_\bullet$, where $U(\alpha_1,\ldots,\alpha_l;\mu,\tau)$ are some combinatorial coefficients.

Gross' isomorphism

Theorem (Gross+BLM)

Suppose that $\dim(X) \leqslant 2$. *Joyce's vertex algebra is isomorphic to the (generalized) lattice vertex algebras*

$$V_\bullet \cong \mathbb{C}[K^0_{sst}(X)] \otimes SSym\big[H^\bullet(X)[\![t]\!]\big]$$

constructed from the symmetric bilinear form on $H^\bullet(X)$ *given by*

$$\chi_{sym}(\gamma, \delta) = \int_X (-1)^{\lfloor \frac{\deg \gamma}{2} \rfloor} \gamma \cdot \delta \cdot \mathrm{td}(X) + \int_X (-1)^{\lfloor \frac{\deg \delta}{2} \rfloor} \delta \cdot \gamma \cdot \mathrm{td}(X)\,.$$

In general, the pairing χ_{sym} can be degenerate so we do not have a natural conformal element on $V_\bullet$. This is solved by considering a larger vertex algebra $V_\bullet^{\mathrm{pa}} \supseteq V_\bullet$ which arises naturally when we adapt Joyce's construction to moduli spaces of pairs.

Conformal element in Joyce's VA

Theorem (Bojko-Lim-M)

Let X be a point, a curve or a surface with $h^{0,2}=0$. Then there is a conformal element ω in the pair vertex algebra $V_\bullet^{\mathrm{pa}}$.
Under the duality between $V_\bullet^{\mathrm{pa}}$ and $\mathbb{D}^{X,\mathrm{pa}}$, the Virasoro fields L_n induced by ω are dual to the pair Virasoro operators $\mathsf{L}_n^{\mathrm{pa}}$ defined in the algebra of descendents $\mathbb{D}^{X,\mathrm{pa}}$.

The construction of ω requires a choice of a maximal isotropic decomposition of the fermionic part, which in our case is

$$H^{\mathrm{odd}}(X) = H^{\bullet,\bullet+1}(X) \oplus H^{\bullet+1,\bullet}(X)\,.$$

Physical states

There is a well-known vertex algebra notion called the subspaces of *physical states*

$$\check{P}_0 \subseteq \check{V}_\bullet\,, \quad P_0^{\mathrm{pa}} \subseteq V_\bullet^{\mathrm{pa}}\,.$$

Corollary (Bojko-Lim-M)

1. *A moduli of sheaves M satisfies the sheaf Virasoro constraints if and only if*
$$[M]^{\mathrm{vir}} \in \check{P}_0$$
is a physical state.
2. *A moduli of pairs P with universal pair $q^*V \to \mathbb{F}$ satisfies the pair Virasoro constraints if and only if*
$$[P]^{\mathrm{vir}}_{(q^*V,\mathbb{F})} \in P_0^{\mathrm{pa}}$$
is a physical state.

Wall-crossing compatibility

Proposition

1. *The subspace* $\breve{P}_0 \subseteq \breve{V}_\bullet$ *is a Lie subalgebra, i.e.*

$$\overline{u}, \overline{v} \in \breve{P}_0 \Rightarrow [\overline{u}, \overline{v}] \in \breve{P}_0 \,.$$

2. *The subspace* $P_0 \subseteq V_\bullet$ *is a Lie algebra subrepresentation of* $\breve{P}_0 \subseteq \breve{V}_\bullet$*, i.e.*

$$\overline{u} \in \breve{P}_0,\ v \in P_0^{\mathrm{pa}} \Rightarrow [\overline{u}, v] \in P_0^{\mathrm{pa}} \,.$$

This proposition translates to a compatibility between the Virasoro constraints and wall-crossing in moduli spaces of sheaves!

Main theorem

Theorem (Bojko-Lim-M)

The Virasoro constraints hold for the following moduli spaces:

1. *Moduli spaces of stable bundles on curves* $M_C(r, d)$*;*
2. *Moduli spaces of stable torsion-free sheaves* $M_S^H(r, \beta, n)$ *on surfaces* S *with* $h^{0,1} = h^{0,2} = 0$ *and a polarization* H*;*
3. *Moduli spaces of stable 1 dimensional sheaves* $M_S^H(\beta, n)$ *on surfaces* S *with* $h^{0,1} = h^{0,2} = 0$ *and a polarization* H *(assuming a technical condition necessary for the proof of the relevant wall-crossing formula).*

Sketch of proof – Bradlow pairs

I will focus on the case of curves, so fix a curve C. The proof relies on wall-crossing for moduli spaces of Bradlow pairs.

Definition

Given $t \in \mathbb{R}_{>0}$ we define the μ_t-slope of a pair $\mathcal{O}_C^{\oplus m} \to F$ by

$$\mu_t(\mathcal{O}_C^{\oplus m} \to F) = \frac{\deg(F) + t \cdot m}{\mathrm{rk}(F)} .$$

A pair $\mathcal{O}_C^{\oplus m} \to F$ is called μ_t-semistable if for every subpair $\mathcal{O}^{\oplus m'} \to F'$ we have

$$\mu_t(\mathcal{O}_C^{\oplus m'} \to F') \leqslant \mu_t(\mathcal{O}_C^{\oplus m} \to F).$$

We denote by $P^t(r, d)$ the moduli space of μ_t-semistable pairs of the form $\mathcal{O}_C \to F$ with $\mathrm{rk}(F) = r, \deg(F) = d$.

Sketch of proof – Wall-crossing

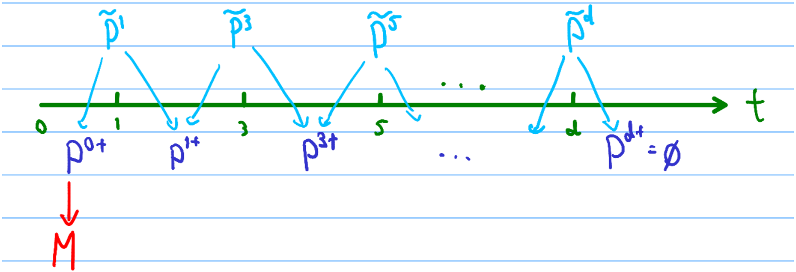
1. When $t \gg 1$ there are no μ_t-semistable pairs, i.e. $P^t(r,d)$ becomes empty.
2. When $0 < t \ll 1$ and $M_C(r,d)$ has no strictly semistables ($\gcd(r,d)=1$), a pair $[\mathcal{O}_C \xrightarrow{s} F]$ is μ_t-stable if and only if F is stable and $s \neq 0$. Assuming d is large,

$$P^{0+}(r,d) \to M(r,d)$$

   is a projective bundle with fibers $\mathbb{P}(H^0(F))$. More generally, $P^{0+}(r,d)$ is the moduli space of Joyce-Song pairs and is used to define $[M_C(r,d)]^{\mathrm{Jo}}$ when $\gcd(r,d) \neq 1$.
3. The moduli space $P^t(r,d)$ changes when we cross a t for which $P^t(r,d)$ has strictly semistable objects. Such t is called a wall.

Wall-crossing formula, $r = 2$ example

For rank 2 we have walls at the odd integers up to d. The moduli spaces $P^t(2,d)$ are related by a sequence of flips:



Joyce's wall-crossing formula is:

$$0 = [P^{d+}(2,d)]^{\mathrm{Jo}} = [P^{0+}(2,d)]^{\mathrm{Jo}} + \sum_{\substack{d_1+d_2=d \\ d_1>d_2\geqslant 0}} \big[[M(1,d_1)]^{\mathrm{Jo}}, [P(1,d_2)]^{\mathrm{Jo}}\big]$$

Sketch of proof – Induction

4. We prove by induction on r that $M(r,d)$ and $P^t(r,d)$ satisfy the sheaf and the pair Virasoro constraints, respectively.
5. In the base case $r = 1$,

$$M(1,d) = \mathrm{Jac}(C) \text{ and } P^t(1,d) = C^{[d]}.$$

   Both cases can be proven "by hand". For surfaces, everything can be reduced to the Hilbert scheme of points which was proven earlier (M-Oblomkov-Okounkov-Pandharipande, M).
6. Using the wall-crossing compatibility and $P^{t\gg 0}(r,d) = 0$, $P^t(r,d)$ satisfies the pair Virasoro constraints for every t. Induction guarantees that all the wall-crossing terms already satisfy Virasoro.

Sketch of proof – Projective bundle compatibility

7. The projective bundle structure can be used to prove that if $P^{0+}(r,d)$ satisfies the pair Virasoro constraints then $M(r,d)$ satisfies the sheaf Virasoro constraints.
8. It is crucial that we include in the induction the cases in which there are strictly semistable sheaves ($\gcd(r,d) \neq 1$), i.e. prove that $[M_C(r,d)]^{\mathrm{Jo}} \in \check{P}_0$.

Thanks!