

D

Nonlinear Programming Solution Techniques

Most mathematical techniques for solving nonlinear programming problems are very complex. In this module two of the more well known but simpler mathematical methods will be demonstrated—the substitution method and the method of Lagrange multipliers.

The Substitution Method

*In the **substitution method** the constraint equation is solved for one variable in terms of another and then substituted into the objective function.*

The least complex method for solving nonlinear programming problems is referred to as *substitution*. This method is restricted to models that contain only equality constraints, and typically only one of these. The method involves solving the constraint equation for one variable in terms of another. This new expression is then substituted into the objective function, effectively eliminating the constraint. In other words, a constrained optimization model is transformed into an unconstrained model.

For an example of the substitution method we will use a profit analysis model. This is a nonlinear model that we introduced in chapter 10 of the text. The demand function is a constraint. The nonlinear programming model is formulated as

$$\text{maximize } Z = vp - c_f - vc_v$$

subject to

$$v = 1,500 - 24.6p$$

The objective function in this model is nonlinear, because both v (volume) and p (price) are variables and multiplying them (i.e., vp) creates a curvilinear relationship.

The constraint has already been solved for one variable (v) in terms of another (p); thus, we can substitute this expression directly into the objective function. This results in the following unconstrained function.

$$Z = 1,500p - 24.6p^2 - c_f - 1,500c_v + 24.6pc_v$$

By substituting the constant values for c_f (\$10,000) and c_v (\$8), we obtain

$$Z = 1,696.8p - 24.6p^2 - 22,000$$

Next, we solve this problem by differentiating the function Z and setting it equal to zero.

$$\begin{aligned}\frac{\partial Z}{\partial p} &= 1,696.8 - 49.2p \\ 0 &= 1,696.8 - 49.2p \\ 49.2p &= 1,696.8 \\ p &= \$34.49\end{aligned}$$

We will present another example as a further illustration of a nonlinear programming problem and the substitution method. In this example the Beaver Creek Pottery Company produces bowls (x_1) and mugs (x_2). We will assume that the profit contribution for each product *declines* as the quantity of each item produced increases. Thus, for bowls the per unit profit contribution is now expressed according to the relationship

$$\$4 - 0.1x_1$$

For mugs, the profit contribution per unit is

$$\$5 - 0.2x_2$$

(These relationships express the fact that production costs for each product increase as the number of units sold increases.)

These profit relationships are on a per unit basis. Thus, the total profit contribution from each product is determined by multiplying these relationships by the number of units produced. For bowls, the profit contribution is

$$(4 - 0.1x_1)x_1$$

or

$$4x_1 - 0.1x_1^2$$

For mugs, the profit contribution is

$$(5 - 0.2x_2)x_2$$

or

$$5x_2 - 0.2x_2^2$$

Total profit is the sum of these two terms.

$$Z = \$4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2$$

In this model, we will consider only a labor constraint, and we will treat it as an equality rather than an inequality.

$$x_1 + 2x_2 = 40 \text{ hr}$$

The complete nonlinear programming model is as follows.

$$\text{maximize } Z = \$4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2$$

subject to

$$x_1 + 2x_2 = 40$$

The first step in the substitution method is to solve the constraint equation for one variable in terms of another. We will arbitrarily decide to solve for x_1 as follows.

$$x_1 = 40 - 2x_2$$

Now wherever x_1 appears in the nonlinear objective function, we will substitute the expression $40 - 2x_2$.

$$\begin{aligned} Z &= 4(40 - 2x_2) - 0.1(40 - 2x_2)^2 + 5x_2 - 0.2x_2^2 \\ &= 160 - 8x_2 - 0.1(1,600 - 1,600x_2 + 4x_2^2) + 5x_2 - 0.2x_2^2 \\ &= 160 - 8x_2 - 160 + 16x_2 - 0.4x_2^2 + 5x_2 - 0.2x_2^2 \\ &= 13x_2 - 0.6x_2^2 \end{aligned}$$

This is an unconstrained optimization function, and we can solve it by differentiating it and setting it equal to zero.

$$\begin{aligned}\frac{\partial Z}{\partial x_2} &= 13 - 1.2x_2 \\ 0 &= 13 - 1.2x_2 \\ 1.2x_2 &= 13 \\ x_2 &= 10.8 \text{ mugs}\end{aligned}$$

To determine x_1 , we can substitute x_2 into the constraint equation.

$$\begin{aligned}x_1 + 2x_2 &= 40 \\ x_1 + 2(10.8) &= 40 \\ x_1 &= 18.4 \text{ bowls}\end{aligned}$$

Substituting the values of x_1 and x_2 into the original objective function gives the total profit, as follows.

$$\begin{aligned}Z &= \$4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2 \\ &= \$4(18.4) - 0.1(18.4)^2 + 5(10.8) - 0.2(10.8)^2 \\ &= \$70.42\end{aligned}$$

Both of the examples presented in this section for solving nonlinear programming problems exhibit the limitations of this approach. The objective functions were not very complex (i.e., the highest order of a variable was a power of two in the second example), there were only two variables, and the single constraint in each example was an equation. This method becomes very difficult if the constraint becomes complex. An alternative solution approach that is not quite as restricted is the method of Lagrange multipliers.

The Method of Lagrange Multipliers

*In the method of **Lagrange multipliers**, constraints are multiples of a multiplier, λ , are subtracted from the objective function, which is then differentiated with respect to each variable and solved.*

The method of Lagrange multipliers is a general mathematical technique that can be used for solving constrained optimization problems consisting of a nonlinear objective function and one or more linear or nonlinear constraint equations. In this method, the constraints as multiples of a Lagrange multiplier, λ , are subtracted from the objective function.

To demonstrate this method, we will use our modified pottery company example developed in the preceding section. This model was formulated as

$$\begin{aligned}\text{maximize } Z &= 4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2 \\ \text{subject to} \\ x_1 + 2x_2 &= 40\end{aligned}$$

The first step is to transform the nonlinear objective function into a *Lagrangian function*. This is accomplished by transforming the constraint equation as follows.

$$x_1 + 2x_2 - 40 = 0$$

Next, this expression is multiplied by λ , the *Lagrangian multiplier*, and subtracted from the objective function to form the Lagrangian function L .

$$L = 4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2 - \lambda(x_1 + 2x_2 - 40)$$

Because the constraint equation now equals zero, the subtraction of the constraint, multiplied by λ , from the objective function does not affect the value of the function. (We will explain the exact meaning of λ after a solution has been determined.)

The Lagrangian function is differentiated with respect to each variable, and the resulting equations are solved simultaneously.

Now we must determine the partial derivatives of the Lagrangian function with respect to each of the three variables, x_1 , x_2 , and λ .

$$\begin{aligned}\frac{\partial L}{\partial x_1} &= 4 - 0.2x_1 - \lambda \\ \frac{\partial L}{\partial x_2} &= 5 - 0.4x_2 - 2\lambda \\ \frac{\partial L}{\partial \lambda} &= -x_1 - 2x_2 + 40\end{aligned}$$

These three equations are all set equal to zero and solved simultaneously to determine the values of x_1 , x_2 , and λ .

$$\begin{aligned}4 - 0.2x_1 - \lambda &= 0 \\ 5 - 0.4x_2 - 2\lambda &= 0 \\ -x_1 - 2x_2 + 40 &= 0\end{aligned}$$

To solve these equations simultaneously, we multiply the first equation by -2 and add it to the second equation, which eliminates λ .

$$\begin{array}{r} -8 + 0.4x_1 + 2\lambda = 0 \\ 5 - 0.4x_2 - 2\lambda = 0 \\ \hline -3 + 0.4x_1 - 0.4x_2 = 0 \end{array}$$

This new equation and the original preceding third equation represent two equations with two unknowns (x_1 and x_2). We multiply the preceding third equation by 0.4 and add it to the new equation in order to eliminate x_1 .

$$\begin{array}{r} -0.4x_1 - 0.8x_2 + 16 = 0 \\ 0.4x_1 - 0.4x_2 - 3 = 0 \\ \hline -1.2x_2 + 13 = 0 \end{array}$$

The resulting equation is solved for x_2 as follows.

$$\begin{aligned}-1.2x_2 + 13 &= 0 \\ -1.2x_2 &= -13 \\ x_2 &= 10.8 \text{ mugs}\end{aligned}$$

Substituting this value back into the previous equations gives the values of both x_1 and λ .

$$\begin{aligned}-x_1 - 2x_2 + 40 &= 0 \\ -x_1 - 2(10.8) &= -40 \\ x_1 &= 18.3 \text{ bowls} \\ 5 - 0.4x_2 - 2\lambda &= 0 \\ 5 - 0.4(10.8) &= 2\lambda \\ \lambda &= 0.33\end{aligned}$$

Substituting the values for x_1 and x_2 into the original objective function yields the total profit.

$$\begin{aligned}Z &= \$4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2 \\ &= \$4(18.3) - 0.1(18.3)^2 + 5(10.8) - 0.2(10.8)^2 \\ &= \$70.42\end{aligned}$$

This result can also be obtained by using the Lagrangian function, L , and multiplier, λ .

$$\begin{aligned} L &= \$4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2 - \lambda(x_1 + 2x_2 - 40) \\ &= \$4(18.3) - 0.1(18.3)^2 + 5(10.8) - 0.2(10.8)^2 - 0.33(0) \\ &= \$70.42 \end{aligned}$$

To summarize, we have $x_1 = 18.3$ bowls, $x_2 = 10.8$ mugs, $Z = \$70.42$, and $\lambda = 0.33$. This is the same answer obtained previously using the substitution method. However, unlike the substitution method, the Lagrange multiplier approach can be used to solve nonlinear programming problems with more complex constraint equations and *inequality constraints*. In addition, it can encompass problems with more than two variables. We will not pursue any examples, though, that demonstrate the complexities involved. The Lagrange multiplier method must be altered to compensate for inequality constraints and additional variables, and the resulting mathematics are very difficult. Even though the Lagrange multiplier method is more flexible than the substitution method, it is practical for solving only small problems. As the size of the problem increases, the mathematics become overwhelmingly difficult.

The Meaning of λ

The Lagrange multiplier, λ , reflects a change in the objective function from a unit change in the right-hand-side value of a constraint.

The Lagrange multiplier, λ , in nonlinear programming problems is analogous to the dual variables in a linear programming problem. It reflects the approximate change in the objective function resulting from a unit change in the quantity (right-hand-side) value of the constraint equation. For our example, we will increase the quantity value in the constraint equation from 40 to 41 hours of labor. (You will recall from the preceding calculations that $\lambda = 0.33$.)

$$x_1 + 2x_2 = 41 \quad \text{or} \quad x_1 + 2x_2 - 41 = 0$$

This constraint equation will result in the following Lagrangian function.

$$L = 4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2 - \lambda(x_1 + 2x_2 - 41)$$

Solving this problem the same way we solved the original model gives the following solution.

$$\begin{aligned} x_1 &= 18.7 \\ x_2 &= 11.2 \\ \lambda &= 0.27 \\ Z &= \$70.75 \end{aligned}$$

This value for Z is \$0.33 greater than the previous Z value of \$70.42. Thus, a one-unit increase in the right-hand side of the constraint equation results in a λ increase in the objective function. More specifically, a unit increase in a resource (labor) results in a \$0.33 increase in profit. Thus, we would be willing to pay \$0.33 for one additional hour of labor. This is the same interpretation as that given for a dual variable in linear programming.

In general, if λ is positive, the optimal objective function value will increase if the quantity (absolute) value in the constraint equation is increased, and it will decrease if the quantity (absolute) value is decreased. On the other hand, if λ is negative, the optimal objective function value will increase if the quantity (absolute) value is decreased, and it will decrease if the quantity (absolute) value is increased.

Problems

1. The Hickory Cabinet and Furniture Company makes chairs. The fixed cost per month of making chairs is \$7,500, and the variable cost per chair is \$40. Price is related to demand according to the following linear equation.

$$v = 400 - 1.2p$$

Develop the nonlinear profit function for this company and determine the price that will maximize profit, the optimal volume, and the maximum profit per month.

2. Graphically illustrate the profit curve developed in problem 1. Indicate the optimal price and the maximum profit per month.
3. The Rainwater Brewery produces beer. The annual fixed cost is \$150,000, and the variable cost per barrel is \$16. Price is related to demand according to the following linear equation.

$$v = 75,000 - 1,153.8p$$

Develop the nonlinear profit function for the brewery and determine the price that will maximize profit, the optimal volume, and the maximum profit per year.

4. The Rolling Creek Textile Mill makes denim. The monthly fixed cost is \$8,000, and the variable cost per yard of denim is \$0.35. Price is related to demand according to the following linear equation.

$$v = 17,000 - 5,666p$$

Develop the nonlinear profit function for the textile mill and determine the optimal price, the optimal volume, and the maximum profit per month.

5. The Grady Tire Company recaps tires. The weekly fixed cost is \$2,500, and the variable cost per tire is \$9. Price is related to demand according to the following linear equation.

$$v = 200 - 4.75p$$

Develop the nonlinear profit function for the tire company and determine the optimal price, the optimal volume, and the maximum profit per week.

6. Andy Mendoza makes handcrafted dolls, which he sells at craft fairs. He is considering mass-producing the dolls to sell in stores. He estimates that the initial investment for plant and equipment will be \$25,000, while labor, material, packaging, and shipping will be about \$10 per doll. He has determined that sales volume is related to price according to the following linear equation.

$$v = 4,000 - 80p$$

Develop the nonlinear profit function for Andy and determine the price that will maximize profit, the optimal volume, and the maximum profit per month.

7. The Rainwater Brewery produces beer, which it sells to distributors in barrels. The brewery incurs a monthly fixed cost of \$12,000, and the variable cost per barrel is \$17. The brewery has developed the following profit function and demand constraint.

$$\text{maximize } Z = vp - \$12,000 - 17v$$

subject to

$$v = 800 - 15p$$

Solve this nonlinear programming model for the optimal price (p) using the substitution method.

8. The Beaver Creek Pottery Company has developed the following nonlinear programming model to determine the optimal number of bowls (x_1) and mugs (x_2) to produce each day.

$$\text{maximize } Z = \$7x_1 - 0.3x_1^2 + 8x_2 - 0.4x_2^2$$

subject to

$$4x_1 + 5x_2 = 100 \text{ hr}$$

Determine the optimal solution to this nonlinear programming model using the substitution method.

9. The Evergreen Fertilizer Company produces two types of fertilizers, Fastgro and Super Two. The company has developed the following nonlinear programming model to determine the optimal number of bags of Fastgro (x_1) and Super Two (x_2) that they must produce each day to maximize profit, given a constraint for available potassium.

$$\text{maximize } Z = \$30x_1 - 2x_1^2 + 25x_2 - 0.5x_2^2$$

subject to

$$3x_1 + 6x_2 = 300 \text{ lb}$$

Determine the optimal solution to this nonlinear programming model using the substitution method.

10. The Rolling Creek Textile Mill produces denim and brushed-cotton cloth. The company has developed the following nonlinear programming model to determine the optimal number of yards of denim (x_1) and brushed cotton (x_2) to produce each day to maximize profit, subject to a labor constraint.

$$\text{maximize } Z = \$10x_1 - 0.02x_1^2 + 12x_2 - 0.03x_2^2$$

subject to

$$0.2x_1 + 0.1x_2 = 40 \text{ hr}$$

Determine the optimal solution to this nonlinear programming model using the substitution method.

11. Solve problem 8 using the method of Lagrange multipliers.
12. Solve problem 9 using the method of Lagrange multipliers.
13. Solve problem 10 using the method of Lagrange multipliers.
14. The Riverwood Paneling Company makes two kinds of wood paneling, Colonial and Western. The company has developed the following nonlinear programming model to determine the optimal number of sheets of Colonial paneling (x_1) and Western paneling (x_2) to produce to maximize profit, subject to a labor constraint.

$$\text{maximize } Z = \$25x_1 - 0.8x_1^2 + 30x_2 - 1.2x_2^2$$

subject to

$$x_1 + 2x_2 = 40 \text{ hr}$$

Determine the optimal solution to this nonlinear programming model using the method of Lagrange multipliers.

15. Interpret the meaning of λ , the Lagrange multiplier, in problem 14.