Evolutionary Dynamics of Collective Action in N-person Stag-Hunt Dilemmas

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ELECTRONIC SUPPLEMENTARY MATERIAL

1. N-PERSON STAG-HUNT IN INFINITE POPULATIONS

The evolutionary dynamics of Cs and Ds in the N-person Stag-Hunt game with a minimum threshold M can be studied by analyzing the sign of $f_C - f_D$ (see Appendix 1). Hence, using the same conventions introduced in the Appendix 1, we shall study in detail the following polynomial

$$Q(x) = f_C - f_D = c\left(\frac{F}{N} - 1\right) - c\frac{F}{N}(1-x)^{N-M}\sum_{k=0}^{M-1}\binom{N-1}{k}\left(1 - M\delta_{k,M-1}\right)x^k(1-x)^{M-1-k}.$$

The roots of $Q(x)$ provide the interior fixed points of the replicator dynamics equation. In what follows, we shall assume that $N \geq 2$. For most of the time, we shall also assume that $1 < M < N$. The degenerate cases will be dealt with at the end. Let us start by recasting $Q(x)$ in a more amenable form. To this end, let $F/N = \lambda$; we may rewrite

$$Q(x) = -c\left\{1 - \lambda + \lambda\left[\sum_{k=0}^{M-1}\binom{N-1}{k}x^k(1-x)^{N-1-k} - M\binom{N-1}{M-1}x^{M-1}(1-x)^{N-M}\right]\right\}.$$

Since

$$1 = 1^{N-1} = (x + 1 - x)^{N-1} = \sum_{k=0}^{N-1}\binom{N-1}{k}x^k(1-x)^{N-1-k},$$

we have that

$$Q(x) = -c\left\{1 - \lambda\left[\sum_{k=M}^{N-1}\binom{N-1}{k}x^k(1-x)^{N-1-k} + M\binom{N-1}{M-1}x^{M-1}(1-x)^{N-M}\right]\right\}.$$

Let

$$\begin{aligned} R(x) &= \sum_{k=M}^{N-1}\binom{N-1}{k}x^k(1-x)^{N-1-k} + M\binom{N-1}{M-1}x^{M-1}(1-x)^{N-M} \\ &= x^{M-1}\left(\sum_{k=M}^{N-1}\binom{N-1}{k}x^{k-M+1}(1-x)^{N-1-k} + M\binom{N-1}{M-1}(1-x)^{N-M}\right) \qquad (1) \end{aligned}$$

Then we have that

$$Q(x) = -c\big(1 - \lambda R(x)\big)$$

Hence, the roots of $Q(x)$ are given by the intersection(s) of the line $1/\lambda \equiv N/F$ with the polynomial $R(x)$. It turns out that Figure 1-a provides examples of $N/R(x)$, such that intersections with the line F identify the interior fixed points. We shall show below various properties of $R(x)$ that capture the possibilities already illustrated in Figure 1, which we now prove are quite general.

Lemma 1

1. $R(0) = 0$*;*

2. $R(1) = 1$*;*

3. $R(x) > 0, \quad x \in (0,1)$*;*

4. Let $x^* = \dfrac{M}{N}$. *Then we have that* $R'(x) > 0$ *for* $0 \le x < x^*$, *and* $R'(x) < 0$ *for* $x^* < x < 1$. *In particular,* $R'(x^*) = 0$ *and* x^* *is a point of maximum of* R *with* $R(x^*) > 1$*;*

Before we prove Lemma 1, let us use it to prove the main result :

Proposition 1

Let $\lambda^* = \dfrac{1}{R(x^*)}$. *We have that* $0 < \lambda^* < 1$. *Moreover,* $Q(x)$ *satisfies:*

a. For $\lambda < \lambda^*$ *there are no roots in* $(0,1)$*;*

b. For $\lambda = \lambda^*$ *there exists one double root at* $x = x^*$*;*

c. For $\lambda^* < \lambda \le 1$ *there are two simple roots* $\{x_1, x_2\}$, *with* $x_1 \in (0, x^*)$ *and* $x_2 \in (x^*, 1]$*;*

d. For $\lambda > 1$ *there is a single root in* $(0, x^*)$.

Proof of Proposition 1

From Lemma 1 we have that $R(x^*) > 1$, thus $0 < \lambda^* < 1$. We then observe that

i. For $\lambda < \lambda^*$, we have that $\lambda R(x) < \lambda^* R(x^*) = 1$. Thus $Q(x) < -c(1-1) = 0$

ii. For $\lambda = \lambda^*$, we compute $Q(x^*) = -c(1 - \lambda^* R(x^*)) = -c(1-1) = 0$.

Also, $Q'(x^*) = cR'(x^*) = 0$ and an easy calculation shows that $R''(x^*) \neq 0$. Hence, x^* is a double root.

iii. For $\lambda^* < \lambda \leq 1$, we first observe that we have $Q(0) = -c$, $Q(1) = -c(1-\lambda) < 0$. Since $1 - \lambda R(x^*) < 0$, we have $Q(x^*) > 0$. By the Intermediate Value Theorem, $Q(x)$ will have at least two roots: one in $(0, x^*)$ and another at $(x^*, 1]$. Moreover, $Q'(x) = cR'(x)$. Thus $Q(x)$ is monotonically increasing in $(0, x^*)$ and monotonically decreasing in $(x^*, 0)$. Thus these roots are unique.

iv. For $\lambda > \lambda^*$, we now have $Q(1) > 0$, and thus there is no root in $(x^*, 1]$. However, the argument for $(0, x^*)$ remains unchanged, and we have the result.

Let us now prove Lemma 1.

Proof of Lemma 1

First, notice that (1), (2) and (3) are straightforward from the form of the polynomial $R(x)$. cf. (Eq. 1). To prove (4), we let $k = N - 1 - k'$, and given that

$$\binom{N-1}{N-1-k'} = \binom{N-1}{k'},$$

we may write

$$\begin{aligned} R(x) &= x^{M-1}\left[\sum_{k'=0}^{N-M-1}\binom{N-1}{k'}x^{N-M-k'}(1-x)^{k'} + M\binom{N-1}{M-1}(1-x)^{N-M}\right] \\ &= x^{N-1}\left[\sum_{k'=0}^{N-M-1}\binom{N-1}{k'}\left(\frac{1-x}{x}\right)^{k'} + M\binom{N-1}{M-1}\left(\frac{1-x}{x}\right)^{N-M}\right] \quad . \end{aligned}$$

Let $z = \dfrac{1-x}{x}$. Then, we have that $z' = -\dfrac{1}{x^2} = -\dfrac{1}{x}(z+1)$.

Thus

$$R(x) = x^{N-1}p(z), \quad p(z) = \sum_{i=0}^{N-M} a_i z^i \ ,$$

where

$$a_i = \binom{N-1}{i}, \ 0 \le i < N - M \ \text{ and } \ a_{N-M} = M\binom{N-1}{M-1}$$

We now compute R':

$$\begin{aligned} R'(x) &= (N-1)x^{N-2}p(z) - x^{N-2}p'(z)(z+1) \\ &= x^{N-2}\left[(N-1)p(z) - p'(z)(z+1)\right] \\ &= x^{N-2}\left[(N-1)\sum_{i=0}^{N-M} a_i z^i - \sum_{i=1}^{N-M} i a_i z^i - \sum_{i=1}^{N-M} i a_i z^{i-1}\right] \\ &= x^{N-2}\left[(N-1)a_0 - a_1 + (N-1)\sum_{i=1}^{N-M} a_i z^i - \sum_{i=1}^{N-M} i a_i z^i - \sum_{i=2}^{N-M} i a_i z^{i-1}\right] \end{aligned}$$

Since $a_0 = 1$ and $a_1 = N-1$, and writing $i = i+1$ in the last sum, we find that

$$R'(x) = x^{N-2}\left[(N-1)\sum_{i=1}^{N-M} a_i z^i - \sum_{i=1}^{N-M} i a_i z^i - \sum_{i=2}^{N-M-1} (i+1) a_{i+1} z^i\right] = x^{N-2}S(z)$$

where

$$S(z) = \sum_{i=1}^{N-M-2}\left[(N-1-i)a_i - (i+1)a_{i+1}\right]z^i + \left[Ma_{N-M-1} - (N-M)a_{N-M}\right]z^{N-M-1} + (M-1)a_{N-M}z^{N-M} \quad .$$

On noting that

$$\binom{L}{j+1} = \frac{L-j}{j+1}\binom{L}{j} \qquad (2)$$

we obtain, for $1 \le i < N - N$, that

$$a_{i+1} = \frac{N-1-i}{i+1} a_i.$$

Hence,

$$\sum_{i=1}^{N-M-2}\left[(N-1-i)a_i - (i+1)a_{i+1}\right]z^i = 0 \quad .$$

Also, we have

$$Ma_{N-M-1} - (N-M)a_{N-M} = M\binom{N-1}{M} - (N-M)\binom{N-1}{M-1},$$

which on calling upon (Eq. 2) yields

$$M\binom{N-1}{M} - (N-M)\binom{N-1}{M-1} = (N-M)\binom{N-1}{M} - (N-M)\binom{N-1}{M-1}$$
$$= -(N-M)(M-1)\binom{N-1}{M-1} \quad .$$

Thus, we can write

$$S(z) = z^{N-M-1}\binom{N-1}{M-1}\left[-(N-M)(M-1) + M(M-1)z\right]$$

which yields

$$R'(x) = x^{M-1}(1-x)^{N-M-1}\binom{N-1}{M-1}\left[-(N-M)(M-1) + M(M-1)z\right] \qquad (3)$$

For $x \in (0,1)$, (Eq. 3) vanishes at

$$z^* = \frac{N-M}{M} = \frac{1-M/N}{M/N} \qquad .$$

Since $z = \dfrac{1-x}{x}$, $x^* = \dfrac{M}{N}$.

Also, from (Eq. 3), we see that

i. For $0 < z < z^*$, $R'(x) < 0$;

ii. For $z > z^*$, $R'(x) > 0$.

Moreover, $z = \dfrac{1-x}{x}$ is monotonically decreasing and maps (0,1) into $(0,\infty)$ (thus reversing the orientation), which yields that $0 < z < z^*$ corresponds to $x^* < x < 1$ and $z > z^*$ corresponds to $0 < x < x^*$. This proves (4).

Next we consider the degenerate cases not included in the proofs above.

Degenerate cases

For the cases, $M = 1$ and $M = N$ the above analysis does not hold, but they can be easily analyzed directly. Since

$$p(z) = \sum_{i=0}^{N-M} \binom{N-1}{i} z^i + (M-1)\binom{N-1}{M-1} z^{N-M}$$

we have for $M = 1$ that

$$R(x) = x^{N-1}(z+1)^{N-1} = 1.$$

Thus $Q(x) = -c(1-\lambda)$, with $\lambda^* = 1$ and then $Q(x) \equiv 0$.

For $M = N$, we have that

$$R(x) = Nx^{N-1} \qquad \text{and} \qquad Q(x) = -c\left(1 - \lambda N x^{N-1}\right) \qquad .$$

Thus Q will have a single root for $\lambda > \lambda^* = 1/N$.

In any case, for $1 \le M \le N$, we have that

$$R(x^*) = \left(\frac{M}{N}\right)^{N-1} \left[\sum_{i=0}^{N-M} \binom{N-1}{i} \left(\frac{N-M}{M}\right)^i + (M-1)\binom{N-1}{M-1} \left(\frac{N-M}{M}\right)^{N-M} \right].$$

Recalling that $\lambda^* = \dfrac{1}{R(x^*)}$ and that $\lambda = \dfrac{F}{N}$, we may write the critical F, F^*, as

$$F^* = N^N \left[\sum_{i=0}^{N-M} \binom{N-1}{i} (N-M)^i M^{N-1-i} + (M-1)\binom{N-1}{M-1} (N-M)^{N-M} M^{-1} \right]^{-1}.$$

2. N-PERSON PRISONER'S DILEMMA IN FINITE POPULATIONS

Here we detail the derivation of $f_C(k) - f_D(k)$ for the N-person Prisoner's Dilemma in finite, well-mixed populations. We may write

$$f_C(k) - f_D(k) = \binom{Z-1}{N-1}^{-1} \sum_{j=0}^{N-1} \left\{ \binom{k-1}{j}\binom{Z-k}{N-j-1} \Pi_C(j+1) - \binom{k}{j}\binom{Z-1-k}{N-1-j} \Pi_D(j) \right\} =$$

$$= c\binom{Z-1}{N-1}^{-1} \sum_{j=0}^{N-1} \left\{ \binom{k-1}{j}\binom{Z-k}{N-j-1} \frac{F}{N}(j+1) - \binom{k}{j}\binom{Z-1-k}{N-1-j} \frac{F}{N} j \right\}$$

Introducing the notation $\widetilde{x} = x - 1$ we may now write

$$f_C(k) - f_D(k) = c\binom{\widetilde{Z}}{\widetilde{N}}^{-1} \sum_{j=0}^{\widetilde{N}} \left\{ \binom{\widetilde{k}}{j}\binom{Z-k}{\widetilde{N}-j} \frac{F}{N}(j+1) - \binom{k}{j}\binom{\widetilde{Z}-k}{\widetilde{N}-j} \frac{F}{N} j \right\} =$$

$$= c(\frac{F}{N} - 1) + \frac{F}{N}\binom{Z-1}{N-1}^{-1} \sum_{j=0}^{\widetilde{N}} j \left\{ \binom{\widetilde{k}}{j}\binom{Z-k}{\widetilde{N}-j} - \binom{k}{j}\binom{\widetilde{Z}-k}{\widetilde{N}-j} \right\}$$

We may readily simplify the complicated sum obtaining the desired result:

$$f_C(k) - f_D(k) = c\left[(\frac{F}{N} - 1) + \frac{F}{N}\left(\frac{\tilde{N}}{\tilde{Z}}\right)(\tilde{k} - k) \right] = c\left[\frac{F}{N}\left(1 - \frac{\tilde{N}}{\tilde{Z}}\right) - 1 \right]$$

$$= c\left[\frac{F}{N}\left(1 - \frac{N-1}{Z-1}\right) - 1 \right] .$$