Chapter IV

LINEAR PROGRAMMING

IV.4. The more-for-less paradox

Let us consider the production problem described in Section I.2.3. and based on the assumption of the minimization of the costs for the given output level. Suppose the firm using n different technologies $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$ has to produce the given amounts $b_1, b_2, \ldots, b_m$ of m different products $P_1, P_2, \ldots, P_m$. The technology T_j is described by the output coefficients a_{ij} giving the amount of good P_i produced by the technology $\mathcal{T}_j$ with unit intensity. The unit cost of technology T_j is described by the coefficient c_j $(j = 1, 2, \ldots, n)$. The decision problem of the firm is: how to combine the different technologies and at which intensities in order to produce the given amounts of the products $P_1, P_2, \ldots, P_m$ at the minimal cost.

For simplicity and without loss of generality suppose only two products $(m = 2)$ and six technologies $(n = 6)$ with the following matrix of output coefficients

$$A = \begin{pmatrix} 3 & 2 & 3 & 2 & 2 & 4 \\ 1 & 1 & 2 & 2 & 3 & 5 \end{pmatrix},$$

with the output requirements $\mathbf{b}' = (30, 40)$ and with the unit cost $\mathbf{c} = (2, 1, 4, 2, 6, 8)$, expressed in Euro. This problem can be formulated as

$$\begin{aligned} \text{minimize } f_0(\mathbf{x}) &= 2x_1 + x_2 + 4x_3 + 2x_4 + 6x_5 + 8x_6 \\ \text{subject to} \quad 3x_1 + 2x_2 + 3x_3 + 2x_4 + 2x_5 + 4x_6 &= 30 \\ x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 + 5x_6 &= 40 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geqq 0. \end{aligned} \tag{IV.22}$$

The optimal solution to problem (IV.22) is $\mathbf{x}^0 = (0, 0, 0, 0, 5, 5)$ with the objective value $f(\mathbf{x}^0) = 70$ Euro. Although the method of solution will be presented only in the next section for the problem (IV.22) it can be easily shown - by means of duality theory from the previous section - that the above solution is optimal.

The dual problem corresponding to the problem (IV.22) has the following form:

$$\begin{aligned} \text{maximize } g(\mathbf{u}) &= 30u_1 + 40u_2 \\ \text{subject to} \quad 3u_1 + u_2 &\leqq 2 \\ 2u_1 + u_2 &\leqq 1 \\ 3u_1 + 2u_2 &\leqq 4 \\ 2u_1 + 2u_2 &\leqq 2 \\ 2u_1 + 3u_2 &\leqq 6 \\ 4u_1 + 5u_2 &\leqq 8 \end{aligned} \tag{IV.23}$$

Due to Theorem 5, which provides the necessary and sufficient conditions for the optimal solution of a linear programming problem we write the following system of equations

$$\begin{aligned} 2u_1^0 + 3u_2^0 &= 6 \qquad (\text{because of } x_5^0 > 0) \\ 4u_1^0 + 5u_2^0 &= 8 \qquad (\text{because of } x_6^0 > 0) \end{aligned}$$

which yields the solution $\mathbf{u}^0 = (-3, 4)$. Because all dual constraints and all complementary slackness conditions are fulfilled this is the optimal solution for the dual problem (IV.23). Substituting $u_1^0 = -3$ and $u_2^0 = 4$ into the dual objective function we get the same value of 70 Euro as for the primal objective function $f(\mathbf{x}^0)$. The reader can easily verify that the graphic method for solving problem (IV.23) gives the same solution.

Suppose we increase the output requirements in the production problem (IV.22) from $(30, 40)$ to $(60, 50)$ i.e. we change the right handside of problem (IV.22) to $(60, 50)$. The optimal solution for the new problem is $x_2^* = 10$ and $x_4^* = 20$ with the objective value 50 Euro[1]. We can see the firm produces more units of both products (100% more of the product P_1 and 25% more of the product P_2) for less total costs (71,4% of the previous cost). Hence we have the more-for-less paradox in this situation.

After the numerical illustration we define the more-for-less paradox generally. Let us consider the following problem of linear programming with all c_j $(j = 1, 2, \ldots, n)$ and b_i $(i = 1, 2, \ldots, m)$ assumed positive:

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) = \mathbf{c}'\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geqq \mathbf{0} \end{aligned} \tag{IV.24}$$

where A is an $m \times n$ matrix $(m < n)$ and of full rank, $\mathbf{b}$ is an $m \times 1$ vector, $\mathbf{x}$ is an $n \times 1$ vector and $\mathbf{c}$ is an $n \times 1$ vector.

The problem (IV.24) has the property more-for-less if we can increase all or some components of $\mathbf{b}$ and reduce the value of the objective function $f_0(\mathbf{x})$ without reducing other components of $\mathbf{b}$, and keeping all c_j fixed. The following theorem provides the necessary and sufficient condition for the more-for-less paradox.

Theorem 6: *The linear programming problem (IV.24) has the property more-for-less if and only if every optimal solution of the corresponding dual problem*

$$\begin{aligned} \textit{maximize} \quad & g(\mathbf{u}) = \mathbf{b}'\mathbf{u} \\ \textit{subject to} \quad & A'\mathbf{u} \leqq \mathbf{c} \end{aligned}$$

has at least one negative dual variable u_i^0.

The proof of this theorem can be found in two independent papers, first by CHOBOT - TURNOVEC (1974), p. 379 and then by CHARNES - DUFFUAA - RYAN (1987), p. 195. The more-for-less paradox in the case of the distribution model was first studied by CHARNES - KLINGMAN (1971) and similar theorems about this paradox was proved by SZWARC (1962), (1971). The Theorem 6 is an extension of the main theorem in CHARNES - KLINGMAN (1971). For illustration we return to our numerical example (IV.22). Because the optimal solution $\mathbf{x}^0 = (0, 0, 0, 0, 5, 5)$ is non-degenerate, the corresponding dual solution $u_1^0 = -3$ and $u_2^0 = 4$ is unique and with one negative component. Due to Theorem 6 there exists a vector $\mathbf{\Delta b} > 0$ (in our example $\Delta b_1 = 30$, $\Delta b_2 = 10$) such that the optimal solution $\mathbf{x}^* = (0, 10, 0, 20, 0, 0)$ for the problem with these new right handsides

[1]The optimality of the solution can be shown in the same way as before. The optimal dual solution for the new problem is $\mathbf{u}^* = (0, 1)$

yields a lower objective function value ($\Delta f_0(\mathbf{x}) = -20$). The optimal solution for the corresponding dual problem is degenerate and nonnegative $\mathbf{u}^* = (0,1)$. The new problem does not have the property more-for-less.

¿From the economic interpretation point of view is this result remarkable. How to explain for a given linear technology that the increasing production of all goods can lead to a decreasing total production cost? Or in other words the reduced level of production is accompanied by increasing total cost.

For the given unit cost c_j $(j = 1, 2, \ldots, n)$ and the technological coefficients a_{ij} there exist a "technological" optimal structure of production described by $\mathbf{s}^0 = (s_1^0, s_2^0, \ldots, s_m^0)$ which can be found by solving the following linear programming problem (CHOBOT - TURNOVEC, (1974), p. 348):

$$\text{minimize } f_0(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j \tag{IV.25}$$

$$\begin{aligned} \text{subject to} \qquad \sum_{j=1}^{n} a_{ij} x_j - s_i &= 0 \qquad (i = 1, 2, \ldots, m) \\ \sum_{i=1}^{m} s_i &= 1 \qquad \qquad \text{(IV.26)} \\ x_j &\geqq 0 \qquad (j = 1, 2, \ldots, n) \\ s_i &\geqq 0 \qquad (i = 1, 2, \ldots, m). \end{aligned}$$

The "technological" optimal structure can deviate considerably from the given structure, described by the coefficients b_i:

$$\mathbf{s} = (s_1, s_2, \ldots, s_n),$$
$$\text{where} \qquad s_i = \frac{b_i}{\sum_{i=1}^{n} b_i}.$$

Because of the equalities constraints in the problem (IV.24) it is not always possible to realize the "technological" optimal structure $\mathbf{s}^0$. If the value of the structure $\mathbf{s}^0$, described by the optimal value of the objective function (IV.25) deviates considerably from the value of the required production structure $\mathbf{s}$, the increasing production in the problem (IV.24) accompanied by changing the production structure towards the "technological" optimal structure $\mathbf{s}^0$ will lead to decreasing production costs.

In order to characterize the "technological" optimal structure analytically we consider the dual problem corresponding to the problem (IV.25) - (IV.26):

$$\text{maximize } g(\mathbf{u}) = u_{m+1} \tag{IV.27}$$

$$\text{subject to} \qquad \sum_{i=1}^{m} a_{ij} u_i \leqq c_j \qquad (j = 1, 2, \ldots, n) \tag{IV.28}$$

$$-u_i + u_{m+1} \leqq 0 \qquad (i = 1, 2, \ldots, m) \tag{IV.29}$$

Assuming $s_i > 0$ the complementary slackness theorem yields:

$$u_{m+1} = u_i \qquad (i = 1, 2, \ldots, m).$$

The dual constraints (IV.28) can be then rewritten as:

$$\sum_{i=1}^{m} a_{ij} u_{m+1} \leqq c_j \qquad (j = 1, 2, \ldots, n)$$

or

$$u_{m+1} \leqq \frac{c_j}{\sum_{i=1}^{m} a_{ij}} \qquad (j = 1, 2, \ldots, n).$$

The optimal solution of the dual problem (IV.27) - (IV.29) is given by

$$u_{m+1}^0 = \min \frac{c_j}{\sum_{ij} a_{ij}} \qquad (j = 1, 2, \ldots, n). \tag{IV.30}$$

Using the complementary slackness theorem the "technological" optimal structure can be easily obtained according to the relation (IV.30).

For illustration we consider our example (IV.22) again. According to (IV.25) - (IV.26) the linear programming problem for the estimation of the "technological" optimal structure is the following:

$$\begin{aligned}
\text{minimize } & f_0(\mathbf{x}) = 2x_1 + x_2 + 4x_3 + 2x_4 + 6x_5 + 8x_6 \\
\text{subject to } & 3x_1 + 2x_2 + 3x_3 + 2x_4 + 2x_5 + 4x_6 - s_1 = 0 \\
& x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 + 5x_6 - s_2 = 0 \\
& s_1 + s_2 = 1 \\
& s_1 \geqq 0,\ s_2 \geqq 0,\ x_j \geqq 0 \qquad (j = 1, \ldots, n).
\end{aligned}$$

The corresponding dual problem

$$\begin{aligned}
\text{maximize } & g(\mathbf{u}) = u_3 \\
\text{subject to } & 3u_1 + u_2 \leqq 2 \\
& 2u_1 + u_2 \leqq 1 \\
& 3u_1 + 2u_2 \leqq 4 \\
& 2u_1 + 2u_2 \leqq 2 \\
& 2u_1 + 3u_2 \leqq 6 \\
& 4u_1 + 5u_2 \leqq 8 \\
& -u_1 + u_3 \leqq 0 \\
& -u_2 + u_3 \leqq 0
\end{aligned}$$

is easily to solve. Assuming $s_1 > 0$, $s_2 > 0$ we obtain $u_3 = u_2 = u_1$. Then the remaining dual constraints reduce to

$$\begin{aligned}
4u_3 &\leqq 2 \\
3u_3 &\leqq 1 \\
5u_3 &\leqq 4 \\
4u_3 &\leqq 2 \\
5u_3 &\leqq 6 \\
9u_3 &\leqq 8.
\end{aligned}$$

Under the maximization of u_3 the optimal solution is $u_3^0 = 1/3$, which corresponds exactly to the relation (IV.30). Due to the complementary slackness theorem the optimal value for the primal variables x_1, x_3, x_4, x_5 and x_6 must be equal zero. Consequently $x_2^0 = 1/3$, $s_1^0 = 2/3$ and $s_2^0 = 1/3$. The "technological optimal" is the technology $\mathcal{T}_2$ with the structure of production $s_1^0 : s_2^0 = 2 : 1$. In our numerical example (IV.22) we changed the initial output requirements $\mathbf{b} = (30, 40)$ to $\mathbf{b}^* = (60, 50)$, which are more closed to the technological optimal structure $\mathbf{s}^0$ and the optimal solution was switching from the technologies T_5 and T_6 to the technologies T_2 and T_4. The reader can easily verify that if we change the output requirements according the structure $\mathbf{s}^0$ (e.g. $b_1^0 = 100$ and $b_2^0 = 50$) the firm will use the technology T_2 only ($x_2^0 = 50$ and the optimal value of the objective function $f(\mathbf{x}^0) = 50$). This is a degenerate optimal solution which implies that the optimal solution of the dual problems is not unique. One of the optimal solutions is $u_1^0 = 1/2$ and $u_2^0 = 0$ (the solution is feasible and the value of the dual objective function $g(\mathbf{u}^0) = f(\mathbf{x}^0) = 50$). Because of the nonnegative dual solution the modified problem (with the right handside vector $\mathbf{b}^*$) has no more the property more-for-less. According to CHARNES - DUFFUAA - RYAN (1987) we obtained an optimal solution which resolves the paradox. By resolve the paradox they mean the increase of the right handside until the total costs starts to increase from the minimum obtained over all possible increases. CHARNES - DUFFUAA - RYAN (1987), p. 196 proved the following:

Theorem 7: *In the more-for-less situation of (IV.24) with increased right handside to resolve the paradox, there exists a degenerate basic optimal solution.*

The first view of the numerical example (IV.22) may imply that the reason for the "technological" optimality of the process $\mathcal{T}_2$ is the lowest unit cost ($c_2 = 1$). In other words, the reduction of the total cost is caused by switch to the technology with lower unit cost. Using slightly modified data of the example (IV.22) it can be shown that "technological" optimal can be even process with the highest coefficient in the objective function. Increasing the production of all goods the total cost can decrease by switching to the technology with higher unit cost defined by coefficient of the objective function. Let us consider the following example:

$$\begin{aligned}
\text{minimize } f_0(\mathbf{x}) = 5x_1 + 4x_2 + 6x_3 + 5x_4 + 6x_5 + 10x_6 \\
\text{subject to} \quad 3x_1 + 2x_2 + 4x_3 + 2x_4 + 2x_5 + 9x_6 = 41 \\
x_1 + x_2 + 2x_3 + 2x_4 + 4x_5 + 16x_6 = 32 \\
x_j \geqq 0 \quad (j = 1, 2, \ldots, 6)
\end{aligned}$$

At the same way as in previous example the reader can verify that the optimal solution is: $x_3^0 = 8$, $x_6^0 = 1$ and $f_0(\mathbf{x}^0) = 58$. The optimal solution of the dual problem is: $u_1^0 = 1,7$ and $u_2^0 = -0,3$, which imply the property "more for less" for the given example. Increasing the first output to $45(\Delta b_1 = 4)$ and the second output to $80(\Delta b_2 = 48)$ the optimal solution is given by $x_6^* = 5$. It is optimal to switch to the most expensive (defined by the coefficients of the objective function) technology. Even in this case the total cost decreases from 58 units to 50 units ($f_0(\mathbf{x}^*) = 50$). According to the relation (IV.30) the process $\mathcal{T}_6$ is "technological" optimal despite the highest coefficient in the objective function ($c_6 = 10$). Thereupon "technological" optimal is the process with the lowest cost per total product (provided by unit intensity).

IV.6. Some applications of linear programming in economics

In Section I.2 we already formulated some models of mathematical programming used in economics. Now we want to analyse some of these models more deeply in order to show how the linear programming can be used as an instrument of qualitative analysis.

6.1. The theory of comparative advantage. One field of economics where the linear programming is very oft applied is the international trade (e.g. GANDOLFO (1998)). A well known example of RICARDO (1817) in the slightly modification by DORFMAN - SAMULESON - SOLOW (1958), pp. 31-32 leads - for England - to the following linear programming model (as formulated in Section I.2.4):

$$\begin{aligned} \text{maximize } \mathcal{Z} &= \frac{p_1}{p_2}x_1 + x_2 \\ \text{subject to } \quad 2x_1 + x_2 &\leqq C \\ x_1 \geqq 0, \quad x_2 &\geqq 0 \end{aligned}$$

where $\mathcal{Z}$ denotes the National Product (NP) of England.

The optimization problem for Portugal has the same structure:

$$\begin{aligned} \text{maximize } \mathcal{Z}' &= \frac{p_1}{p_2}x_1' + x_2' \\ \text{subject to } \quad x_1' + x_2' &\leqq C' \\ x_1' \geqq 0, \quad x_2' &\geqq 0 \end{aligned}$$

where $\mathcal{Z}'$ denotes the National Product of Portugal.

The graphical representation of the feasible set or "production - possibility" curve for England is given in Fig. 4.3, for Portugal in Fig. 4.4.

Fig.4.3.
Fig.4.4

We can see that the decision about the production of food and clothing in England and in Portugal depends on the slope of objective function, in other words on the international price ratio p_1/p_2. If there exists a price ration p_1/p_2 somewhere between 1 and 2 it is optimal for England to produce only clothing and for Portugal to produce only food. Although Portugal needs less (or no more) input for both products, the best production pattern for this country involves zero clothing production and complete specialization on food. Portugal will export food in exchange for clothing imports from England which will specialize completely in clothing. Portugal has comparative advantage in the food production (it can convert one unit of food into one unit of clothing but the price for food is higher than for clothing), England in the clothing production (it can convert one unit of food into two units of clothing, but it get for one unit of food less than for two units of clothing). Both countries will be better off than if they do not specialize. The world will in fact be at the Ricardo point where $1 \leqq$ marginal cost $(MC) \leqq 2$. The reader may easily verify that for $p_1/p_2 = 2$, the optimal solution for England is not unique (the contour line for NP, isoincome line is parallel with the "production - possibility" curve). The best production pattern for Portugal in this situation is a complete specialization on food.

When $p_1/p_2 = 1$, the optimal solution for Portugal is not unique.

When $p_1/p_2 < 1$ (or > 2), both countries will specialize completely in clothing (in food).

In the next step we want to generalize this model for m commodities and n countries (GANDOLFO (1998), Chapter 2 and Appendix to Chapter 2; TAKAYAMA (1972), Chapter 6). The following notation is introduced ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$):

x_{ij} = quantity of good i produced in country j,

l_{ij} = constant labour - input coefficient in the production of good i in country j,

L_j = total quantity of labour available in country j,

p_i = given international price of good i.

In the previous simple model with two countries and two commodities we have formulated the problem in terms of maximization of the (value of) National Product of each country separately considered. Now we will formulate the problem directly in terms of maximization of world output. It can be shown (TAKAYAMA (1972), pp. 172-173) that world output will be maximized, if and only if, each country maximizes its own national output.

The problem of maximizing the value of world output under the constraints that the amount of labour employed in each country cannot exceed the amount disposable and under the nonnegativity constraints for outputs leads to the following linear programming model:

$$\begin{aligned} \text{maximize } & p_1 \left(\sum_{j=1}^{n} x_{1j} \right) + p_2 \left(\sum_{j=1}^{n} x_{2j} \right) + \cdots + \\ & + p_m \left(\sum_{j=1}^{n} x_{mj} \right) \\ \text{subject to } & \sum_{i=1}^{m} l_{ij} x_{ij} \leqq L_j \qquad (j = 1, 2, \ldots, n) \\ & x_{ij} \geqq 0 \qquad \begin{pmatrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n \end{pmatrix}. \end{aligned} \tag{IV.42}$$

We are assuming that the resources within each country are completely substitutable so that there is only a single resource limitation (labour) in each country. Further, we postulate constant returns to scale in each country.

The solution to the primal problem (IV.42) yields the allocation of m products between n countries. In order to find it we now ask a different question: what will be the value of labour in each country?

For this purpose we consider the dual problem to problem (IV.42):

$$\begin{aligned} &\text{minimize } \sum_{j=1}^{n} w_j L_j && && \text{(IV.43)}\\ &\text{subject to} \quad l_{ij} w_j \geqq p_i && (i = 1, 2, \ldots, m) &&\\ &\qquad\qquad\quad w_j \geqq 0 && (j = 1, 2, \ldots, n) && \end{aligned}$$

where the shadow price of labour in country j, w_j is interpreted as the money wage rate. Therefore the dual problem (IV.43) consists in minimizing the world total labour reward (world production cost) subject both to the constraint that the value of the resources used will be at least as great as the value of the goods produced and to the nonnegativity constraint on the wage rate.

Given $p_i > 0$ $(i = 1, 2, \ldots, m)$ and $l_{ij} > 0$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ it follows from the constraints of the dual problem (IV.43) that the optimal wage rate w_j^0 must be positive in every country. Due to the complementary slackness theorem

$$\text{if } \ w_j^0 > 0, \quad \text{then } \sum_{i=1}^{m} l_{ij} x_{ij}^0 = L_j \qquad (j = 1, 2, \ldots, n)$$

where x_{ij}^0 denotes the optimal solution to the primal problem (IV.42). Because the optimal money wage rate is positive in the j-th country, all of the labour available in that country must be fully utilized. Consequently - due to the nonnegativity of outputs and positivity of labour - input coefficients l_{ij} - at least one good must be produced in each country. Because the optimal solution to the problem (IV.42) must be basic solution, it consists n positive components. Assuming

$$\frac{p_i}{l_{ij}} \neq \frac{p_k}{l_{kj}} \qquad \begin{matrix} (i, k = 1, 2, \ldots, m) \\ (j = 1, 2, \ldots, n) \end{matrix} \tag{IV.44}$$

each country will specialize on its best product. To find it we rewrite the constraint in the dual problem (IV.43) as:

$$w_j \geqq \frac{p_i}{l_{ij}} \qquad \begin{matrix} (i = 1, 2, \ldots, m) \\ (j = 1, 2, \ldots, n) \end{matrix} .$$

$$\text{Then} \qquad w_j^0 = \max \frac{p_i}{l_{ij}} \qquad (i = 1, 2, \ldots, m).$$

Because of the assumption (IV.44) the optimal wage rate in country j is unique and determined by the maximal ratio of the given international price p_i to the labour input coefficient l_{ij} for $i = 1, 2, \ldots, m$. For this good (say k) is the dual constraint in (IV.43) fulfil as an equality whereas

$$w_j^0 > \frac{p_i}{l_{ij}}$$

or

$$w_j^0 l_{ij} > p_i \qquad \text{for} \quad i = 1, 2, \ldots, n \text{ and } \quad i \neq k.$$

According to the complementary slackness theorem if the unit cost of good i ($i = 1, 2, \dots, n;\ i \neq k$) in country j is greater than the price of this good, then good i will not be produced in country j. In other words, country j will specialize on product k.

This reasoning can be applied for all countries ($j = 1, 2, \dots, n$) in order to estimate the best product for each country.

The answer to the question whether the best product k is different from the best product of all other countries or many countries will specialize on the same product depends on the international relative prices for the goods.

Due to Theorem 4 for the optimal quantities x_{ij}^0 and the optimal wage rates w_j^0 the values of the objective functions are equal:

$$\sum_{i=1}^{m} p_i \sum_{i=1}^{n} x_{ij}^0 = \sum_{i=1}^{n} w_j^0 L_j.$$

In economic terms, the value of world output coincides with total factor income of the world.

The reader will verify that the application of this general model to the simple Ricardian example from Section I.2.4 leads to the same results as the graphical solution at Fig. 4.3 and Fig. 4.4.

6.3. Leontief pollution model

In the Section I.2.8 the basic Leontief's input-output model has been introduced. With the increasing pollution as a by-product of regular economic activities arises the need to incorporate environmental effects in an input-output framework. In the well known paper LEONTIEF (1970) extended the input-output model in two ways. Firstly, he added rows to show the output of pollutants by industries. Secondly, he introduced a pollution abatement "industry" with a specific technology for the elimination of each pollutant. With respect to the exogenously given level of tolerated pollution two formulations of the model can be found in his paper.

In the first version an exogenously given vector of tolerated level of pollutants or environmental standards is treated as a negative variable on the right-hand side of the model (see also LOWE (1979), MILLER - BLAIR (1985)). It consists of the following equations:

$$(E - A_{11})\mathbf{x}_1 - A_{12}\mathbf{x}_2 = \mathbf{y}_1 \tag{IV.49a}$$

$$-A_{21}\mathbf{x}_1 + (E - A_{22})\mathbf{x}_2 = -\mathbf{y}_2 \tag{IV.49b}$$

where

$\mathbf{x}_1$ is the n-dimensional vector of gross industrial outputs;

$\mathbf{x}_2$ is the k-dimensional vector of abatement (or anti-pollution) activity levels;

A_{11} is the $n \times n$ matrix of conventional input coefficients, showing the input of good i per unit of the output of good j (produced by sector j);

A_{12} is the $n \times k$ matrix with a_{ig} representing the input of good i per unit of the eliminated pollutant g (eliminated by abatement activity g);

A_{21} is the $k \times n$ matrix that shows the output of pollutant g per unit of good i (produced by sector i);

A_{22} is the $k \times k$ matrix that shows the output of pollutant g per unit of eliminated pollutant h (eliminated by abatement activity h);

E is the identity matrix;
$\mathbf{y}_1$ is the n-dimensional vector of final demands for economic commodities;
$\mathbf{y}_2$ is the k-dimensional vector of the net generation of pollutants which remain untreated. The g-th element of this vector represents the environmental standard of pollutant g and indicates the tolerated level of net pollution.

¿From equation (IV.49a) we can see that one part of the industrial output is used as an input in the other sectors of the economy ($A_{11}\mathbf{x}_1$), another part as an input for the abatement activities ($A_{12}\mathbf{x}_2$) and one part is devoted for the final demand ($\mathbf{y}_1$). The balance equations for the pollutants or for the undesirable outputs are given by (IV.49b). The total amount of pollution consists of pollution generated by production of desirable goods ($A_{21}\mathbf{x}_1$) and by the abatement activities themselves ($A_{22}\mathbf{x}_2$). One part of the gross pollution will be eliminated ($\mathbf{x}_2$) and the amount $\mathbf{y}_2$ remain untreated because it is tolerated.

The solution of equations (IV.49) for given levels of final demand $\mathbf{y}_1$ and given pollution standards $\mathbf{y}_2$ can be obtained by inverting the augmented Leontief matrix such that

$$\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} E - A_{11} & -A_{12} \\ -A_{21} & E - A_{22} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{y}_1 \\ -\mathbf{y}_2 \end{pmatrix}.$$

The sufficient conditions for the existence of non-negative solution of the systems (IV.49) are given by LUPTÁČIK - BÖHM (1994).

The price model corresponding to the model (IV.49) has the following form

$$\mathbf{p}'(E - A_{11}) - \mathbf{r}'A_{21} = \mathbf{v}_1' \qquad \text{(IV.50a)}$$

$$-\mathbf{p}'A_{12} + \mathbf{r}'(E - A_{22}) = \mathbf{v}_2' \qquad \text{(IV.50b)}$$

with $\mathbf{p}'$ the $(1 \times n)$ vector of commodity prices, and $\mathbf{r}'$ the $(1 \times k)$ vector of prices (= cost per unit) for eliminating pollutants. $\mathbf{v}_1'$ and $\mathbf{v}_2'$ are the exogenously given $(1 \times n)$ and $(1 \times k)$ vectors of primary inputs values per unit of production and per unit level of abatement activities respectively.

Equation (IV.50a) shows that the commodity prices $\mathbf{p}'$ must be such they cover the costs of inputs from other sectors of the economy ($\mathbf{p}'A_{11}$), the costs of primary factors $\mathbf{v}_1'$ and the pollution costs ($\mathbf{r}'A_{21}$). Equation (IV.50b) determines the prices of pollutant $\mathbf{r}'$ from abatement cost ($\mathbf{p}'A_{12}$), costs of primary inputs per unit level of abatement activities $\mathbf{v}_2'$ and the pollution costs of the abatement activities themselves ($\mathbf{r}'A_{22}$).

The solution of the price or of the dual model is then

$$(\mathbf{p}', \mathbf{r}') = (\mathbf{v}_1', \mathbf{v}_2') \begin{pmatrix} E - A_{11} & -A_{12} \\ -A_{21} & E - A_{22} \end{pmatrix}^{-1}.$$

In the second version of the Leontief pollution model the environmental standard has been defined - in the one pollutant case - as the ratio of eliminated pollution x_{2g} to the gross pollution which is the sum of net pollution and abatement activity $(x_{2g} + y_{2g})$. Denoting the $(k \times k)$ diagonal matrix of proportions of abated gross pollutants by $\hat{S}$, we have

$$\hat{S}(\mathbf{x}_2 + \mathbf{y}_2) = \mathbf{x}_2.$$

Then the quantity model can be formulated as (see also STENGE (1978), LOWE (1979) and ARROUS (1994))

$$(E - A_{11})\mathbf{x}_1 - A_{12}\mathbf{x}_2 = \mathbf{y}_1 \tag{IV.51a}$$

$$-\hat{S}A_{21}\mathbf{x}_1 + (E - \hat{S}A_{22})\mathbf{x}_2 = \mathbf{0}. \tag{IV.51b}$$

The equation (IV.51b) determines the level of abatement activity $\mathbf{x}_2$ as the sum of abated pollution generated by the production ($\hat{S}A_{21}\mathbf{x}_1$) and by the antipollution activities themselves ($\hat{S}A_{22}\mathbf{x}_2$). Obviously, if $\mathbf{y}_2 = \mathbf{0}$, then $\hat{S} = E$. This is the case of complete abatement (no pollution is tolerated), where the models (IV.49) and (IV.51) coincide.

The corresponding price model is

$$\mathbf{p}_s'(E - A_{11}) - \mathbf{r}_s'\hat{S}A_{21} = \mathbf{v}_1' \tag{IV.52a}$$

$$-\mathbf{p}_s'A_{12} + \mathbf{r}_s'(E - \hat{S}A_{22}) = \mathbf{v}_2'. \tag{IV.52b}$$

Note that prices in this model are subscripted by "s".

According to (IV.52a) the commodity prices $\mathbf{p}_s'$ include the costs of intermediate inputs ($\mathbf{p}_s'A_{11}$) and of the primary inputs ($\mathbf{v}_1'$), and the pollution abatement costs ($\mathbf{r}_s'\hat{S}A_{21}$). The interpretation of the equation (IV.52b) for the pollutant prices $\mathbf{r}_s'$ is similar.

The solution of the price model (IV.52) is

$$(\mathbf{p}_s', \mathbf{r}_s') = (\mathbf{v}_1', \mathbf{v}_2')\begin{pmatrix} E - A_{11} & -A_{12} \\ -\hat{S}A_{21} & E - \hat{S}A_{22} \end{pmatrix}^{-1}.$$

The given environmental standards or the tolerated level of the net pollution $\mathbf{y}_2$, the corresponding elements of the diagonal matrix $\hat{S}$ can be chosen such that the models (IV.49) and (IV.51) share the same solution (for the levels of production and abatement). Even in this case the commodity prices $\mathbf{p}_s$ and the prices for eliminating pollutants $\mathbf{r}_s$ are smaller than or equal to the prices determined by the model (IV.50) for any nonnegative vector $(\mathbf{v}_1', \mathbf{v}_2')$ when some net pollution is left untreated (LUPTÁČIK - BÖHM (1999), Theorem 1, p. 267).

Already in the paper by LOWE (1979) the price solutions of both models were compared. He showed that only the prices $\mathbf{p}_s$ and $\mathbf{r}_s$ were the appropriate industrial prices and effluent charges because they were consistent with financial viability and national income-expenditure balance. That means, all chosen activities could be met from revenue. At the other side the prices determinated by model (IV.50) only possessed the property of opportunity costs of environmental restriction in terms of extra value-added or lost final demand. Differences in the two sets of prices appear, because of untreated net pollution is discharged free to final consumers. The question that arises by imposing emissions charges (effluent taxes) for untreated pollution is: how to estimate the level of emissions charges in both models, such that the prices for both models - providing the same level of production and of net pollution - are the same?

For this purpose the augmented Leontief model (IV.49) is formulated as an optimization model with the net generation of pollutants $\mathbf{y}_2$ as endogenous variables which are limited to specified amounts $\overline{\mathbf{y}}_2$. However, untreated pollutants are not discharged free in a receiving medium but the polluters have to pay effluent charge on every untreated unit. Denoting by $\mathbf{t}$ the vector of effluent taxes levied per unit of residual pollutants, the environmental costs $\mathbf{t}'\mathbf{y}_2$ will be added to the costs of primary factors required by industrial production $\mathbf{x}_1$ and abatement activities $\mathbf{x}_2$. The gross national product (GNP) at factor costs, including the environmental costs should be minimized for a given level of final demand $\overline{\mathbf{y}}_1$. The resulting optimization model (with a possibility of including alternative techniques of industrial production and pollution abatement), denoted as Model I, (LUPTÁČIK - BÖHM, 1999, p. 269), is then

$$\text{minimize} \quad V(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_2) = \mathbf{v}_1'\mathbf{x}_1 + \mathbf{v}_2'\mathbf{x}_2 + \mathbf{t}'\mathbf{y}_2 \tag{IV.53}$$

$$\text{subject to} \quad (E - A_{11})\mathbf{x}_1 - A_{12}\mathbf{x}_2 \geqq \overline{\mathbf{y}}_1 \tag{IV.54}$$

$$-A_{21}\mathbf{x}_1 + (E - A_{22})\mathbf{x}_2 + \mathbf{y}_2 \geqq \mathbf{0} \tag{IV.55}$$

$$-\mathbf{y}_2 \geqq -\overline{\mathbf{y}}_2 \tag{IV.56}$$

$$\mathbf{x}_1 \geqq \mathbf{0}, \quad \mathbf{x}_2 \geqq \mathbf{0}, \quad \mathbf{y}_2 \geqq \mathbf{0}. \tag{IV.57}$$

The inequalities (IV.54) express the requirement that a given bill of goods $\overline{\mathbf{y}}_1$ for final demand must be provided. According to expressions (IV.55) and (IV.56) the actual amount of pollutants $\mathbf{y}_2$ which remain untreated after abatement activity does not exceed the environmental standards $\overline{\mathbf{y}}_2$.

The subject of our interest is the dual or price model corresponding to model (IV.53) - (IV.57), i.e.

$$\text{maximize} \quad W(\mathbf{p}, \mathbf{r}, \mathbf{s}) = \mathbf{p}'\overline{\mathbf{y}}_1 - \mathbf{s}'\overline{\mathbf{y}}_2 \tag{IV.58}$$

$$\text{subject to} \quad \mathbf{p}'(E - A_{11}) - \mathbf{r}'A_{21} \leqq \mathbf{v}_1' \tag{IV.59}$$

$$-\mathbf{p}'A_{12} + \mathbf{r}'(E - A_{22}) \leqq \mathbf{v}_2' \tag{IV.60}$$

$$\mathbf{r}' - \mathbf{s}' \leqq \mathbf{t}' \tag{IV.61}$$

$$\mathbf{p}' \geqq \mathbf{0}, \quad \mathbf{r}' \geqq \mathbf{0}, \quad \mathbf{s}' \geqq \mathbf{0} \tag{IV.62}$$

where $\mathbf{s}'$ is a $(1 \times k)$ vector of dual variables related to the environmental constraints (IV.56).

For positive levels of gross industrial outputs $\mathbf{x}_1$ and of abatement activities $\mathbf{x}_2$ the constraints (IV.59) and (IV.60) are fulfilled as equalities (due to the complementary slackness theorem)

$$\mathbf{p}' = \mathbf{p}'A_{11} + \mathbf{v}_1' + \mathbf{r}'A_{21} \tag{IV.63}$$

$$\mathbf{r}' = \mathbf{p}'A_{12} + \mathbf{v}_2' + \mathbf{r}'A_{22}. \tag{IV.64}$$

These equations correspond to the equations (IV.50a) - (IV.50b) and provide the economic foundation to the "polluter pays principle".

A positive level of net pollution $\mathbf{y}_2 > \mathbf{0}$ implies an equality in expression (IV.61): $\mathbf{r}' - \mathbf{s}' = \mathbf{t}'$ or $\mathbf{r}' = \mathbf{t}' + \mathbf{s}'$, where s_g indicates the increase of GNP at factor costs, by tightening the environmental standard $\bar{y}_{2g} \quad (g = 1, 2, \ldots, k)$ by a small unit. If the amount of untreated pollutant g is below the tolerated level $\bar{y}_{2g}$, then the corresponding dual variable s_g is equal zero and the price of pollutants g is determined by the effluent tax t_g levied on residual pollutant. When the environmental constraint

(IV.56) is binding, the shadow price $\mathbf{s}'$ can be positive, and the prices of pollutants $\mathbf{r}'$ and commodity prices $\mathbf{p}'$ will rise to include the additional environmental costs $\mathbf{s}'$ caused by the obligation to meet the standards. The higher environmental quality is paid for by increasing commodity prices and prices for eliminating pollutants. As shown already by LEONTIEF (1970), a tightening the environmental standard $\overline{\mathbf{y}}_2$ implies higher industrial production (because of the inputs for the anti-pollution activities) and higher GNP.

The modification of the model (IV.51) by imposing effluent taxes $\mathbf{t}'_s$ per unit of untreated pollution leads to the following optimization model, denoted in LUPTÁČIK - BÖHM (1999) p.271 as Model II:

$$\text{minimize} \quad V_s(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_2) = \mathbf{v}'_1\mathbf{x}_1 + \mathbf{v}'_2\mathbf{x}_2 + \mathbf{t}'_s\mathbf{y}_2 \tag{IV.65}$$

$$\text{subject to} \quad (E - A_{11})\mathbf{x}_1 - A_{12}\mathbf{x}_2 \geqq \overline{\mathbf{y}}_1 \tag{IV.66}$$

$$-\hat{S}A_{21}\mathbf{x}_1 + (E - \hat{S}A_{22})\mathbf{x}_2 \geqq \mathbf{0} \tag{IV.67}$$

$$-(E - \hat{S})A_{21}\mathbf{x}_1 - (E - \hat{S})A_{22}\mathbf{x}_2 + \mathbf{y}_2 \geqq \mathbf{0} \tag{IV.68}$$

$$\mathbf{x}_1 \geqq \mathbf{0}, \quad \mathbf{x}_2 \geqq \mathbf{0}, \quad \mathbf{y}_2 \geqq \mathbf{0}. \tag{IV.69}$$

Note that we use subscript "s" to distinguish the variables or parameters of both model. The objective function (IV.65), apart from the possible differences in the level of effluent taxes $\mathbf{t}'_s$ and $\mathbf{t}'$, respectively is the same as objective function (IV.53). Furthermore, there is no difference in the constraints (IV.54) and (IV.66). Expression (IV.67) requires that the levels of abatement activities $\mathbf{x}_2$ must at least meet the given proportions of gross pollution. Because of the objective function (IV.65) in the optimal solution of Model II., condition (IV.68) will be fulfilled under equality. Thereupon, constraint (IV.68) describes the levels of untreated pollution $\mathbf{y}_2$ for which the specific effluent taxes are levied.

Again, the subject of our analysis is the price model of Model II., i.e.

$$\text{maximize} \quad W_s(\mathbf{p}, \mathbf{r}, \mathbf{s}) = \mathbf{p}'_s\overline{y}_1 \tag{IV.70}$$

$$\text{subject to} \quad \mathbf{p}'_s(E - A_{11}) - \mathbf{r}'_s\hat{S}A_{21} - \mathbf{s}'_s(E - \hat{S})A_{21} \leqq \mathbf{v}'_1 \tag{IV.71}$$

$$-\mathbf{p}'_sA_{12} + \mathbf{r}'_s(E - \hat{S}A_{22}) - \mathbf{s}'_s(E - \hat{S})A_{22} \leqq \mathbf{v}'_2 \tag{IV.72}$$

$$\mathbf{s}'_s \leqq \mathbf{t}'_s \tag{IV.73}$$

$$\mathbf{p}'_s \geqq \mathbf{0}, \quad \mathbf{r}'_s \geqq \mathbf{0}, \quad \mathbf{s}'_s \geqq \mathbf{0}. \tag{IV.74}$$

Assuming again positive levels of industrial production $\mathbf{x}_1$ and of abatement activities $\mathbf{x}_2$, constraints (IV.71) and (IV.72) can be written as equalities:

$$\mathbf{p}'_s = \mathbf{p}'_sA_{11} + \mathbf{v}'_1 + \mathbf{r}'\hat{S}A_{21} + \mathbf{s}'_s(E - \hat{S})A_{21} \tag{IV.75}$$

$$\mathbf{r}'_s = \mathbf{p}'_sA_{12} + \mathbf{v}'_2 + \mathbf{r}'\hat{S}A_{22} + \mathbf{s}'_s(E - \hat{S})A_{22}. \tag{IV.76}$$

For positive levels of untreated pollution $\mathbf{y}_2$, the dual variables $\mathbf{s}'_s$ are equal to the effluent taxes $\mathbf{t}'_s$. Then, price equations (IV.75) and (IV.76) get a clear economic meaning. Compared with the price equation (IV.63) the price equation (IV.75) takes into account not only the costs of intermediate inputs $\mathbf{p}'_s A_{11}$ and the primary inputs $\mathbf{v}'_1$ but the pollution abatement costs $\mathbf{r}'\hat{S}A_{21}$ and the charges for untreated pollution, given by $\mathbf{t}'_s(E-\hat{S})A_{22}$ too. The interpretation of the equation (IV.76) for the pollutants prices $\mathbf{r}'_s$ is similar.

The answer to the question how to avoid the differences in the prices for models (IV.50) and (IV.52) is given in the following

Proposition 1 (LUPTÁČIK - BÖHM (1999) p. 272 including the proof) **:** *If* $\mathbf{t}'_s = \mathbf{t}' + \mathbf{s}'$ *for given levels of the effluent taxes* $\mathbf{t}'$, *and if model (IV.53)-(IV.57) and model (IV.65)-(IV.69) share the same optimal solution, then the commodity prices and the prices for eliminating pollutants for both programmes, i.e. (IV.58)-(IV.62) and (IV.70)-(IV.74), respectively, are equal.*

In this way the prices are consistent with financial viability and can be interpreted as opportunity cost variables. The shadow prices $\mathbf{s}$ provide the appropriate rates for the effluent taxes to be charged on untreated pollutions. If these charges are lower than the shadow prices of environmental standards, then production and abatement activities will exactly meet the standards. For the effluent taxes higher than the shadow prices, the pollutants will be clean up completely. It is cheaper to abate than to pay taxes. The switch between completely protected economy ($\mathbf{y}_2 = \mathbf{0}$) and polluting up to the standards ($\mathbf{y}_2 = \overline{\mathbf{y}}_2$) follows from the linearity of the model.