Converting Differential Equations into First Order Systems

An nth order differential equation can be converted into an n-dimensional system of first order equations. There are various reasons for doing this, one being that a first order system is much easier to solve numerically (using computer software) and most differential equations you encounter in "real life" (physics, engineering etc) don't have nice exact solutions.

If the equation is of order n and the unknown function is y, then set $x_1 = y, x_2 = y', \ldots, x_n = y^{(n-1)}$. Note (and then note again) that we only go up to the $(n-1)$st derivative in this process.

Examples:

(1) $y^{(4)} - 3y'y''' + \sin(ty'') - 7ty^2 = e^t$. Set

$$x_1 = y, x_2 = y', x_3 = y'', x_4 = y'''$$

and then we have

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = x_4 \\ x_4' &= y^{(4)} = 3y'y''' - \sin(ty'') + 7ty^2 + e^t = 3x_2x_4 - \sin(tx_3) + 7tx_1^2 + e^t \end{aligned}$$

(2) $y''' + 2y'' - y' - 2y = 0$. Set

$$x_1 = y, x_2 = y', x_3 = y''$$

and then we have

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = 2y + y' - 2y'' = 2x_1 + x_2 - 2x_3 \end{aligned}$$

Observe that the linear (homogeneous) equation is converted to a linear (homogeneous) system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

Note further that

$$\begin{aligned} \det(A - \lambda I) &= -(\lambda^3 + 2\lambda^2 - \lambda - 2\lambda) \\ &= -(\lambda - 1)(\lambda + 1)(\lambda + 2) \end{aligned}$$

so the eigenvalues of A are $\lambda = 1, -1$ and 2 which are the same as the roots of the characteristic equation for the original 3rd order differential equation. This is always the case for linear equations with constant coefficients.