# Systems of First Order Linear Differential Equations

We will now turn our attention to solving systems of simultaneous homogeneous first order linear differential equations. The solutions of such systems require much linear algebra (Math 220). But since it is not a prerequisite for this course, we have to limit ourselves to the simplest instances: those systems of two equations and two unknowns only. But first, we shall have a brief overview and learn some notations and terminology.

A system of $n$ linear first order differential equations in $n$ unknowns (an $n \times n$ system of linear equations) has the general form:

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + g_1 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + g_2 \\
x_3' &= a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n + g_3 \qquad (*) \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + g_n
\end{aligned}
$$

Where the coefficients $a_{ij}$'s, and $g_i$'s are arbitrary functions of $t$. If every term $g_i$ is constant zero, then the system is said to be homogeneous. Otherwise, it is a nonhomogeneous system if even one of the $g$'s is nonzero.

The system (*) is most often given in a shorthand format as a matrix-vector equation, in the form:

$$\boxed{\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}}$$

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ \vdots \\ x_n' \end{bmatrix}}_{\boldsymbol{x}'} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}}_{\boldsymbol{x}} + \underbrace{\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ \vdots \\ g_n \end{bmatrix}}_{\boldsymbol{g}}$$

Where the matrix of coefficients, $\boldsymbol{A}$, is called the *coefficient matrix* of the system. The vectors $\boldsymbol{x}'$, $\boldsymbol{x}$, and $\boldsymbol{g}$ are

$$\boldsymbol{x}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \qquad \boldsymbol{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{bmatrix}.$$

For a homogeneous system, $\boldsymbol{g}$ is the zero vector. Hence it has the form

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}.$$

 

*Fact*: Every $n$-th order linear equation is equivalent to a system of $n$ first order linear equations. (This relation is not one-to-one. There are multiple systems thus associated with each linear equation, for $n > 1$.)

*Examples*:

(i) The mechanical vibration equation $m u'' + \gamma u' + k u = F(t)$ is equivalent to

$$x_1' = x_2$$

$$x_2' = \frac{-k}{m} x_1 - \frac{\gamma}{m} x_2 + \frac{F(t)}{m}$$

Note that the system would be homogeneous (respectively, nonhomogeneous) if the original equation is homogeneous (respectively, nonhomogeneous).

(ii) $y''' - 2y'' + 3y' - 4y = 0$ is equivalent to

$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = 4x_1 - 3x_2 + 2x_3$$

This process can be easily generalized. Given an $n$-th order linear equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \ldots + a_2 y'' + a_1 y' + a_0 y = g(t).$$

Make the substitutions: $x_1 = y$, $x_2 = y'$, $x_3 = y''$, … , $x_n = y^{(n-1)}$, and $x_n' = y^{(n)}$. The first $n - 1$ equations follow thusly. Lastly, substitute the $x$'s into the original equation to rewrite it into the $n$-th equation and obtain the system of the form:

$$\begin{array}{ccl}
x_1' & = & x_2 \\
x_2' & = & x_3 \\
x_3' & = & x_4 \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
x_{n-1}' & = & x_n \\
 & & \\
x_n' & = & \dfrac{-a_0}{a_n} x_1 - \dfrac{a_1}{a_n} x_2 - \dfrac{a_2}{a_n} x_3 - \ldots - \dfrac{a_{n-1}}{a_n} x_n + \dfrac{g(t)}{a_n}
\end{array}$$

*Note*: The reverse is also true (mostly)*. Given an $n \times n$ system of linear equations, it can be rewritten into a single $n$-th order linear equation.

* The exceptions being the systems whose coefficient matrices are diagonal matrices. However, our Eigenvector method will nevertheless be able to solve them without any modification.

 

*Exercises D-1.1*:

1 – 3 Convert each linear equation into a system of first order equations.

1. $$y'' - 4y' + 5y = 0$$

2. $$y''' - 5y'' + 9y = t\cos 2t$$

3. $$y^{(4)} + 3y''' - \pi y'' + 2\pi y' - 6y = 11$$

4. Rewrite the system you found in (a) *Exercise* 1, and (b) *Exercise* 2, into a matrix-vector equation.

5. Convert the third order linear equation below into a system of 3 first order equation using (a) the usual substitutions, and (b) substitutions in the reverse order: $x_1 = y''$, $x_2 = y'$, $x_3 = y$. Deduce the fact that there are multiple ways to rewrite each $n$-th order linear equation into a linear system of $n$ equations.

$$y''' + 6y'' + y' - 2y = 0$$

*Answers D-1.1*:

1. $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -5x_1 + 4x_2 \end{aligned}$$

2. $$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -9x_1 + 5x_3 + t\cos 2t \end{aligned}$$

3. $$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= 6x_1 - 2\pi x_2 + \pi x_3 - 3x_4 + 11 \end{aligned}$$

4. (a) $$\boldsymbol{x}' = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix} \boldsymbol{x}$$

(b) $$\boldsymbol{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & 0 & 5 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ t\cos 2t \end{bmatrix}$$

5. (a) $$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 2x_1 - x_2 - 6x_3 \end{aligned}$$

(b) $$\begin{aligned} x_1' &= -6x_1 - x_2 + 2x_3 \\ x_2' &= x_1 \\ x_3' &= x_2 \end{aligned}$$

# A Crash Course in (2 × 2) Matrices

Several weeks worth of matrix algebra in an hour… (Relax, we will only study the simplest case, that of 2 × 2 matrices.)

**Review topics:**

1. What is a matrix (*pl*. matrices)?

A matrix is a rectangular array of objects (called *entries*). Those entries are usually numbers, but they can also include functions, vectors, or even other matrices. Each entry's position is addressed by the row and column (in that order) where it is located. For example, $a_{52}$ represents the entry positioned at the 5th row and the 2nd column of the matrix $\boldsymbol{A}$.

2. The size of a matrix

The size of a matrix is specified by 2 numbers

$$[\textit{number of rows}] \times [\textit{number of columns}].$$

Therefore, an $m \times n$ matrix is a matrix that contains $m$ rows and $n$ columns. A matrix that has equal number of rows and columns is called a *square matrix*. A square matrix of size $n \times n$ is usually referred to simply as a square matrix of size (or order) $n$.

Notice that if the number of rows or columns is 1, the result (respectively, a $1 \times n$, or an $m \times 1$ matrix) is just a vector. A $1 \times n$ matrix is called a *row vector*, and an $m \times 1$ matrix is called a *column vector*. Therefore, vectors are really just special types of matrices. Hence, you will probably notice the similarities between many of the matrix operations defined below and vector operations that you might be familiar with.

3. Two special types of matrices

*Identity matrices* (square matrices only)

The $n \times n$ identity matrix is often denoted by $\boldsymbol{I}_n$.

$$\boldsymbol{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ etc.}$$

Properties (assume $\boldsymbol{A}$ and $\boldsymbol{I}$ are of the same size):

$$\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$$
$$\boldsymbol{I}_n \boldsymbol{x} = \boldsymbol{x}, \qquad \boldsymbol{x} = \text{any } n \times 1 \text{ vector}$$

*Zero matrices* – matrices that contain all-zero entries.

Properties:

$$\boldsymbol{A} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{A} = \boldsymbol{A}$$
$$\boldsymbol{A}\,\boldsymbol{0} = \boldsymbol{0} = \boldsymbol{0}\,\boldsymbol{A}$$

4. Arithmetic operations of matrices

(i) <u>Addition / subtraction</u>

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \pm \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$$

(ii) Scalar Multiplication

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}, \qquad \text{for any scalar } k.$$

(iii) Matrix multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

The matrix multiplication $\boldsymbol{AB} = \boldsymbol{C}$ is defined only if there are as many rows in $\boldsymbol{B}$ as there are columns in $\boldsymbol{A}$. For example, when $\boldsymbol{A}$ is $m \times k$ and $\boldsymbol{B}$ is $k \times n$. The product matrix $\boldsymbol{C}$ is going to be of size $m \times n$, and whose $ij$-th entry, $c_{ij}$, is equal to the vector dot product between the $i$-th row of $\boldsymbol{A}$ and the $j$-th column of $\boldsymbol{B}$. Since vectors are matrices, we can also multiply together a matrix and a vector, assuming the above restriction on their sizes is met. The product of a $2 \times 2$ matrix and a 2-entry column vector is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}.$$

*Note 1:* Two square matrices of the same size can always be multiplied together. Because, obviously, having the same number of rows and columns, they satisfy the size requirement outlined above.

*Note 2*: In general, $\boldsymbol{AB} \neq \boldsymbol{BA}$. Indeed, depending on the sizes of $\boldsymbol{A}$ and $\boldsymbol{B}$, one product might not even be defined while the other product is.

5. Determinant (square matrices only)

For a $2 \times 2$ matrix, its determinant is given by the formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

*Note*: The determinant is a function whose domain is the set of all square matrices of a certain size, and whose range is the set of all real (or complex) numbers.

6. Inverse matrix (of a square matrix)

Given an $n \times n$ square matrix $\boldsymbol{A}$, if there exists a matrix $\boldsymbol{B}$ (necessarily of the same size) such that

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}_n,$$

then the matrix $\boldsymbol{B}$ is called the *inverse matrix* of $\boldsymbol{A}$, denoted $\boldsymbol{A}^{-1}$. The inverse matrix, if it exists, is unique for each $\boldsymbol{A}$. A matrix is called *invertible* if it has an inverse matrix.

*Theorem*: For any $2 \times 2$ matrix $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,
its inverse, if exists, is given by

$$\boldsymbol{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

*Theorem*: A square matrix is invertible if and only if its determinant is nonzero.

*Examples*: Let $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$.

(i) $2\boldsymbol{A} - \boldsymbol{B} = 2\begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2-2 & -4-(-3) \\ 10-(-1) & 4-4 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 11 & 0 \end{bmatrix}$

(ii) $\boldsymbol{AB} = \begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2+2 & -3-8 \\ 10-2 & -15+8 \end{bmatrix} = \begin{bmatrix} 4 & -11 \\ 8 & -7 \end{bmatrix}$

On the other hand:

$\boldsymbol{BA} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2-15 & -4-6 \\ -1+20 & 2+8 \end{bmatrix} = \begin{bmatrix} -13 & -10 \\ 19 & 10 \end{bmatrix}$

(iii) $\det(\boldsymbol{A}) = 2 - (-10) = 12$, $\quad \det(\boldsymbol{B}) = 8 - 3 = 5$.

*Since neither is zero, as a result, they are both invertible matrices*.

(iv) $\boldsymbol{A}^{-1} = \dfrac{1}{2-(-10)}\begin{bmatrix} 2 & 2 \\ -5 & 1 \end{bmatrix} = \dfrac{1}{12}\begin{bmatrix} 2 & 2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/6 \\ -5/12 & 1/12 \end{bmatrix}$

7. Systems of linear equations (also known as *linear systems*)

A system of linear (algebraic) equations, $\boldsymbol{Ax} = \boldsymbol{b}$, could have zero, exactly one, or infinitely many solutions. (Recall that each linear equation has a line as its graph. A solution of a linear system is a common intersection point of all the equations' graphs − and there are only 3 ways a set of lines could intersect.)

If the vector $\boldsymbol{b}$ on the right-hand side is the zero vector, then the system is called homogeneous. A homogeneous linear system always has a solution, namely the all-zero solution (that is, the origin). This solution is called the *trivial solution* of the system. Therefore, a homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ could have either exactly one solution, or infinitely many solutions. There is no other possibility, since it always has, at least, the trivial solution. If such a system has $n$ equations and exactly the same number of unknowns, then the number of solution(s) the system has can be determined, without having to solve the system, by the determinant of its coefficient matrix:

> *Theorem*: If $\boldsymbol{A}$ is an $n \times n$ matrix, then the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has exactly one solution (the trivial solution) if and only if $\boldsymbol{A}$ is invertible (that is, it has a nonzero determinant). It will have infinitely many solutions (the trivial solution, plus infinitely many nonzero solutions) if $\boldsymbol{A}$ is not invertible (equivalently, has zero determinant).

8. Eigenvalues and Eigenvectors

Given a square matrix $\boldsymbol{A}$, suppose there are a constant $r$ and a <u>nonzero</u> vector $\boldsymbol{x}$ such that

$$\boldsymbol{A}\boldsymbol{x} = r\boldsymbol{x},$$

then $r$ is called an *Eigenvalue* of $\boldsymbol{A}$, and $\boldsymbol{x}$ is an *Eigenvector* of $\boldsymbol{A}$ corresponding to $r$.

Do eigenvalues/vectors always exist for any given square matrix? The answer is yes. How do we find them, then?

Rewrite the above equation, we get $\boldsymbol{A}\boldsymbol{x} - r\boldsymbol{x} = \boldsymbol{0}$. The next step would be to factor out $\boldsymbol{x}$. But doing so would give the expression

$$(\boldsymbol{A} - r)\boldsymbol{x} = \boldsymbol{0}.$$

Notice that it requires us to subtract a number from an $n \times n$ matrix. That's an undefined operation. Hence, we need to further refined it by rewriting the term $r\boldsymbol{x} = r\boldsymbol{I}\boldsymbol{x}$, and then factoring out $\boldsymbol{x}$, obtaining

$$(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}.$$

This is an $n \times n$ system of homogeneous linear (algebraic) equations, where the coefficient matrix is $(\boldsymbol{A} - r\boldsymbol{I})$. We are looking for a nonzero solution $\boldsymbol{x}$ of this system. Hence, by the theorem we have just seen, the necessary and sufficient condition for the existence of such a nonzero solution, which will become an eigenvector of $\boldsymbol{A}$, is that the coefficient matrix $(\boldsymbol{A} - r\boldsymbol{I})$ must have zero determinant. Set its determinant to zero and what we get is a degree $n$ polynomial equation in terms of $r$. The case of a $2 \times 2$ matrix is as follow:

$$\boldsymbol{A} - r\boldsymbol{I} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a - r & b \\ c & d - r \end{bmatrix}.$$

Its determinant, set to 0, yields the equation

$$\det\begin{bmatrix} a-r & b \\ c & d-r \end{bmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + (ad-bc) = 0$$

It is a degree 2 polynomial equation of $r$, as you can see.

This polynomial on the left is called the *characteristic polynomial* of the (original) matrix $\boldsymbol{A}$, and the equation is the *characteristic equation* of $\boldsymbol{A}$. The root(s) of the characteristic polynomial are the eigenvalues of $\boldsymbol{A}$. Since any degree $n$ polynomial always has $n$ roots (real and/or complex; not necessarily distinct), any $n \times n$ matrix always has at least one, and up to $n$ different eigenvalues.

Once we have found the eigenvalue(s) of the given matrix, we put each specific eigenvalue back into the linear system $(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$ to find the corresponding eigenvectors.

*Examples*: $\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$

$$\boldsymbol{A} - r\boldsymbol{I} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} - r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-r & 3 \\ 4 & 3-r \end{bmatrix}.$$

Its characteristic equation is

$$\det\begin{bmatrix} 2-r & 3 \\ 4 & 3-r \end{bmatrix} = (2-r)(3-r) - 12 = r^2 - 5r - 6 = (r+1)(r-6) = 0$$

The eigenvalues are, therefore, $r = -1$ and 6.

Next, we will substitute each of the 2 eigenvalues into the matrix equation $(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = \boldsymbol{0}$.

For $r = -1$, the system of linear equations is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = (\boldsymbol{A} + \boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 2+1 & 3 \\ 4 & 3+1 \end{bmatrix} x = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Notice that the matrix equation represents a degenerated system of 2 linear equations. Both equations are constant multiples of the equation $x_1 + x_2 = 0$. There is now only 1 equation for the 2 unknowns, therefore, there are infinitely many possible solutions. <u>This is always the case when solving for eigenvectors</u>. Necessarily, there are infinitely many eigenvectors corresponding to each eigenvalue.

Solving the equation $x_1 + x_2 = 0$, we get the relation $x_2 = -x_1$. Hence, the eigenvectors corresponding to $r = -1$ are all nonzero multiples of

$$k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Similarly, for $r = 6$, the system of equations is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = (\boldsymbol{A} - 6\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 2-6 & 3 \\ 4 & 3-6 \end{bmatrix} x = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Both equations in this second linear system are equivalent to $4x_1 - 3x_2 = 0$. Its solutions are given by the relation $4x_1 = 3x_2$. Hence, the eigenvectors corresponding to $r = 6$ are all nonzero multiples of

$$k_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

*Note*: Every nonzero multiple of an eigenvector is also an eigenvector.

**Two short-cuts to find eigenvalues:**

1. If $\boldsymbol{A}$ is a diagonal or triangular matrix, that is, if it has the form

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \text{ or } \qquad \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \text{ or } \qquad \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}.$$

Then the eigenvalues are just the main diagonal entries, $r = a$ and $d$ in all 3 examples above.

2. If $\boldsymbol{A}$ is any $2 \times 2$ matrix, then its characteristic equation is

$$\det \begin{bmatrix} a-r & b \\ c & d-r \end{bmatrix} = r^2 - (a+d)r + (ad - bc) = 0$$

If you are familiar with terminology of linear algebra, the characteristic equation can be memorized rather easily as

$$\boxed{r^2 - \text{Trace}(\boldsymbol{A})\, r + \det(\boldsymbol{A}) = 0.}$$

*Note*: For any square matrix $\boldsymbol{A}$, Trace($\boldsymbol{A}$) = [*sum of all entries on the main diagonal (running from top-left to bottom-right)*]. For a $2 \times 2$ matrix $\boldsymbol{A}$, Trace($\boldsymbol{A}$) = $a + d$.

**A short-cut to find eigenvectors (of a 2 × 2 matrix):**

Similarly, there is a trick that enables us to find the eigenvectors of any 2 × 2 matrix without having to go through the whole process of solving systems of linear equations. This short-cut is especially handy when the eigenvalues are complex numbers, since it avoids the need to solve the linear equations which will have complex number coefficients. (*Warning*: *This method does not work for any matrix of size larger than 2 × 2.*)

We first find the eigenvalue(s) and then write down, for each eigenvalue, the matrix ($\boldsymbol{A} - r\boldsymbol{I}$) as usual. Then we take *any* row of ($\boldsymbol{A} - r\boldsymbol{I}$) that is not consisted of entirely zero entries, say it is the row vector ($\alpha$ , $\beta$). We put a minus sign in front of one of the entries, for example, ($\alpha$ , $-\beta$). Then an eigenvector of the matrix A is found by switching the two entries in the above vector, that is, $\boldsymbol{k} = (-\beta , \alpha)$.

*Example*: Previously, we have seen $\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$.

The characteristic equation is

$$r^2 - \mathrm{Trace}(\boldsymbol{A})\, r + \det(\boldsymbol{A}) = r^2 - 5r - 6 = (r + 1)(r - 6) = 0,$$

which has roots $r = -1$ and 6. For $r = -1$, the matrix ($\boldsymbol{A} - r\boldsymbol{I}$) is $\begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$.

Take the first row, (3, 3), which is a non-zero vector; put a minus sign to the first entry to get (−3, 3); then switch the entry, we now have $\boldsymbol{k}_1 = (3, -3)$. It is indeed an eigenvector, since it is a nonzero constant multiple of the vector we found earlier.

On very rare occasions, both rows of the matrix ($\boldsymbol{A} - r\boldsymbol{I}$) have all zero entries. If so, the above algorithm will not be able to find an eigenvector. Instead, under this circumstance any non-zero vector will be an eigenvector.

*Exercises D-1.2*:

$$\text{Let } \boldsymbol{C} = \begin{bmatrix} -5 & -1 \\ 7 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{D} = \begin{bmatrix} 2 & 0 \\ -2 & -1 \end{bmatrix}.$$

1. Compute: (*i*) $\boldsymbol{C} + 2\boldsymbol{D}$ and (*ii*) $3\boldsymbol{C} - 5\boldsymbol{D}$.

2. Compute: (*i*) $\boldsymbol{CD}$ and (*ii*) $\boldsymbol{DC}$.

3. Compute: (*i*) $\det(\boldsymbol{C})$, (*ii*) $\det(\boldsymbol{D})$, (*iii*) $\det(\boldsymbol{CD})$, (*iv*) $\det(\boldsymbol{DC})$.

4. Compute: (*i*) $\boldsymbol{C}^{-1}$, (*ii*) $\boldsymbol{D}^{-1}$, (*iii*) $(\boldsymbol{CD})^{-1}$, (*iv*) show $(\boldsymbol{CD})^{-1} = \boldsymbol{D}^{-1}\boldsymbol{C}^{-1}$.

5. Find the eigenvalues and their corresponding eigenvectors of $\boldsymbol{C}$ and $\boldsymbol{D}$.

*Answers D-1.2*:

1. (*i*) $\begin{bmatrix} -1 & -1 \\ 3 & 1 \end{bmatrix}$, (*ii*) $\begin{bmatrix} -25 & -3 \\ 31 & 14 \end{bmatrix}$

2. (*i*) $\begin{bmatrix} -8 & 1 \\ 8 & -3 \end{bmatrix}$, (*ii*) $\begin{bmatrix} -10 & -2 \\ 3 & -1 \end{bmatrix}$

3. (*i*) −8, (*ii*) −2, (*iii*) 16, (*iv*) 16

4. (*i*) $\begin{bmatrix} -3/8 & -1/8 \\ 7/8 & 5/8 \end{bmatrix}$, (*ii*) $\begin{bmatrix} 1/2 & 0 \\ -1 & -1 \end{bmatrix}$, (*iii*) $\begin{bmatrix} -3/16 & -1/16 \\ -1/2 & -1/2 \end{bmatrix}$, (*iv*) The equality is not a coincidence. In general, for any pair of invertible matrices $\boldsymbol{C}$ and $\boldsymbol{D}$, $(\boldsymbol{CD})^{-1} = \boldsymbol{D}^{-1}\boldsymbol{C}^{-1}$.

5. (*i*) $r_1 = 2$, $k_1 = \begin{bmatrix} s \\ -7s \end{bmatrix}$; $r_2 = -4$, $k_2 = \begin{bmatrix} s \\ -s \end{bmatrix}$ ; $s$ = any nonzero number

(*ii*) $r_1 = 2$, $k_1 = \begin{bmatrix} s \\ -2s/3 \end{bmatrix}$; $r_2 = -1$, $k_2 = \begin{bmatrix} 0 \\ s \end{bmatrix}$ ; $s$ = any nonzero number

## Solution of 2 × 2 systems of first order linear equations

Consider a system of 2 simultaneous first order linear equations

$$x_1' = a x_1 + b x_2$$
$$x_2' = c x_1 + d x_2$$

It has the alternate matrix-vector representation

$$\boldsymbol{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \boldsymbol{x}.$$

Or, in shorthand $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, if $\boldsymbol{A}$ is already known from context.

We know that the above system is equivalent to a second order homogeneous linear differential equation. As a result, we know that the general solution contains two linearly independent parts. As well, the solution will be consisted of some type of exponential functions. Therefore, assume that $\boldsymbol{x} = \boldsymbol{k}\, e^{rt}$ is a solution of the system, where $\boldsymbol{k}$ is a vector of coefficients (of $x_1$ and $x_2$). Substitute $\boldsymbol{x}$ and $\boldsymbol{x}' = r\, \boldsymbol{k}\, e^{rt}$ into the equation $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$, and we have

$$r\, \boldsymbol{k}\, e^{rt} = \boldsymbol{A}\, \boldsymbol{k}\, e^{rt}.$$

Since $e^{rt}$ is never zero, we can always divide both sides by $e^{rt}$ and get

$$r\, \boldsymbol{k} = \boldsymbol{A}\, \boldsymbol{k}.$$

We see that this new equation is exactly the relation that defines eigenvalues and eigenvectors of the coefficient matrix $\boldsymbol{A}$. In other words, in order for a function $\boldsymbol{x} = \boldsymbol{k}\, e^{rt}$ to satisfy our system of differential equations, the number $r$ must be an eigenvalue of $\boldsymbol{A}$, and the vector $\boldsymbol{k}$ must be an eigenvector of $\boldsymbol{A}$ corresponding to $r$. Just like the solution of a second order homogeneous linear equation, there are three possibilities, depending on the number of distinct, and the type of, eigenvalues the coefficient matrix $\boldsymbol{A}$ has.

 

The possibilities are that $\boldsymbol{A}$ has

I. Two distinct real eigenvalues
II. Complex conjugate eigenvalues
III. A repeated eigenvalue

A related note, (from linear algebra,) we know that eigenvectors that each corresponds to a different eigenvalue are always linearly independent from each others. Consequently, if $r_1$ and $r_2$ are two different eigenvalues, then their respective eigenvectors $\boldsymbol{k}_1$ anf $\boldsymbol{k}_2$, and therefore the corresponding solutions, are always linearly independent.

**Case I** Distinct real eigenvalues

If the coefficient matrix $\boldsymbol{A}$ has two distinct real eigenvalues $r_1$ and $r_2$, and their respective eigenvectors are $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Then the $2 \times 2$ system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a general solution

$$\boxed{x = C_1 k_1 e^{r_1 t} + C_2 k_2 e^{r_2 t}.}$$

*Example*: $$\boldsymbol{x}' = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} \boldsymbol{x}.$$

We have already found that the coefficient matrix has eigenvalues $r = -1$ and 6. And they each respectively has an eigenvector

$$k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad k_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Therefore, a general solution of this system of differential equations is

$$x = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{6t}$$

*Example*: $\boldsymbol{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \boldsymbol{x}$, $\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The characteristic equation is $r^2 - r - 2 = (r + 1)(r - 2) = 0$. The eigenvalues are $r = -1$ and 2. They have, respectively, eigenvectors

For $r = -1$, the system is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = (\boldsymbol{A} + \boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 3+1 & -2 \\ 2 & -2+1 \end{bmatrix} x = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the bottom equation of the system: $2x_1 - x_2 = 0$, we get the relation $x_2 = 2x_1$. Hence,

$$k_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

For $r = 2$, the system is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = (\boldsymbol{A} - 2\,\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} x = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the first equation of the system: $x_1 - 2x_2 = 0$, we get the relation $x_1 = 2x_2$. Hence,

$$k_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore, a general solution is

$$x = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}.$$

Apply the initial values,

$$x(0) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{0} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{0} = \begin{bmatrix} C_1 + 2C_2 \\ 2C_1 + C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

That is

$$\begin{array}{cccc} C_1 & + & 2C_2 & = 1 \\ 2C_1 & + & C_2 & = -1 \end{array}.$$

We find $C_1 = -1$ and $C_2 = 1$, hence we have the particular solution

$$x = -\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} -e^{-t} + 2e^{2t} \\ -2e^{-t} + e^{2t} \end{bmatrix}.$$

**Case II** Complex conjugate eigenvalues

If the coefficient matrix $\boldsymbol{A}$ has two distinct complex conjugate eigenvalues $\lambda \pm \mu i$. Also suppose $\boldsymbol{k} = \boldsymbol{a} + \boldsymbol{b}\, i$ is an eigenvector (necessarily has complex-valued entries) of the eigenvalue $\lambda + \mu i$. Then the $2 \times 2$ system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a <u>real-valued</u> general solution

$$\boxed{x = C_1 e^{\lambda t}\big(a\cos(\mu t) - b\sin(\mu t)\big) + C_2 e^{\lambda t}\big(a\sin(\mu t) + b\cos(\mu t)\big)}$$

*A little detail*: Similar to what we have done before, first there was the complex-valued general solution in the form

$$x = C_1 k_1\, e^{(\lambda+\mu i)t} + C_2 k_2\, e^{(\lambda-\mu i)t}.$$

We "filter out" the imaginary parts by carefully choosing two sets of coefficients to obtain two corresponding real-valued solutions that are also linearly independent:

$$u = e^{\lambda t}\big(a\cos(\mu t) - b\sin(\mu t)\big)$$
$$v = e^{\lambda t}\big(a\sin(\mu t) + b\cos(\mu t)\big)$$

The real-valued general solution above is just $\boldsymbol{x} = C_1\,\boldsymbol{u} + C_2\,\boldsymbol{v}$. In particular, it might be useful to know how $\boldsymbol{u}$ and $\boldsymbol{v}$ could be derived by expanding the following complex-valued expression (the front half of the complex-valued general solution):

$$k_1\, e^{(\lambda+\mu i)t} = (a+bi)e^{\lambda t}e^{(\mu t)i} = e^{\lambda t}(a+bi)(\cos(\mu t) + i\sin(\mu t))$$
$$= e^{\lambda t}(a\cos(\mu t) + ia\sin(\mu t) + ib\cos(\mu t) + i^2 b\sin(\mu t))$$
$$= e^{\lambda t}(a\cos(\mu t) - b\sin(\mu t)) + i\, e^{\lambda t}(a\sin(\mu t) + b\cos(\mu t))$$

Then, $\boldsymbol{u}$ is just the real part of this complex-valued function, and $\boldsymbol{v}$ is its imaginary part.

*Example*:
$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$$

The characteristic equation is $r^2 + 1 = 0$, giving eigenvalues $r = \pm\, i$. That is, $\lambda = 0$ and $\mu = 1$.

Take the first (the one with positive imaginary part) eigenvalue $r = i$, and find one of its eigenvectors:

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the first equation of the system: $(2 - i)x_1 - 5x_2 = 0$, we get the relation $(2 - i)x_1 = 5x_2$. Hence,

$$k = \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = \underset{\boldsymbol{a}}{\begin{bmatrix} 5 \\ 2 \end{bmatrix}} + \underset{\boldsymbol{b}}{\begin{bmatrix} 0 \\ -1 \end{bmatrix}} i = a + bi$$

Therefore, a general solution is

$$x = C_1 e^{0t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\cos(t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(t)\right) + C_2 e^{0t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\sin(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(t)\right)$$
$$= C_1\begin{pmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{pmatrix} + C_2\begin{pmatrix} 5\sin(t) \\ 2\sin(t) - \cos(t) \end{pmatrix}$$

*Example*: $\boldsymbol{x}' = \begin{bmatrix} -1 & -6 \\ 3 & 5 \end{bmatrix} \boldsymbol{x}$, $\boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

The characteristic equation is $r^2 - 4r + 13 = 0$, giving eigenvalues $r = 2 \pm 3i$. Thus, $\lambda = 2$ and $\mu = 3$.

Take $r = 2 + 3i$ and find one of its eigenvectors:

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} -1-(2+3i) & -6 \\ 3 & 5-(2+3i) \end{bmatrix} x = \begin{bmatrix} -3-3i & -6 \\ 3 & 3-3i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the second equation of the system: $3x_1 + (3 - 3i)x_2 = 0$, we get the relation $x_1 = (-1 + i)x_2$. Hence,

$$k = \begin{bmatrix} -1+i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i = a + bi$$

The general solution is

$$x = C_1 e^{2t}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\cos(3t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(3t)\right) + C_2 e^{2t}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\sin(3t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(3t)\right)$$

$$= C_1 e^{2t}\begin{pmatrix} -\cos(3t) - \sin(3t) \\ \cos(3t) \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} \cos(3t) - \sin(3t) \\ \sin(3t) \end{pmatrix}$$

Apply the initial values to find $C_1$ and $C_2$:

$$x(0) = C_1 e^0 \left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cos(0) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(0) \right) + C_2 e^0 \left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin(0) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(0) \right)$$

$$= C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -C_1 + C_2 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Therefore, $C_1 = 2$ and $C_2 = 2$. Consequently, the particular solution is

$$x = 2e^{2t} \begin{pmatrix} -\cos(3t) - \sin(3t) \\ \cos(3t) \end{pmatrix} + 2e^{2t} \begin{pmatrix} \cos(3t) - \sin(3t) \\ \sin(3t) \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} -4\sin(3t) \\ 2\cos(3t) + 2\sin(3t) \end{pmatrix}$$

**Case III** Repeated real eigenvalue

Suppose the coefficient matrix $\boldsymbol{A}$ has a repeated real eigenvalues $r$, there are 2 sub-cases.

(i) If $r$ has two linearly independent eigenvectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Then the $2 \times 2$ system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a general solution

$$\boxed{\boldsymbol{x} = C_1 \boldsymbol{k}_1 e^{rt} + C_2 \boldsymbol{k}_2 e^{rt}.}$$

*Note*: For $2 \times 2$ matrices, this possibility only occurs when the coefficient matrix $\boldsymbol{A}$ is a scalar multiple of the identity matrix. That is, $\boldsymbol{A}$ has the form

$$\alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \qquad \text{for any constant } \alpha.$$

*Example*:
$$\boldsymbol{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \boldsymbol{x}.$$

The eigenvalue is $r = 2$ (repeated). There are 2 sets of linearly independent eigenvectors, which could be represented by any 2 nonzero vectors that are not constant multiples of each other. For example

$$k_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad k_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, a general solution is

$$x = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}.$$

(ii) If $r$, as it usually does, only has one linearly independent eigenvector $\boldsymbol{k}$. Then the $2 \times 2$ system $\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$ has a general solution

$$\boxed{\boldsymbol{x} = C_1 \boldsymbol{k} e^{rt} + C_2(\boldsymbol{k}\, t e^{rt} + \boldsymbol{\eta}\, e^{rt}).}$$

Where the second vector $\boldsymbol{\eta}$ is any solution of the nonhomogeneous linear system of algebraic equations

$$\boxed{(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{\eta} = \boldsymbol{k}.}$$

*Example*: $\quad \boldsymbol{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$

The eigenvalue is $r = -3$ (repeated). The corresponding system is

$$(\boldsymbol{A} - r\boldsymbol{I})\,\boldsymbol{x} = \begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} x = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Both equations of the system are $4x_1 - 4x_2 = 0$, we get the same relation $x_1 = x_2$. Hence, there is only one linearly independent eigenvector:

$$k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, solve for $\boldsymbol{\eta}$:

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \boldsymbol{\eta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It has solution in the form $\boldsymbol{\eta} = \begin{bmatrix} \frac{1}{4} + \eta_2 \\ \eta_2 \end{bmatrix}$.

Choose $\eta_2 = 0$, we get $\boldsymbol{\eta} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$.

A general solution is, therefore,

$$x = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + C_2 \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} e^{-3t} \right)$$

Apply the initial values to find $C_1 = 1$ and $C_2 = -12$. The particular solution is

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} - 12 \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{-3t} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} e^{-3t} \right) = \begin{bmatrix} -12t - 2 \\ -12t + 1 \end{bmatrix} e^{-3t}$$

## <u>Summary</u>: Solving a Homogeneous System of Two Linear First Order Equations in Two Unknowns

Given:

$$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}.$$

First find the two eigenvalues, $r$, and their respective corresponding eigenvectors, $\boldsymbol{k}$, of the coefficient matrix $\boldsymbol{A}$. Depending on the eigenvalues and eigenvectors, the general solution is:

I. Two distinct real eigenvalues $r_1$ and $r_2$:

$$x = C_1 k_1 e^{r_1 t} + C_2 k_2 e^{r_2 t}.$$

II. Two complex conjugate eigenvalues $\lambda \pm \mu i$, where $\lambda + \mu i$ has as an eigenvector $\boldsymbol{k} = \boldsymbol{a} + \boldsymbol{b}\, i$:

$$x = C_1 e^{\lambda t}\left(a\cos(\mu t) - b\sin(\mu t)\right) + C_2 e^{\lambda t}\left(a\sin(\mu t) + b\cos(\mu t)\right)$$

III. A repeated real eigenvalue $r$:

(i) When two linearly independent eigenvectors exist –

$$\boldsymbol{x} = C_1 \boldsymbol{k}_1 e^{rt} + C_2 \boldsymbol{k}_2 e^{rt}.$$

(ii) When only one linearly independent eigenvector exist –

$$\boldsymbol{x} = C_1 \boldsymbol{k} e^{rt} + C_2(\boldsymbol{k}\, t\, e^{rt} + \boldsymbol{\eta}\, e^{rt}).$$

*Note*: Solve the system $(\boldsymbol{A} - r\boldsymbol{I})\boldsymbol{\eta} = \boldsymbol{k}$ to find the vector $\boldsymbol{\eta}$.

*Exercises D-1.3*:

1. Rewrite the following second order linear equation into a system of two equations.

$$y'' + 5y' - 6y = 0$$

Then: (a) show that both the given equation and the new system have the same characteristic equation. (b) Find the system's general solution.

2 – 7 Find the general solution of each system below.

2. $\boldsymbol{x}' = \begin{bmatrix} 2 & 7 \\ -5 & -10 \end{bmatrix}\boldsymbol{x}.$

3. $\boldsymbol{x}' = \begin{bmatrix} -3 & 6 \\ -3 & 3 \end{bmatrix}\boldsymbol{x}.$

4. $\boldsymbol{x}' = \begin{bmatrix} 8 & -4 \\ 1 & 4 \end{bmatrix}\boldsymbol{x}.$

5. $\boldsymbol{x}' = \begin{bmatrix} -3 & 2 \\ -1 & -5 \end{bmatrix}\boldsymbol{x}.$

6. $\boldsymbol{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\boldsymbol{x}.$

7. $\boldsymbol{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -4 \end{bmatrix}\boldsymbol{x}.$

8 – 15 Solve the following initial value problems.

8. $\boldsymbol{x}' = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\boldsymbol{x},$ $\quad \boldsymbol{x}(0) = \begin{bmatrix} -4 \\ 2 \end{bmatrix}.$

9. $\boldsymbol{x}' = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}\boldsymbol{x},$ $\quad \boldsymbol{x}(3) = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$

10. $\boldsymbol{x}' = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\boldsymbol{x},$ $\quad \boldsymbol{x}(1) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$

11. $\boldsymbol{x}' = \begin{bmatrix} 6 & 8 \\ 2 & 6 \end{bmatrix}\boldsymbol{x},$ $\quad \boldsymbol{x}(0) = \begin{bmatrix} 8 \\ 0 \end{bmatrix}.$

12. $\boldsymbol{x}' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\boldsymbol{x},$ $\quad \boldsymbol{x}(0) = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$

13. $\boldsymbol{x}' = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix} \boldsymbol{x}$, $\qquad \boldsymbol{x}(-55) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

14. $\boldsymbol{x}' = \begin{bmatrix} 6 & 3 \\ -2 & 1 \end{bmatrix} \boldsymbol{x}$, $\qquad \boldsymbol{x}(20) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

15. $\boldsymbol{x}' = \begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix} \boldsymbol{x}$, $\qquad \boldsymbol{x}(243) = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$.

16. For each of the initial value problems #8 through #15, how does the solution behave as $t \to \infty$?

17. Find the general solution of the system below, and determine the possible values of $\alpha$ and $\beta$ such that the initial value problem has a solution that tends to the zero vector as $t \to \infty$.

$$\boldsymbol{x}' = \begin{bmatrix} -5 & -1 \\ 7 & 3 \end{bmatrix} \boldsymbol{x}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

*Answers D-1.3*:

1. (a) $r^2 + 5r - 6 = 0$, (b) $x = C_1 \begin{bmatrix} -1 \\ 6 \end{bmatrix} e^{-6t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$

2. $x = C_1 \begin{bmatrix} 7 \\ -5 \end{bmatrix} e^{-3t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t}$

3. $x = C_1 \begin{bmatrix} \cos(3t) + \sin(3t) \\ \cos(3t) \end{bmatrix} + C_2 \begin{bmatrix} -\cos(3t) + \sin(3t) \\ \sin(3t) \end{bmatrix}$

4. $x = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{6t} + C_2 \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} t e^{6t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{6t} \right)$

5. $x = C_1 e^{-4t} \begin{bmatrix} \cos(t) - \sin(t) \\ -\cos(t) \end{bmatrix} + C_2 e^{-4t} \begin{bmatrix} \cos(t) + \sin(t) \\ -\sin(t) \end{bmatrix}$

6. $x = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

7. $x = C_1 \begin{bmatrix} 5 \\ 1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}$

8. $x = e^{-t} \begin{bmatrix} -4\cos t - 2\sin t \\ 2\cos t - 4\sin t \end{bmatrix}$

9. $x = \begin{bmatrix} 5e^{-4t+12} \\ -2e^{-4t+12} \end{bmatrix}$

10. $x = \begin{bmatrix} 2e^{5t-5} - 2e^{-t+1} \\ 2e^{5t-5} + e^{-t+1} \end{bmatrix}$

11. $x = \begin{bmatrix} 4e^{2t} + 4e^{10t} \\ -2e^{2t} + 2e^{10t} \end{bmatrix}$

12. $x = \begin{bmatrix} 5 + e^{2t} \\ 5 - e^{2t} \end{bmatrix}$

13. $x = \begin{bmatrix} 9 - 6e^{-6t+330} \\ 3 + 2e^{-6t+330} \end{bmatrix}$

14. $x = \begin{bmatrix} 5e^{3t-60} - 6e^{4t-80} \\ -5e^{3t-60} + 4e^{4t-80} \end{bmatrix}$

15. $x = e^{2t-486} \begin{bmatrix} 5\cos(3t-729) + 25\sin(3t-729) \\ 14\cos(3t-729) - 8\sin(3t-729) \end{bmatrix}$

16. For #8 and 9, $\lim_{t\to\infty} x(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. For #10, 11, 12, 14, and 15, the limits do not exist, as $\boldsymbol{x}(t)$ moves infinitely far away from the origin. For #13, $\lim_{t\to\infty} x(t) = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$

17. $x = C_1 \begin{bmatrix} 1 \\ -7 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$; the particular solution will tend to zero as $t \to \infty$ provided that $C_1 = 0$, which can be achieved whenever the initial condition is such that $\alpha = -\beta$ (i.e., $\alpha + \beta = 0$, including the case $\alpha = \beta = 0$).

# The Laplace Transform Method of Solving Systems of Linear Equations (Optional topic)

The method of Laplace transforms, in addition to solving individual linear differential equations, can also be used to solve systems of simultaneous linear equations. The same basic steps of transforming, simplifying, and taking the inverse transform of the solution still apply.

*Example*: $\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}\mathbf{x}$, $\quad \mathbf{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Before we start, let us rewrite the problem into the explicit form of individual linear equations:

$$x_1' = x_1 - 4x_2 \qquad x_1(0) = -2$$
$$x_2' = 4x_1 - 7x_2 \qquad x_2(0) = 1$$

We then first transform both equations using the usual rules of Laplace transform:

$$s\mathcal{L}\{x_1\} - x_1(0) = s\mathcal{L}\{x_1\} + 2 = \mathcal{L}\{x_1\} - 4\mathcal{L}\{x_2\} \tag{1}$$

$$s\mathcal{L}\{x_2\} - x_2(0) = s\mathcal{L}\{x_2\} - 1 = 4\mathcal{L}\{x_1\} - 7\mathcal{L}\{x_2\} \tag{2}$$

Partially simplifying both equations

$$(s-1)\mathcal{L}\{x_1\} + 4\mathcal{L}\{x_2\} = -2 \tag{1*}$$

$$-4\mathcal{L}\{x_1\} + (s+7)\mathcal{L}\{x_2\} = 1 \tag{2*}$$

Then multiply eq. (1*) by −4 and eq. (2*) by $s - 1$.

$$-4(s-1)\mathcal{L}\{x_1\} - 16\mathcal{L}\{x_2\} = 8 \qquad (1^{**})$$

$$-4(s-1)\mathcal{L}\{x_1\} + (s-1)(s+7)\mathcal{L}\{x_2\} = s-1 \qquad (2^{**})$$

Subtract eq. (1**) from eq. (2**)

$$[(s-1)(s+7) - (-16)]\mathcal{L}\{x_2\} = s-9$$

$$(s^2 + 6s + 9)\mathcal{L}\{x_2\} = s-9$$

Therefore,

$$\mathcal{L}\{x_2\} = \frac{s-9}{(s+3)^2} = \frac{1}{s+3} - \frac{12}{(s+3)^2}$$

$$\rightarrow \quad x_2 = e^{-3t} - 12te^{-3t}$$

Similarly, multiply eq. (1*) by $s + 7$ and eq. (2*) by 4.

$$(s-1)(s+7)\mathcal{L}\{x_1\} + 4(s+7)\mathcal{L}\{x_2\} = -2(s+7) \qquad (3)$$

$$-16\mathcal{L}\{x_1\} + 4(s+7)\mathcal{L}\{x_2\} = 4 \qquad (4)$$

Subtract eq. (4) from eq. (3)

$$[(s-1)(s+7) + 16]\mathcal{L}\{x_1\} = -2s - 18$$

$$(s^2 + 6s + 9)\mathcal{L}\{x_1\} = -2s - 18$$

$$\mathcal{L}\{x_1\} = \frac{-2s-18}{(s+3)^2} = \frac{-2}{s+3} - \frac{12}{(s+3)^2}$$

$$\rightarrow \quad x_1 = -2e^{-3t} - 12te^{-3t}$$

Therefore,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -12te^{-3t} - 2e^{-3t} \\ -12te^{-3t} + e^{-3t} \end{bmatrix} = \begin{bmatrix} -12t - 2 \\ -12t + 1 \end{bmatrix} e^{-3t} .$$

This agrees with the solution we have found earlier using the eigenvector method.

The method above can also, without any modification, be used to solve nonhomogeneous systems of linear differential equations. It gives us a way to solve nonhomogeneous linear systems without having to learn a separate technique. In addition, it also allows us to tackle linear systems with discontinuous forcing functions, if necessary.

*Example*: $\quad \boldsymbol{x}' = \begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} -t \\ 2t-1 \end{bmatrix}, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$

Rewrite the problem explicitly and transform:

$$\begin{aligned} x_1' &= -4x_1 - 2x_2 - t & \qquad x_1(0) &= 3 \\ x_2' &= 3x_1 + x_2 + 2t - 1 & \qquad x_2(0) &= -5 \end{aligned}$$

$$s\mathcal{L}\{x_1\} - 3 = -4\mathcal{L}\{x_1\} - 2\mathcal{L}\{x_2\} - \frac{1}{s^2} \qquad (5)$$

$$s\mathcal{L}\{x_2\} + 5 = 3\mathcal{L}\{x_1\} + \mathcal{L}\{x_2\} + \frac{2}{s^2} - \frac{1}{s} \qquad (6)$$

Simplify:

$$(s+4)\mathcal{L}\{x_1\} + 2\mathcal{L}\{x_2\} = -\frac{1}{s^2} + 3 = \frac{3s^2 - 1}{s^2} \qquad (5^*)$$

$$-3\mathcal{L}\{x_1\} + (s-1)\mathcal{L}\{x_2\} = \frac{2}{s^2} - \frac{1}{s} - 5 = \frac{-5s^2 - s + 2}{s^2} \qquad (6^*)$$

Multiplying eq. (5*) by $s - 1$ and eq. (6*) by 2, then subtract the latter from the former. We eliminate $\mathcal{L}\{x_2\}$, to find $\mathcal{L}\{x_1\}$.

$$(s^2 + 3s + 2)\mathcal{L}\{x_1\} = \frac{3s^3 + 7s^2 + s - 3}{s^2}$$

Therefore,

$$\mathcal{L}\{x_1\} = \frac{3s^3 + 7s^2 + s - 3}{s^2(s+1)(s+2)} = \frac{1}{4(s+2)} - \frac{3}{2s^2} + \frac{11}{4s}$$

$$\rightarrow \quad x_1 = \frac{1}{4}e^{-2t} - \frac{3}{2}t + \frac{11}{4}$$

Likewise, multiplying eq. (5*) by 3 and eq. (6*) by $s + 4$, then add them together. We find $\mathcal{L}\{x_2\}$.

$$(s^2 + 3s + 2)\mathcal{L}\{x_2\} = \frac{-5s^3 - 12s^2 - 2s + 5}{s^2}$$

Therefore,

$$\mathcal{L}\{x_2\} = \frac{-5s^3 - 12s^2 - 2s + 5}{s^2(s+1)(s+2)} = \frac{-1}{4(s+2)} + \frac{5}{2s^2} - \frac{19}{4s}$$

$$\rightarrow \quad x_2 = \frac{-1}{4}e^{-2t} + \frac{5}{2}t - \frac{19}{4}$$

 

Finally,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{-2t} - \frac{3}{2}t + \frac{11}{4} \\ \frac{-1}{4}e^{-2t} + \frac{5}{2}t - \frac{19}{4} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} e^{-2t} - 6t + 11 \\ -e^{-2t} + 10t - 19 \end{bmatrix}.$$

*Exercise D-1.4*:

Use Laplace transforms to solve each nonhomogeneous linear system.

1. $\boldsymbol{x}' = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 5e^{-3t} \\ -6e^{-3t} \end{bmatrix}$, $\quad \boldsymbol{x}(0) = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$.

2. $\boldsymbol{x}' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} -1 \\ 8 \end{bmatrix}$, $\quad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. $\boldsymbol{x}' = \begin{bmatrix} 3 & 0 \\ 5 & -2 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} -4\sin 2t \\ \cos 2t \end{bmatrix}$, $\quad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

4. $\boldsymbol{x}' = \begin{bmatrix} 2 & -8 \\ -1 & 4 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 3t+1 \\ -6t-2 \end{bmatrix}$, $\quad \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

*Answers D-1.4*:

1. $x = \begin{bmatrix} \frac{13}{10}e^{-3t} + \frac{51}{5}e^{2t} - \frac{3}{2}e^{-5t} \\ \frac{-17}{5}e^{-3t} + \frac{17}{5}e^{2t} + 3e^{-5t} \end{bmatrix}$

2. $x = \begin{bmatrix} -e^{2t}\cos t + 4e^{2t}\sin t + 2 \\ 4e^{2t}\cos t + e^{2t}\sin t - 3 \end{bmatrix}$

3. $x = \begin{bmatrix} \frac{18}{13}e^{3t} + \frac{8}{13}\cos 2t + \frac{12}{13}\sin 2t \\ \frac{18}{13}e^{3t} - \frac{117}{52}e^{-2t} - \frac{7}{52}\cos 2t + \frac{113}{52}\sin 2t \end{bmatrix}$

4. $x = \begin{bmatrix} \frac{-55}{12}e^{6t} - 3t^2 - \frac{7}{2}t + \frac{55}{12} \\ \frac{55}{24}e^{6t} - \frac{3}{4}t^2 + \frac{1}{4}t + \frac{41}{24} \end{bmatrix}$