Logistic, Richards, Morgan-Mercer-Flodin

Gompertz: $y = a\,\exp\left(-b\,\mathrm{e}^{-gx}\right)$ formula only

Weibull-type: $y = a - b\,\exp\left(-gx^{d}\right)$ formula only

Logistic model

$$\boxed{y = \frac{a}{1 + b\,\exp(-gx)}} \qquad \{1\}$$

$$E_x = \exp(-gx) \qquad \{2\}$$

$$\boxed{y = \frac{a}{1 + bE_x}} \qquad \{3\}$$

Beginning and end

$$y_0 = \frac{a}{1+b} \qquad\qquad y_\infty = a \qquad \{4\}$$

1.st derivative

$$\begin{aligned} y' &= -a\,(1 + bE_x)^{-2}\frac{\mathrm{d}}{\mathrm{d}\,x}(1 + bE_x) = -a\,(1 + bE_x)^{-2}(-bgE_x) = \\ &= a\,bg\,(1 + bE_x)^{-2}E_x = bg\,\frac{a}{1 + bE_x} = \\ &= y\,bg\,\frac{E_x}{1 + bE_x} = y^2\,\frac{bg}{a}\,E_x \end{aligned} \qquad \{5\}$$

$$\boxed{y' = y^2\,\frac{bg}{a}\,E_x} \qquad \{6\}$$

2.nd derivative

$$\begin{aligned} y'' &= \frac{\mathrm{d}}{\mathrm{d}\,x}\left(y^2\,\frac{bg}{a}\,E_x\right) = \frac{bg}{a}\,\frac{\mathrm{d}}{\mathrm{d}\,x}\left(y^2E_x\right) = \frac{bg}{a}\left[2yy'E_x + y^2(-g)E_x\right] = \\ &= y\,\frac{bg}{a}\,E_x\,(2y' - gy) = y^2\,\frac{bg}{a}\,E_x\left(2\frac{y'}{y} - g\right) = \\ &= y^2\,\frac{bg^2}{a}\,E_x\left(2y\frac{b}{a}E_x - 1\right) = y'\left(2y\frac{b}{a}E_x - 1\right) \end{aligned} \qquad \{7\}$$

$$\boxed{y'' = y'g\left(2y\frac{b}{a}E_x - 1\right)} \qquad \{8\}$$

Inflection point

$$2y\frac{b}{a}E_x = 1 \quad \{9\}$$

$$\frac{a}{2b} = yE_x = \frac{a}{1+bE_x}E_x \quad \{10\}$$

$$1 + bE_x = 2bE_x \quad \{11\}$$

$$\boxed{x^* = \frac{1}{g}\ln b} \quad \{12\}$$

Function at inflection point

$$\begin{aligned} y^* &= \frac{a}{1 + b\exp\left(-gx^*\right)} = \\ &= \frac{a}{1 + b\exp\left(-g\frac{1}{g}\ln b\right)} = \frac{a}{1 + b\exp\left(\ln b^{-1}\right)} = \\ &= \frac{a}{1 + bb^{-1}} = \frac{a}{2} \end{aligned} \quad \{13\}$$

$$y^* = \frac{a}{2} \quad \{14\}$$

Derivative at inflection point

$$y' = \frac{a^2}{4}\frac{bg}{a}\exp\left(\ln b^{-1}\right) = \frac{ag}{4} \quad \{15\}$$

$$y'^* = \frac{ag}{4} \quad \{16\}$$

Parameters from problem data

$$a = y_\infty \qquad b = \frac{y_\infty}{y_0} - 1 \quad \{17\}$$

$$g = \frac{\ln b}{x^*} \quad \{18\}$$

Example: $y_0 = 5, y_\infty = 100, x^* = 0.6$:

$$\begin{bmatrix} a \\ b \\ g \end{bmatrix} = \begin{bmatrix} y_\infty \\ \dfrac{y_\infty}{y_0} - 1 \\ \dfrac{\ln b}{x^*} \end{bmatrix} = \begin{bmatrix} 100 \\ 19 \\ 4.9 \end{bmatrix} \qquad \{19\}$$

Richards model

$$\boxed{y = a\left[1 + b\exp(-gx)\right]^l} \qquad \{20\}$$

$$E_x = \exp(-gx) \qquad \{21\}$$

$$\boxed{y = a\left(1 + bE_x\right)^l} \qquad \{22\}$$

Beginning and end

$$y_0 = a\left(1 + b\right)^l \qquad\qquad y_\infty = a \qquad \{23\}$$

1.st derivative

$$\begin{aligned} y' &= a\,l\left(1 + bE_x\right)^{l-1}\frac{\mathrm{d}}{\mathrm{d}\,x}\left(1 + bE_x\right) = a\,l\left(1 + bE_x\right)^{l-1}\left(-bgE_x\right) = \\ &= -abg\,l\left(1 + bE_x\right)^{l-1}E_x = -y\ bg\,l\,\frac{E_x}{1 + bE_x} \end{aligned} \qquad \{24\}$$

$$\boxed{y' = -y\left(bg\,l\right)\frac{E_x}{1 + bE_x}} \qquad \{25\}$$

2.nd derivative

$$\begin{aligned} \frac{y''}{-bg\,l} &= \frac{\mathrm{d}}{\mathrm{d}\,x}\left(y\frac{E_x}{1 + bE_x}\right) = \frac{\mathrm{d}}{\mathrm{d}\,x}\left[yE_x\left(1 + bE_x\right)^{-1}\right] = \\ &= y'E_x\left(1 + bE_x\right)^{-1} - gyE_x\left(1 + bE_x\right)^{-1} - yE_x\left(1 + bE_x\right)^{-2}\left(-bg\right)E_x \end{aligned} \qquad \{26\}$$

$$\begin{aligned} \frac{y''}{-bg\,l} &= -y\left(bg\,l\right)\left(\frac{E_x}{1 + bE_x}\right)^2 - gy\frac{E_x}{1 + bE_x} + \left(bg\right)y\left(\frac{E_x}{1 + bE_x}\right)^2 = \\ &= \left(bg\right)\left(1 - l\right)y\left(\frac{E_x}{1 + bE_x}\right)^2 - gy\frac{E_x}{1 + bE_x} \end{aligned} \qquad \{27\}$$

$$\frac{y''}{-bg\,l}\left(\frac{1+bE_x}{E_x}\right)^2 = bg\,(1-l\,)y - gy\left(\frac{1+bE_x}{E_x}\right) = \quad = gy\left[b\,(1-l\,) - \left(\frac{1+bE_x}{E_x}\right)\right] \tag{28}$$

$$\boxed{\frac{y''}{-bg^2\,l\;y}\left(\frac{1+bE_x}{E_x}\right)^2 = b\,(1-l\,) - \left(\frac{1+bE_x}{E_x}\right)} \tag{29}$$

Inflection point

$$b\,(1-l\,)E_x = 1 + bE_x \tag{30}$$

$$b\,(-l\,)E_x = 1 \tag{31}$$

$$\exp(-gx^*) = -\frac{1}{b\,l} \tag{32}$$

$$\boxed{x^* = \frac{1}{g}\ln(-b\,l\,)} \tag{33}$$

Function at inflection point

$$y^* = a\left(1 + b\,\frac{-1}{b\,l}\right)^l \tag{34}$$

$$y^* = a\left(1 - \frac{1}{l}\right)^l \tag{35}$$

Derivative at inflection point

$$y' = a\left(1-\frac{1}{l}\right)^l (bg\,l\,)\frac{1/(b\,l\,)}{1-b/(b\,l\,)} = ag\left(1-\frac{1}{l}\right)^l \frac{1}{1-\frac{1}{l}} = \quad = ag\left(1-\frac{1}{l}\right)^{l-1} \tag{36}$$

$$y'^* = ag\left(1-\frac{1}{l}\right)^{l-1} \tag{37}$$

Parameters from problem data

$$a = y_\infty \qquad\qquad l = \frac{\ln(y_0/y_\infty)}{\ln(1+b\,)} \tag{38}$$

Example: $y_0 = 5$, $y_\infty = 100$, $x^* = 0.6$:

$$\begin{bmatrix} a \\ b \\ g \\ l \end{bmatrix} = \begin{bmatrix} y_\infty \\ \\ \\ 3 \end{bmatrix} = \begin{bmatrix} 100 \\ 19 \\ 4.9 \end{bmatrix} \qquad \{39\}$$

Morgan-Mercer-Flodin model

$$\boxed{y = \frac{bg + ax^d}{g + x^d}} \qquad \{40\}$$

Beginning and end ($d < 0$)

$$y_0 = b \qquad\qquad y_\infty = a \qquad \{41\}$$

1.st derivative, with $a > b$ and $d, g < 0$

$$\begin{aligned} y' &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(bg + ax^d\right)\left(g + x^d\right)^{-1}\right] = \\ &= adx^{d-1}\left(g + x^d\right)^{-1} - \left(bg + ax^d\right)\frac{dx^{d-1}}{\left(g + x^d\right)^2} = \\ &= \frac{adx^{d-1}\left(g + x^d\right) - \left(bg + ax^d\right)dx^{d-1}}{\left(g + x^d\right)^2} = gd\,\frac{a - b}{\left(g + x^d\right)^2}\,x^{d-1} \end{aligned} \qquad \{42\}$$

$$\boxed{y' = gd\,\frac{a - b}{\left(g + x^d\right)^2}\,x^{d-1}} \qquad \{43\}$$

2.nd derivative

$$\begin{aligned} \frac{y''}{gd\,(a - b)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(g + x^d\right)^2 x^{d-1}\right] = 2\left(g + x^d\right)dx^{d-1}\,x^{d-1} + \left(g + x^d\right)^2 (d - 1)x \\ &= \left(g + x^d\right)x^{d-2}\left[2dx^{2d-2} + \left(g + x^d\right)(d - 1)\right] = \\ &= \left(g + x^d\right)x^{d-2}\left[2dx^{2d-2} - (1 - d)\left(g + x^d\right)\right] \end{aligned} \qquad \{44\}$$

$$\boxed{\frac{y''}{gd\,(a - b)} = \left(g + x^d\right)x^{d-2}\left[2dx^{2d-2} - (1 - d)\left(g + x^d\right)\right]} \qquad \{45\}$$

Inflection point

$$2dx^{2d-2} = (1 - d)\left(g + x^d\right) = g + x^d - gd - dx^d \qquad \{46\}$$

$$\boxed{(1 - d)x^d - 2dx^{2d-2} + g\,(1 - d) = 0} \qquad \{47\}$$

Function at inflection point

{48}

{49}

Derivative at inflection point

{50}

{51}

Parameters from problem data

$$a = y_\infty \qquad\qquad b = y_0 \tag{52}$$

$$g = \frac{\ln b}{x^*} \tag{53}$$

Example: $y_0 = 5$, $y_\infty = 100$, $x^* =$ (?) 0.6:

$$\begin{bmatrix} a \\ b \\ g \\ d \end{bmatrix} = \begin{bmatrix} y_\infty \\ y_0 \\ ? \\ ? \end{bmatrix} = \begin{bmatrix} 100 \\ 5 \\ ? \\ ? \end{bmatrix} \tag{54}$$

❖