Proper values and proper vectors

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The calculation of the (distinct) proper values and of the proper vectors of a general, real matrix is presented. The characteristic polynomial is calculated by the method of A. M. Danilevskii, with the roots of the polynomial, which are the proper values, calculated by the method of Durand and Kerner. The method then permits the direct calculation of the proper vectors. This methodology is made available on the Internet.

Keywords: *proper values, proper vectors, characteristic polynomial, Danilevskii.*

1. Fundamentals and scope

The calculation of the *proper values* and *proper vectors* (also extensively known as *eigenvalues* and *eigenvectors*) is necessary in innumerable applications. The calculation of these entities is generally a difficult numerical task for matrices of high order, indeed three or more. Here, a method is presented to solve this problem for a real general, i.e., symmetrical or non-symmetrical square matrices, with the exception of identical proper values. For a (real) symmetrical matrix, its proper values are real, but a non-symmetrical matrix can have complex ones. These will, notwithstanding, be conjugate pairs and, for odd matrix order, at least one real. In what follows, no other distinction is made about the nature (real or complex) of the variables involved.

The scope of this stud is to obtain the aforementioned entities. We will start with a real matrix, and apply the method of A. M. Danilevskii to obtain its characteristic polynomial. From this polynomial, we calculate its roots, which are the proper values, by the method of Durand and Kerner. From these, the method of Danilevskii directly supplies the proper vectors.

2. The method of A. M. Danilevskii

Let A be a user given real square matrix, or order n. The method of A. M. Danilevskii is based on the conversion of A into Frobenius form by a series of similarity transformations. These preserve the proper values and alter the proper values in a simple way.

The matrix A will be successively left- and right-multiplied by $n-1$ matrices M^{-1} and M that differ from the identity matrix only in their row $n-1$. Supposing, for the moment, that it is $a_{n,n-1} \neq 0$ (the opposite being treated further below), this row is given by

$$\left(M_{n-1}\right)_{n-1,j} = \begin{bmatrix} -\dfrac{a_{n1}}{a_{n,n-1}} & -\dfrac{a_{n2}}{a_{n,n-1}} & \cdots & \dfrac{1}{a_{n,n-1}} & -\dfrac{a_{n\,n}}{a_{n,n-1}} \end{bmatrix} \qquad \{1\}$$

for $j = 1..n$. As is known, $\left(M_{n-1}\right)^{-1}$, the inverse of the matrix in Eq. {1}, has its row $n-1$ given simply by

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$$\left(M_{n-1}\right)^{-1}_{n-1,j} = \begin{bmatrix} a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{n\,n} \end{bmatrix} \qquad \{2\}$$

Thus, *A* is converted to another matrix, *B*, by the mentioned multiplications:

$$B = \left(M_{n-1}\right)^{-1} A\, M_{n-1} \qquad \{3\}$$

We will simplify the procedure using the following convention (of successive updates of *A*, instead of creating a succession of matrices).

$$A := \left(M_{n-1}\right)^{-1} A\, M_{n-1} \qquad \{4\}$$

In the end, we will have

$$A := \left(M_1\right)^{-1}\left(M_2\right)^{-1} \ldots \left(M_{n-2}\right)^{-1}\left(M_{n-1}\right)^{-1} A\, M_{n-1} M_{n-2} \ldots M_2 M_1 \qquad \{5\}$$

or

$$A := \left(M_{n-1} M_{n-2} \ldots M_2 M_1\right)^{-1} A \left(M_{n-1} M_{n-2} \ldots M_2 M_1\right) \qquad \{6\}$$

The product of the matrices *M* will be necessary for the calculation of the proper vectors (while the inverses will have no use). As done above for *A*, we can simplify the procedure using the following convention:

$$B := B\, M_{n-1} \qquad \{7\}$$

The method transforms *A* into the Frobenius normal form (ones to the left of the diagonal), i.e.,

$$A = \begin{bmatrix} p_1 & p_2 & \cdots & p_{n-1} & p_n \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \qquad \{8\}$$

with determinant

$$D(l) = \det \begin{bmatrix} p_1 - l & p_2 & \cdots & p_{n-1} & p_n \\ 1 & -l & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & -l & 0 \\ 0 & 0 & \ldots & 1 & -l \end{bmatrix} \qquad \{9\}$$

with the advantage of immediately giving the characteristic polynomial

$$D(l) = (-1)\left(l^n - p_1 l^{n-1} - p_2 l^{n-2} - \ldots - p_{n-1} l - p_n\right) \qquad \{10\}$$

The roots of the characteristic polynomial in Eq. {10}, the proper values, can be found by any convenient method. (Here, the method of Durand and Kener will be proposed.) Once the proper values, l_i, $i = 1..n$, are known, the present method permits the calculation of the proper vectors easily. Let *Y* be the following matrix.

$$Y = \begin{bmatrix} l_1^{n-1} & l_2^{n-1} & \dots & l_{n-1}^{n-1} & l_n^{n-1} \\ l_1^{n-2} & l_2^{n-2} & \dots & l_{n-1}^{n-2} & l_n^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ l_1^{1} & l_2^{1} & \dots & l_{n-1}^{1} & l_n^{1} \\ l_1^{0} & l_2^{0} & \dots & l_{n-1}^{0} & l_n^{0} \end{bmatrix} \qquad \{11\}$$

Then, it is

$$X = M\ Y \qquad \{12\}$$

In the next section, we present a computer algorithm for the method.

3. Algorithm of Danilveskii and application

The method of A. M. Danilevskii to calculate the proper values of matrix A can be summarized as follows.

1) Let $M_a = I$, the identity matrix. Do steps 2 to 3 from $n-1$ down to 1.

2) *a)* Let M^{-1} be an elementary matrix, I except in its row $n-1$, equal to row n of A and *b)* let M be its inverse, another elementary matrix, which is I except also in its row $n-1$, given by $-\dfrac{a_{n.j}}{a_{n,n-1}}$ for $j \neq n$–1 and $\dfrac{1}{a_{n,n-1}}$ for $j = n$–1 (if $a_{n,n-1} = 0$, see below).

3) Let $A := M^{-1}AM$ and $M_a := M_a M$.

4) Finally, the first row of A contains the coefficients of the characteristic polynomial with reversed sign, apart from the coefficient of l^n, which is 1. The roots will be the proper values.

5) The proper values will be $X = M_a Y$, where Y is a matrix each column of which contains the decreasing powers of l_i, from $n-1$ down to 0.

If, in step 2, the element $a_{n,n-1}$ is zero, search an alternative non-zero element in the same row, to its left, say $a_{n,\,k}$ ($k < n-1$), if it exists, and exchange rows and columns n and k (similarity). If there is no non-zero element, the original problem has split into two problems, one already solved and the remaining soluble by the same technique.

The algorithm is shown in Table 1 as applied to an illustrative case. It is available on the Internet by Casquilho [2010a].

The solution to the characteristic polynomial

$$l^4 - 3l^3 - 9l^2 + 28l - 6 = 0 \qquad \{13\}$$

—here, by the method of Durand and Kerner— yields the following roots

$$? = [3.3525, 2.4943, \text{-}\,3.08\,0.\,233] \qquad \{14\}$$

which are the proper values. Thus, with the intermediate matrix Y,

Table 1

$A =$

1	2	-1	2
2	1	0	-1
-1	0	0	1
2	-1	**2**	1

$M^{-1} =$

1	0	0	0
0	1	0	0
2	**-1**	**2**	**1**
0	0	0	1

$M =$

1	0	0	0
0	1	0	0
-1	0,5	0,5	-0,5
0	0	0	1

$M^{-1}A =$

1	2	-1	2
2	1	0	-1
0	2	0	8
2	-1	2	1

$A := M^{-1}A\,M =$

2	1,5	-0,5	2,5
2	1	0	-1
0	**2**	0	8
0	0	1	0

$M_a := M_a M$

1	0	0	0
0	1	0	0
-1	0,5	0,5	-0,5
0	0	0	1

$M^{-1} =$

1	0	0	0
0	**2**	**0**	**8**
0	0	1	0
0	0	0	1

$M =$

1	0	0	0
0	0,5	0	-4
0	0	1	0
0	0	0	1

$M^{-1}A =$

2	1,5	-0,5	2,5
4	2	8	-2
0	2	0	8
0	0	1	0

$A := M^{-1}A\,M =$

2	0,75	-0,5	-3,5
4	1	8	-10
0	1	0	0
0	0	1	0

$M_a := M_a M$

1	0	0	0
0	0,5	0	-4
-1	0,25	0,5	-2,5
0	0	0	1

$M^{-1} =$

4	**1**	**8**	**-10**
0	1	0	0
0	0	1	0
0	0	0	1

$M =$

0,25	-0,25	-2	2,5
0	1	0	0
0	0	1	0
0	0	0	1

$M^{-1}A =$

12	12	-4	-24
4	1	8	-10
0	1	0	0
0	0	1	0

$A := M^{-1}A\,M =$

3	9	-28	6
1	0	0	0
0	1	0	0
0	0	1	0

$M_a := M_a M$

0,25	-0,25	-2	2,5
0	0,5	0	-4
-0,25	0,5	2,5	-5
0	0	0	1

and from the algorithm of Durand-Kerner (below), it is

$Y =$

37,68	15,518	-29,21	0,0126
11,239	6,2215	9,4846	0,0543
3,3525	2,4943	-3,08	0,233
1	1	1	1

and the proper vectors, X, are

$X = M_a\, Y =$

2,4051	-0,164	-1,014	2,0236
1,6196	-0,889	0,7423	-3,973
-0,419	0,4669	-0,655	-4,394
1	1	1	1

4. Algorithm of Durand-Kerner and application

The proper values are the roots (complex, in the general case) of the characteristic polynomial. The algorithm of Durand-Kerner [Durand, 1960; Kerner, 1966] is an elegant method to find the roots. Only the case of distinct roots will be addressed here. (In usual applications, the roots are distinct, or the problem can be perturbed to make them distinct enough.) The method is iterative, as follows (all in complex arithmetic).

1) Let X be initial guesses for L.

2) Then, $x_i := x_i - \dfrac{f(X)}{\prod_{j \neq i} (x_i - x_j)}$, until convergence.

The initial guess must be neither a real number nor a root of unity. With p_i the coefficients of l^{n-i} ($i = 1..n$, remembering that the polynomial is monic), suggested values are ($i = 1..n$):

$$x_i = r \operatorname{cis} q_i \qquad \{15\}$$

with

$$r = 1 + \sum_{i=1}^{n} p_i \mathrel{\hat{=}} \text{same}/2 \qquad q_i = 2p\,\frac{i-1}{n} \qquad \{16\}$$

The algorithm was shown in Table 1 as applied to the illustrative case. It is available on the Internet by Casquilho [2010b].

5. Conclusion

The proper values of a general square matrix were obtained by the method of A. M. Danilevskii, which applies successive similarity transformations to the original matrix. Afterwards, the proper vectors were obtained from the proper values, calculated by the Durand-Kerner method, from a matrix with powers of the proper values and a matrix that is the product of the succession of the right-hand side similarity matrices.

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