CHAPTER 14

FRAME RELATED RESTRICTIONS, CORES AND SHAPLEY VALUES

1. Introduction

Using linear programming methods Charnes and Kortanek 1967 followed Scarf 1967 in showing that the existence (or not) of a non empty core to a characteristic function N person game can be characterized as the solution to an appropriately specified extremal problem. In the paper just cited Charnes and Kortanek also used linear programming methods to characterize the nucleolus. (See Schmeidler 1969, Maschler 1992.) All of these approaches assume given characteristic functions for an N person game, where N is fixed exogenously.

In this chapter I use a framing idea to obtain four new and interrelated results. First I develop an explicitly frame related linear programming charac-terisation of the emptiness or nonemptiness of the core to an S person characteristic function game, for which S≤N is endogenously determined. In that context the framing parameters will relate directly to potential rewards to relatively external alter-natives and so to decisions by marginal players on whether or not to start to and/or to continue to play. Secondly, I show that the coalition building feature, which is implicit in this explicitly frame-related formulation, can be related directly to the equi-probable coalition building interpretation of the Shapley value in Shapley 1953. The third class of results interrelates Shapley values derived from a primal linear programming characterisation and Shapley related conditions stemming from its dual and shows that the latter are equivalent to those derived by Hart and Mas Collel 1989 using their idea of *potential*. Finally I consider conditions under which the Shapley value is or is not in the core.

The main results here stem from the explicit use of an optimizing framework to consider the marginal conditions determining decisions by N+1st players *not* to join another N persons in an N+1 person game. In that way the emptiness or nonemptiness of the core, the N player Shapley value and the Hart and Mas Collel potential for the resulting N player game will be uniquely determined *only given critical values for framing parameters* within an explicitly optimizing framework interrelating both for the N players actually playing and the N+1st players not playing at the margin determined by those parameters. From that perspective: i) Shapley's determination of unconditionally unique values for payoffs to players and; ii) Hart and Mas Collel's determination of a unique potential and; iii) Scarf's and Charnes' and Kortanek's extremal characterisation of the emptiness or nonemptiness of the core of an N person characteristic function game, all become classes of cases *implicitly* assuming correspondingly optimizing values for framing parameters.

The structure of the chapter is as follows: In Section 2 I introduce the framing idea, first for linear programs in general and then for the two person constant sum game, to illustrate a distinction between preemptively and nonpreemtively framed specifications which will be useful in the subsequent Sections. In contrast to the wider applications in Section 2, those subsequent Sections all focus on coalition building characterisations of the Shapley value and on extremal characterisations of the emptiness or nonemptiness of the core. Thus, in Section 3 I consider an explicitly frame related characterisation of the emptiness or nonemptiness of the core and in Sections 4 and 5 I turn to a frame contingent coalition building based process derivation of the Shapley Value. In Section 6 I consider associated frame related starting, continuation and stopping criteria for that case. Finally in Section 7 I provide conditions under which the Shapley Value is (or is not) in the core.

2. Linear Programming and Framing

The standard linear program and its dual can be written as:

$$\text{Max } z = \sum_j c_j x_j \qquad\qquad \text{Min } g = \sum_i \mu_i b_i$$

$$\text{st} \quad \sum_j a_{ij} x_j \le b_i \quad \text{(I)} \qquad \text{st} \quad \sum_i \mu_i a_{ij} \ge c_j \quad \text{(I)'}$$

$$x_j \ge 0 \qquad\qquad\qquad \mu_i \ge 0$$

In this form (I),(I)' may not have feasible solutions. And/or optimal solutions to (I),(I)' may be unbounded. Accordingly consider extensions (II),(II)' of (I),(I)' which always yield feasible solutions and are respectively explicitly bounded by M^+, $-M^-$, where M^+, $-M^-$ are of orders lexicograp-hically greater than all other coefficients in the objectives of (II),(II)':

$$\text{Max } z = \sum_j c_j x_j - \sum_i c_i^+ s_i^+ - \sum_i c_i^- s_i^- - \phi M^-$$

$$\text{st} \quad \sum_i a_{ij} x_j + s_i^+ - s_i^- = (1-\phi) b_i \quad \text{(II)}$$

$$\sum_j c_j x_j - c_i^+ s_i^+ - c_i^- s_i^- \le M^+$$

$$-d_{ij}^+ \le x_j \le d_j$$

$$s_i^+, s_i^- \ge 0$$

$$\text{Minimize } g = \sum_i \mu_i b_i - d_i^+ r_i^+ - d_j^- r_j^- + \lambda M^+$$

$$\text{st} \quad \sum_j \mu_i a_{ij} + r_j^+ - r_j^- = (1-\lambda) c_j \quad \text{(II)'}$$

$$\sum_j \mu_i b_i - d_i^+ r_i^+ - d_j^- r_j^- \ge -M^-$$

$$-c_i^+ \le \mu_i \le c_i^-$$

$$r^+, r_j^- \ge 0$$

Clearly (II),(II)' *always* have feasible solutions (e.g. $s_i^+ = (1-\phi) b_i$ if $(1-\phi) b_i > 0$, $s_i^+ = (1-\phi) b_i$ if $(1-\phi) b_i < 0$ in (II) and $r_j^+ = (1-\lambda) c_j$ if $(1-\lambda) c_j > 0$, $r_j^- = (1-\lambda) c_j$ if $(1-\lambda) c_j < 0$, in (II)'. It follows by the Extended Dual Theorem (see Charnes and Cooper 1961) that (II),(II)' always have bounded optimal solutions.

DEFINITION

s_i^+, s_i^-, ϕ and r_j^+, r_j^-, λ are *framing variables* and c_i^+, c_i^+, M^- and d_i^+, d_j^+, M^+ are corresponding *framing parameters* for (II),(II)'.

THEOREM 1 *(Framing)*

If a dual pair of linear programs (I),(I)' is to yield bounded optimal solutions then these programs are necessarily either *implicitly* framed in a manner consistent with program (II) (resp (II)') with $\phi = 0$, $s_i^+ = s_i^- = 0$ all i (resp with $\lambda = 0$, $r_j^+ = r_j^- = 0$ all j), or *explicitly framed* in a manner consistent with the

extension of (I) (resp(I)') to give (II) (resp(II)') with $\phi \ne 0$ and/or $s_i^+ \ne 0, s_i^- \ne 0$ some i (resp $\lambda \ne 0$, and/or $r_j^+ \ne 0$, $r_j^- \ne 0$ all j) *otherwise*.

PROOF

Any bounded optimal solution to (I) (resp (I)') is necessarily bounded feasible and any bounded feasible solution to (I) (resp (I)') can be expressed as a solution to the correspondingly *explicitly framed* system (II) (resp (II)'). It follows that such a solution is necessarily *either* such that $\phi = 0$, $s_i^+ = s_i^- = 0$ all i (resp $\lambda = 0$, $r_j^+ = r_j^- = 0$ all j), i.e. is *implicitly framed*, *or* such that $\phi \ne 0$ and/or $s_i^+ \ne 0, s_i^- \ne 0$ some i (resp $\lambda \ne 0$, and/or $r_j^+ \ne 0$, $r_j^- \ne 0$ all j) i.e. *explicitly framed*.

COROLLARY

An implicitly framed linear program in the sense of Theorem 1 is equivalent to a *preemptively* framed program in which M^-, c_i^- (resp M^+, d_i^+) are sufficiently large to determine that *any* optimal solution to (II) (resp (II)') will be such that $\phi = 0$, $s_i^- = 0$ all i (resp $\lambda = 0$, $r_j^+ = 0$ all j).

It follows from this corollary that even a dual pair of linear programs of the form of (II),(II)') with optimal solutions such $\phi = 0$, $s_i^+ = s_i^- = 0$ all i (resp $\lambda = 0$, $r_j^+, r_j^- = 0$ all j) can be interpreted as *implicitly* framed via a prior process which, among other things, determines binding constraints $\sum a_{ij} x_j = b_i$ (resp $\sum \mu_i a_{ij} = c_j$) on the ranges of x_j, μ_i. More generally, at least some of the parameters associated with framing variables may be non-preemptive and optimal solutions to (II), (II)' will be *explicitly* framed.

One large class of examples here are *feasibility programs* corresponding to the specialization of (II),(II)' with $\phi = 0$, $s_i^- = 0$ all i (resp $\lambda = 0$, $r_j^+ = 0$ all j) to yield (I),(I)' as if via arbitrarily large framing parameters M^-, c_j^- in (II) and M^+, d_i^+ in (II)'. Another corresponds to *goal programs* (see Charnes and Cooper 1961, Shogan 1988) in which some or all of c_i^+, c_i^+ and d_i^+, d_j^+ in (II),(II)' may be non preemptive.

AN EXAMPLE

To illustrate these ideas consider a specialization of the explicitly framed systems (II),(II)' which is contingently consistent with a maximin-minimax formulation of a *constrained* two person constant sum game:

$\text{Max } z = \rho q^* - \Sigma c_i^+ s_i^+ - \Sigma c_i^- s_i^- - c_0^+ s_0^+ - c_0^- s_0^- - \phi M^-$

$q_j \qquad st \quad \Sigma \pi_{ij} p_j + s_i^+ - s_i^- = \rho$

$\mu \qquad\qquad \Sigma p_j + s_0^+ - s_0^- = p^*(1-\phi) \qquad (III)$

$\lambda \qquad \rho q^* - \Sigma M s_i^+ - \Sigma c_i^- s_i^- - c_0^+ s_0^+ - c_0^- s_0^- \le M^+$

$r_j^+, r_j^- \qquad -(1-\phi)d_j^+ \le p_j \le (1-\phi)d_j^-$

$r_0^+, r_0^- \qquad -(1-\phi)d_0^- \le \rho \le (1-\phi)d_0^+$

$\qquad\qquad\qquad s_i^+, s_i^- \ge 0$

$\text{Min } g = \mu p^* + \Sigma d_j^+ r_j^+ + \Sigma d_j^- r_j^- + d_0^+ r_0^+ + d_0^- r_0^- + \lambda M^+$

$p_j \qquad st \quad \Sigma q_i \pi_{ij} + r_j^+ - r_j^- = \mu$

$\rho \qquad\qquad \Sigma q_i + r_0^+ - r_0^- = q^*(1-\lambda) \qquad (III)'$

$\phi \qquad \mu p^* - \Sigma d_j^- r_j^- + \Sigma M r_j^+ + d_0^+ r_0^+ + d_0^- r_0^- \ge -M^-$

$s_i^+, s_i^- \qquad -(1-\lambda)c_i^- \le q_i \le (1-\lambda)c_i^+$

$s_0^+, s_i^- \qquad -(1-\lambda)c_0^- \le \mu \le (1-\lambda)c_0^-$

$\qquad\qquad\qquad r_j^+, r_j^- \ge 0$

For comparison the standard representation of a two person constant sum game with strategies p_j, q_i and contingent payoffs π_{ij} is:

$$\text{Max } z = \rho q^*$$
$$st \quad \Sigma \pi_{ij} p_j - s_i^- = \rho$$
$$\Sigma p_j = p^* \qquad (IV)$$
$$0 \le p_j \le M$$
$$-M \le \rho \le M$$
$$s_i^+, s_i^- \ge 0$$

$$\text{Min } g = \mu p^*$$
$$st \quad \Sigma q_i \pi_{ij} + r_j^+ = \mu$$
$$\Sigma q_i = q^* \qquad (IV)'$$
$$0 \le q_i \le M$$
$$-M \le \mu \le M$$
$$r_j^+, r_j^- \ge 0$$

Clearly (IV) is equivalent to (III) (resp (IV)' is equivalent to (III)') iff *implicit* framing conditions obtain such that: i) optimally $\lambda = -\phi = 0$ so that (III) (resp (IV)) is bounded; ii) c_0^+, c_0^- (resp d_0^+, d_0^-) are effectively arbitrarily large so that $s_0^+ = s_0^- = 0$ (resp $r_0^+ = r_0^- = 0$) at an optimum and; iii) the magnitudes of d_j^-, c_i^+ and d_j^+, c_i^- are respectively such that p_j, q_i are nonnegative and that there are no restrictive upper bounds on the probabilities with which players may play particular strategies.

SOME ECONOMIC INTERPRETATIONS

One interpretation of the constant sum game is as a farmer's game against nature where p_j are proportions of land given over to crops j, q_i are nature's weather related strategies and π_{ij} are crop and weather contingent payoffs. In that case framing condition i), that (IV) is bounded,

amounts to an assumption that the farmer's net expected payoff will be bounded. Framing condition ii) asserts that the farmer will confine attention only to planting on already available land (land will not be left fallow $s_0^+ = 0$, nor will additional land be rented in $s_0^- = 0$). Framing condition ii) also implies that a minimax hypothesis will be adopted by the farmer via framing conditions such that optimally $s_i^+ = 0$ as if in response to the ascription to nature of a maximin objective via $r_j^+ = 0$ all i and a restricted range of strategies such that $\Sigma q_i = q^*$ via conditions $r_0^+ = r_0^- = 0$. Finally, framing condition iii) implies that (III), (III) is not a *constrained game*. That is: the magnitudes of c_i^+, c_i^- (resp d_i^+, d_i^-) are such that there are effectively no externally imposed lower or upper bounds on the proportions of certain crops for the farmer or on probabilities of particular weather patterns. (Notice that in (III) s_i^+, s_i^- are marginal evaluations of prior information - e.g. of elements of a prior weather forecast - for the constrained game case. For more on this and on other constrained game specifications see Charnes 1951, Charnes et al 1993, Ryan 1994,1998b,1998c.)

The demonstration that (IV) can yield an explicitly framed analogue of an implicitly (i.e. exogenously) framed two person constant sum game of the standard type clearly has wider implications for the two person case. I have considered these further in Ryan 1994,1998b and 1998c. The main purpose of introducing this example here is to illustrate the point that in the explicitly framed systems (IV),(IV)' there are explicit *processes* both determining and providing dual evaluators of elements of relevant boundary conditions. In the next and subsequent Sections I will consider different game related applications of the framing idea which provide extremal representations of the emptiness or non-emptiness of the core as well as of the Shapley Value for N person game characteristic function games.

3. Frame related characterisations of the emptiness or nonemptiness of the core

Charnes and Kortanek 1967 followed Scarf 1967 in showing that, for an N person superadditive characteristic function game $\Gamma(N,V)$ with transferable utility, emptiness or nonemptiness of the core can be characterized respectively by means of optimal solutions $z^* = \text{Min}\Sigma x_i = V_{NN}$ and

$z^* = \text{Min}\Sigma x_I > V_{NN}$ to (V), where V_{rS} is the contingent payoff to a set rS of cardinality S with members r. ($V_{r1} \geq 0$ is assumed throughout):

$$\text{Min} \sum_{i\varepsilon N} x_i$$

$$\text{st} \quad \sum_{i\varepsilon S} x_i \geq V_{rS} \quad rS \subset NN \quad (V)$$

$$\sum_{i\varepsilon N} x_i \geq V_{NN}$$

REMARK

If the core is nonempty then $\Sigma x_i = V_{NN}$ and x_i constitute *imputations* of the value of the grand coalition NN to its members.

Now consider an extension (VI) of (V) which is explicitly framed in a manner analogous to the extension of (I) to (II). (Notice that (VI) implicitly omits the bounding conditions corresponding to the final two constraints in (II). I will return to this with the wider context of framing parameters and coalition building processes in the next Section.)

$$\text{Min} \sum_{i\varepsilon N} x_i + \sum_{rS \subset N} c_{rS}{}^+ x_{rS}{}^+ + \sum_{rS \subset N} c_{rS}{}^- x_{rS}{}^- + c_{NN}{}^+ x_{NN}{}^+ + c_{NN}{}^- x_{NN}{}^-$$

$$\text{st} \quad \sum_{i\varepsilon S} x_i + x_{rS}{}^+ - x_{rS}{}^- = V_{rS} \quad rS \subset NN \quad (VI)$$

$$\sum_{i\varepsilon N} x_i + x_{NN}{}^+ - x_{NN}{}^- = V_{NN}$$

$$x_{rS}{}^+, x_{rS}{}^-, x_{NN}{}^+, x_{NN}{}^- \geq 0$$

THEOREM 2 (Framing and contingent equivalence between (V) and (VI))

Any optimal solution to (V) can be represented as a frame contingent optimal solution to (VI).

PROOF

If $c_{rS}{}^+ = M$, $c_{NN}{}^+ = M$ and $c_{rS}{}^- = 0$, $c_{NN}{}^- = 0$, $rS \subseteq NN$ then for these values of these framing parameters *any* optimal solution to (VI) will be consistent with $x_{NN}{}^+ = x_{NN}{}^- = 0$, $x_{rS}{}^+ = 0$, $x_{rS}{}^- \geq 0$ and thence with equivalence between (V) and (VI).

Optimal solutions to (V) correspond to imputations x_i on a coalition of cardinality N only if the core of that game is non empty. Now consider a parameterization of (VI) which yields an imputation on a coalition hT of cardinality T, whether or not the core a coalition is non empty.

DEFINITION 1 *(Core of a coalition)*

The core of a coalition hT of cardinality $T \leq N$ in an N person characteristic function game is non empty if there exists an imputation x_i, $i\varepsilon hT$ on that coalition such that:

$$\sum_{i\varepsilon S} x_i - x_{rS}{}^- = V_{rS} \quad rS \subset hT \text{ where } hT \subseteq NN$$

$$\sum_{i\varepsilon hT} x_i = V_{hT} \quad (Va)$$

$$x_{rS}{}^- \geq 0$$

THEOREM 3 *(Imputations, framing and emptiness or nonemptiness of the core)*

Any superadditive characteristic function game $\Gamma(N,V)$, $N>1$: i) has at least one coalition rS with $S \geq 2$ and a nonempty core and; ii) the imputation corresponding to that core can be found as an appropriately framed class of optimal solutions to (VI).

PROOF

i) From Definition 1 the core of a coalition $hT \subseteq NN$ in a characteristic function game $\Gamma(N,V)$ is nonempty if there exists a feasible solution to (VI) consistent with (Va).

But, if $\Gamma(N,V)$ is superadditive, conditions $\Sigma x_i = V_{h2}$ with $x_i - x_{rS}{}^- = V_{r1}$ $x_{r1} \geq 0$ $i\varepsilon h2 \subseteq N$ are consistent with frame contingent optimal solutions to (VI) for which $x_{h2}{}^+ = x_{hT}{}^+ = 0$ and $x_{r1} = 0$, $r\varepsilon 1h2$, so that, as if via preemptively large weights $c_{h2}{}^+ = c_{h2}{}^+ = M$, $c_{r1}{}^+ = M$, the core of at least one two player coalition is nonempty.

ii) If a nonempty core of a coalition hT, $2 < T \leq N$ exists it can be found as an optimal solution to (VI) with $x_{hT}{}^+ = x_{hT}{}^- = 0$ as if via preemptively large values $c_{hT}{}^+, c_{hT}{}^-$ *and* $x_{rS}{}^+ = 0$, $x_{sS}{}^- \geq 0$ as if respectively via preemptively large weights $c_{rT}{}^+, c_{sT}{}^-$ and sufficiently large values $c_{rS}{}^+$ and sufficiently small values $c_{rS}{}^-$.

REMARKS

• If a non empty core exists for $hT \subset NN$ then players hT and players N-hT constitute a *partition* of the set of players N.
• Solutions consistent with the corollary to Theorem 3 correspond to disjoint core related imputations to "playing" players and opportunity cost related payments to "nonplaying" players. In that way these players constitute a partition of N respectively joining and not joining a coalition $hT \subseteq NN$ with a nonempty core.

This second remark in turn suggests that optimal solutions to (VI), which may correspond to an empty or to a nonempty core for a coalition N, may be potentially *variously* interpreted as:

i) corresponding to *nonempty* core solutions for r players in which the framing parameters are consistent with decisions by marginal players N-r, $|N-r| \geq 0$ *not* to join a coalition rS with a nonempty

core and/or;

ii) corresponding to *empty* core solutions for r players in which the framing parameters are consistent with decisions by a marginal player to join a coalition r-1S-1 *even though* that decision would result in an empty core for coalition rS.

4. Duality, partitions and frame related coalition building restrictions

Associating dual variables λ_{rS}, λ_{NN} with the constraints of (VI) its dual is:

$$\text{Max} \quad \lambda_{NN}V_{NN} + \Sigma\lambda_{rS}V_{rS}$$
$$\underset{rS \subset NN}{}$$
$$\text{(VI)'}$$
$$\text{st} \quad \lambda_{NN} + \underset{i\varepsilon rS}{\Sigma\lambda_{rS}} \leq 1$$
$$-c_{rS}^{-} \leq \lambda_{rS} \leq c_{rS}^{+}$$
$$-M^{-} \leq \lambda_{NN} \leq M^{+}$$

If an optimum to (VI) is consistent with the emptiness (or nonemptiness) of the core of a coalition $hT \subseteq N$ then the framing parameters of (VI) and (VI)' must be consistent with the requirements of the proof of Theorem 3, viz: such that $x_{hT}^{+}=x_{hT}^{-}=0$ and $x_{rS}^{-}=0$, $rS \subset hT$ as if via premptively large weights $c_{hT}^{+}=c_{hT}^{-}=c_{rS}^{-}=M$. If these latter conditions obtain and, if the game $\Gamma(N,V)$ is superadditive, then $x_i>0$ at least one i. For all such $x_i>0$, by complementary slackness via (VI)':

$$x_i>0 \Rightarrow \qquad \lambda_{NN} + \Sigma\lambda_{rS} =1 \qquad (6.1)$$

Now consider an interpretation of λ_{rS} as *probabilities* of formation of coalitions rS. For particular players $i\varepsilon rS \subseteq NN$ conditions (6.1) appear to be consistent with a probabilistic interpretation where, by (6.1), for any player i the sum of probabilities λ_{rS} $i\varepsilon rS \subseteq NN$ of formation of subsets of players (including singleton subsets) of which that player could be a member is equal to 1. Further, by complementary slackness, from (VI):

$$\lambda_{rS}>0 \Rightarrow \quad \underset{i\varepsilon rS}{\Sigma x_i} + x_{rS}^{+} - x_{rS}^{-}=V_{rS} \qquad (6.2)$$

That is: if a coalition rS forms with probability $\lambda_{rS}>0$ then:

- **Either** (6.2) is *conditionally* consistent with imputations x_i to members of rS such that $x_{rS}^{+}=x_{rS}^{-}$

$=0$ and $\Sigma x_i=V_{rS}$ where, by complementary slackness, $x_{rS}^{+}= x_{rS}^{-}=0$ are consistent in turn with conditions such that $-c_{rS}^{-}<\lambda_{rS}<c_{rS}^{+}$

- **And/or** at least one of $x_{rS}^{+},x_{rS}^{-}>0$ in (6.2) so that $\Sigma x_i \neq V_{rS}$ (as would be so for at least one subcoalition in the empty core case). Then, by complementary slackness, $x_{rS}^{+}>0 \Rightarrow \lambda_{rS}=c_{rS}^{+}$ and if $x_{rS}^{-}>0 \Rightarrow -c_{rS}^{-}=\lambda_{rS}$.

THREE EXAMPLES

1. First consider the *nonempty* core case for an N person game via (VI) and (VI)'. In that case, via ii) of Theorem 3, the optimal solution to (VI) is consistent with framing conditions as if $c_{NN}^{+}=M$, $c_{NN}^{-}=M$, $c_{rS}^{+}\geq0$ and $c_{rS}^{-}=0$. These conditions are consistent in turn with interpretations: i) that a coalition of cardinality N forms with probability $\lambda_{NN}=1$ via $x_{NN}^{+}=x_{NN}^{-}=0$ and complementary slackness conditions: $-M=_{def}c_{NN}^{-}<\lambda_{NN}=1<c_{NN}^{+}=_{def}M$ *and/or*; ii) that subsets rS form with $\lambda_{rS}\geq0$ some $rS\subseteq NN$ via $x_{NN}^{+}=0$, $x_{NN}^{-}=0$ and complementary slackness conditions such that $0=c_{rS}^{-}\leq\lambda_{rS}<c_{rS}^{+}$ all $rS\subseteq NN$.

2. Second consider the *empty* core case for an N person game via (VI) and (VI)'. In that case there may be an imputation on N via $x_{NN}^{+}=x_{NN}^{-}=0$ so that by complementary slackness: $-M=_{def}c_{NN}^{-}<\lambda_{NN}<c_{NN}^{+}=_{def}M$. But, in the empty core case there *must* also be at least one coalition $rS\subset NN$ for which $x_{rS}^{+}>0$ so that by complementary slackness $\lambda_{rS}=c_{rS}^{+}$ at least one $rS\subset NN$ in (6.1).

3. Thirdly, again consider the *nonempty* core case for an N person game via (VI) and (VI)' as in Example 1, but now with $c_{rS}^{-}>0$ all $rS\subset NN$. In that case, via ii) of Theorem 3, the optimal solution to (VI) is consistent with framing conditions as if $c_{NN}^{+}=M$, $c_{NN}^{-}=M$, $c_{rS}^{+}=M$ and $c_{rS}^{-}>0$ all $rS\subset NN$. But, if the core is strictly nonempty (i.e. if $c_{rS}^{-}>0$ at least one $rS\subset NN$), then these conditions are *no longer* consistent with an interpretation that a coalition of cardinality N forms with probability $\lambda_{NN}=1$ via $x_{NN}^{+}=x_{NN}^{-}=0$ together with the complementary slackness conditions $-M=_{def}c_{NN}^{-}<\lambda_{NN}=1<c_{NN}^{+}=_{def}M$. This is because, by complementary slackness, $c_{rS}^{-}>0$ at least one $rS\subset NN \Rightarrow 0>-c_{rS}^{-}=\lambda_{rS}$ at least one $rS\subseteq NN$.

INTERPRETATIONS

1. The nonempty core cases in Example 1 can be interpreted straightforwardly as implying that, if the core is nonempty then *either* the grand coalition NN forms with probability 1 *or* partitions of it yielding imputations equivalent to imputations on the grand coalition NN form with finite

probabilities. An equivalent interpretation of Example 1 is that, in the empty core case, the grand coalition NN may optimally form with probability $\lambda_{NN}=1$ and all coalitions constituting subsets of NN may form with probability zero via $\lambda_{rS}=c_{rS}^+=0$ all rS⊂NN, *or vice versa*. That is: *In Example 1 conditions $c_{rS}^+=0$ all rS⊂NN are equivalent to indifference on the part of players to being paid their imputations x_i as if via a grand coalition or as if via subcoalitions rS⊂NN together constituting partitions of NN.*

2. The empty core case of Example 2 also appears straightforward. It has the interpretation that, if the core is empty, then $x_{rS}^+>0$ at least one rS⊂NN, so that by complementary slackness $\lambda_{rS}>0$ at least one rS⊂NN. This in turn implies that $\lambda_{NN}<1$ and the grand coalition NN will form with probability less than 1. That is: in the empty core case in general optimizing imputations to the N players must be attained by means *other than* by the formation of a grand coalition NN with probability 1 and so by means of the formation of partitions of that grand coalition with finite probability. (Notice that, if a coalition rS forms with finite probability $\lambda_{rS}>0$ and if there are N players, then complementary coalitions together of cardinality N-S must correspondingly form with finite probability $\lambda_{rS}>0$.)

3. By contrast with Example 1 the nonempty core cases in Example 3 assume $c_{rS}^+>0$ so that if the core is strictly nonempty then $x_{rS}^+>0$ at least one rS⊂NN and by complementary slackness $\lambda_{rS}=-c_{rS}^-<0$ at least one rS⊂NN. That is: λ_{rS} is then itself *negative* for at least one rS.

If λ_{rS} are interpreted as probabilities a *negative* value as in Example 3 appears, at least, counterintuitive. However, on closer examination this negative value can be seen as an indication that in Example 3 the probability distribution is properly understood as defined over a larger set of players than rS. Consider this in stages: Notice first that Example 1 is equivalent to conditions for *indifference* between the formation of NN with probability 1 and the formation of a partition of it with probability 1. Secondly; Example 2 is consistent with a *preference* on the part of players N, in the sense of a higher value for the optimal solution to (VI), for an optimum with a positive probability $\lambda_{rS}>0$ for at least one coalition rS *smaller* than NN. (That is: they prefer at least one *subset* rS to NN). Thirdly; Example 3 is consistent with a *preference* on the part of players N, in the sense of a higher value for the optimal solution to (VI), for a solution with a higher payoff than V_{rS}.

(Given superadditivity this implies a preference for a positive probability of coalitions *larger* than rS, i.e. a preference for any *superset* containing rS).

With the context of interpretations in the preceding paragraph, one interpretation of the relation $\lambda_{rS}=-c_{rS}^-<0$ in Example 3 is that in which $\lambda_{rS}<0$ is a predictor of a relatively larger range for the probability distribution. Specifically: in the context of Example 3 and an N player game, rather than being normalized over outcomes relating only to a coalition rS and subsets (subcoalitions) of rS, the relevant probability distribution becomes normalized over outcomes relating to at *least one superset of rS for rS⊂NN*.

More generally, λ_{hT}, the probability of an imputation on a coalition hT, can always be represented as in (6.3), where $\lambda_{rS}^+\geq0$ represent probabilities of outcomes for coalitions of cardinality *less* than hT (where rS⊂hT) and λ_{rS}^- represent probabilities of outcomes for coalitions of cardinality *greater than* than hT (where T⊂rS), viz:

$$\lambda_{hT} + \Sigma\lambda_{rS}^+ = 1 + \Sigma\lambda_{rS}^- \qquad (6.3)$$

With i) $\lambda_{rS}=_{def}\lambda_{rS}^+$ when $\lambda_{rS}>0$; ii) $\lambda_{rS}=_{def}\lambda_{rS}^-$ when $\lambda_{rS}<0$ and; iii) $\lambda_{rS}=_{def}\lambda_{rS}^+=_{def}\lambda_{rS}^-$ when $\lambda_{rS}=0$, (6.3) becomes potentially simultaneously consistent with the constraints of (VI)', with the optimality conditions (6.1) and with the conditions and interpretations of Examples 1-3.

From another perspective Examples 1-3 can be related directly to issues of *degeneracy*. In particular, in Example 1 the wide variety of alternate optimal solutions coincide with conditions of degeneracy since in that case the single condition $\lambda_{NN}=1$ solves all N equations (one for each player in the game $\Gamma(N,V)$). More generally, if for all rS⊂NN $c_{rS}^+=\varepsilon_{rS}^+$ and $c_{rS}^-=\varepsilon_{rS}^+$ with $c_{NN}^+=_{def}M$, $c_{NN}^-=_{def}M$, then $x_{rS}^+>0 => \lambda_{rS}=\varepsilon_{rS}^+$ and $x_{rS}^->0 => \lambda_{rS}=-\varepsilon_{rS}^-$.

With these values for its parameters (VI) becomes equivalent to:

$$\text{Min } \sum_{i\varepsilon N}x_i + \sum_{rS\subset N}\varepsilon_{rS}^{+}x_{rS}^{+}+\sum_{rS\subset N}\varepsilon_{rS}^{-}x_{rS}^{-} + Mx_{NN}^{+}+ Mx_{NN}^{-}$$

$$\text{st} \quad \sum_{i\varepsilon rS}x_i + x_{rS}^{+}- x_{rS}^{-}=V_{rS} \quad rS\subset NN \quad \text{(VIa)}$$

$$\sum_{i\varepsilon N}x_i + x_{NN}^{+}- x_{NN}^{-}=V_{NN}$$

$$x_{rS}^{+},x_{rS}^{-},x_{NN}^{+},x_{NN}^{-}\geq 0$$

For the nonempty core case (VI) and (VIa) are equivalent if $\varepsilon_{rS}^{-}=0$. In that way both (V) and (VIa) can be seen as a special cases of (VI). More particularly, Example 1 above can be seen as a limiting case in which $\varepsilon_{rS}^{+}\to 0$, $\varepsilon_{rS}^{-}\to 0$ all rS $rS\subset NN$ in (VIa) and contrasted both with Example 2 interpreted as implying $\varepsilon_{rS}^{+}>0$ some rS in (VIa) and with Example 3 interpreted as implying $\varepsilon_{rS}^{-}>0$ some rS in (VIa). In that context conditions $\varepsilon_{rS}^{+}\to 0$ and $\varepsilon_{rS}^{-}\to 0$ in (VIa) are consistent with the interpretations that in (V),(VI) and/or (VIa) probabilities of optimal coalition sizes respectively smaller than or larger than N go to zero *whether or not the resulting imputation is in the core*. [It also follows that, for ε_{rS}^{-} sufficiently large, a nonempty core solution is always attainable. This is an idea which can be related directly to Kannai's definition of the *strong core* (Kannai 1992). For more on this, as well as on potential relationships between (VI) and an extremal characterisation of the nucleolus, see Ryan 1998a.]

Even if the core is nonempty there may be more than one solution in the core and so it may be necessary to use supplementary criteria to select between them. In any case, if the core is empty, then clearly, unless in effect payoffs can be supplemented (as implicitly in the case of the strong core) nonemptiness of the core is an insufficient criterion to determine a solution to the N person characteristic function game. In that case, as well as in the nonempty core case, in general other solution criteria will be required, with associated implications for values of parameters in (VI) and/o for variables and restrictions in addition to those of (VI), if a unique solution is to be attained. One such criterion which has particular implications for values of parameters and for additional variables and constraints in addition to those of (VI),(VI)' is the Shapley Value.

Accordingly, in the next Section attention will be

focused on a coalition building motivation for the Shapley value of an N person characteristic function game and associated modifications of (VI) and (VI)' which, among other things, determine the framing conditions under which the Shapley Value is (or is not) in the core of that game.

With the context of a coalition building interpretation of the Shapley Value it is particularly significant that, while in (6.3) λ_{rS}^{+} explicitly suggests probabilities of complementary outcomes *within* N, the notation λ_{rS}^{-} correspondingly suggests probabilities of complementary outcomes *beyond* N. The latter interpretations in turn suggest explicit consideration of processes according to which the potential size of coalitions formed by a given number of players (and thence the probability of formation of relatively larger coalitions for that given number of players) is *growing*. This idea motivates the explicitly coalition building approach to the determination of the Shapley Value in the next Section and a correspondingly constructive proof of Shapley's result.

5. Shapley values and coalition building

In the previous Section I noted that (VI)' implicitly omits restrictions corresponding to the final constraints of the explicitly framed system (III)', which in turn relate to further framing conditions on (III). Clearly further framing conditions on a core related solution will lead to further restrictions on that core related solution and thence to special cases in which those further conditions may (or may not) be consistent with nonemptiness of the core. In this Section I develop this idea with specific reference to Shapley's coalition building interpret-ation of the Shapley Value.

Shapley 1953 concluded his paper with a section showing that the Shapley value emerges from a bargaining model as follows (op cit p78):

> The players constituting a finite carrier N agree to play the game v in a grand coalition formed in the following way: 1. Starting with a single member the coalition adds one player at a time until everyone has been admitted. 2. The order in which the players are to join is determined by chance, with all arrangements equally probable. 3. Each player on his admission, demands and is promised an amount

which his adherence contributes to the value of the coalition (as determined by the function v). The grand coalition then plays the game "efficiently" so as to obtain the amount v(N) - exactly enough to meet all promises.

In this Section I show how 1, 2 and 3 - and thence the Shapley value - follow from particular kinds of probabilistic restrictions on a system analogous to (II)'.

First recall that λ_{rS} in (6.1) and (VI)' may be interpreted as probabilities of formation of coalitions rS. Now, with the perspective of Shapley-like coalition building processes, consider coalition building sequences with members i entering coalitions one at a time. In that case, if it forms, a coalition rS may be composed by adding a player i to a coalition r-iS-1 in any of r ways. Thus the probability λ_{rS} of formation of a coalition rS r-iS-i is such that:

$$\lambda_{rS}+\lambda_{rS}{}^{+}=\Sigma\lambda_{r-iS-i} \qquad i\varepsilon r, r\geq 2 \qquad (5.1)$$

Here $\lambda_{rS}{}^{+}$ is the probability that combination r does *not* form - that is the probability with which these particular coalition building sequences stop with coalitions (combinations) of cardinality r-1.

Next define: i) λ_{irS} as the probability with which player i is chosen as the incoming member into a combination r and enters; ii) $\lambda_{irS}{}^{+}$ as the probability with which player i enters as an ongoing member under these conditions. So, if - as Shapley assumes - the identity of the incoming member i to a coalition of cardinality S is selected equiprobably, then:

$$\lambda_{irS}=1/S\lambda_{rS} \qquad i\varepsilon rS, S\geq 1 \qquad (5.2)$$
$$\text{and} \qquad \lambda_{irS}{}^{+}=(S-1)/S\lambda_{rS} \ i\varepsilon rS, S\geq 1 \quad (5.3)$$

Conditions (5.2), (5.3) are consistent with the fact that for singleton "coalitions" S=1, $\lambda_{ir1}=\lambda_{r1}$, $\lambda_{ir1}{}^{+}=0$ and players are necessarily incoming members. Conditions (5.2) and (5.3) are also consistent with the fact that, if either potentially incoming or potentially ongoing members *do not* choose a coalition r of cardinality S>1, then $\lambda_{irS}=0$, $\lambda_{irS}{}^{+}=0$ and so $\lambda_{rS}=0$. i.e. the probability of formation of that combination r will then also be zero. (Alternatively if preemptively $\lambda_{rS}{}^{+}=\Sigma\lambda_{r-1S-1}>0$ then $\lambda_{rS}=0$ and by implication $\lambda_{irS}=0$, $\lambda_{irS}{}^{+}=0$.)

Because players must be either singleton players, or incoming or continuing members of coalitions, for each player i:

$$\Sigma\lambda_{irS} +\Sigma\lambda_{irS}{}^{+} =1 \qquad i\varepsilon rS\subseteq H \qquad (5.4)$$

Now consider (VII)', which in effect augments (VI)' with the four Shapley related constraints (5.1).. .(5.4) and correspondingly additional terms in the objective:

$$\text{Max } \Sigma\lambda_{irS}V_{rS} +\Sigma e_{rS}{}^{+}\lambda_{rS}{}^{+}$$
$$\qquad\quad rS\subseteq H \qquad rS\subseteq H$$

x_i	st	$\lambda_{irS} +\Sigma\lambda_{irS}{}^{+} =1$	$i\varepsilon rS\subseteq H$
$x_{irs}{}^{+-}$, $x_{irs}{}^{++}$		$-c_{irS}{}^{+-}\leq\lambda_{irS}{}^{+}\leq c_{irS}{}^{++}$	
$x_{irs}{}^{-}$, $x_{irs}{}^{+}$		$-c_{irS}{}^{-}\leq\lambda_{irS}\leq c_{irS}{}^{+}$	(VII)'
φ_{irS}		$\lambda_{irS} =1/S\lambda_{rS}$	$S\geq 1$
$\varphi_{irS}{}^{+}$		$\lambda_{irS}{}^{+} = (S-1)/S\lambda_{rS}$	$S\geq 1$
ψ_{rS}		$\lambda_{rS} +\lambda_{rS}{}^{+} = \Sigma\lambda_{r-iS-i}$	$S\geq 2$

Associating the indicated dual variables with the constraints of (VII)', for H>N the corresponding primal problem becomes an explicitly framed and Shapley process related coalition building extension of (VI). [If H=N (VII) corresponds to an unframed Shapley value related extension of (VI)]:

$$Min\Sigma x_i+\Sigma c_{irS}{}^{+}x_{irS}{}^{+}+\Sigma c_{irS}{}^{-}x_{irS}{}^{-}+\Sigma c_{irS}{}^{++}x_{irS}{}^{++}+\Sigma c_{irS}{}^{+-}x_{irS}{}^{+-}$$
$$\quad i\varepsilon H \quad\ rS\subseteq H \qquad rS\subseteq H \qquad rS\subseteq H \qquad rS\subseteq H$$

λ_{irS}	st	$x_i + \varphi_{irS} + x_{irS}{}^{+}-x_{irS}{}^{-} \geq V_{rS} \ S\geq 1$
$\lambda_{irS}{}^{+}$		$x_i + \varphi_{irS}{}^{+} + x_{irS}{}^{++}-x_{irS}{}^{+-} \geq 0 \ S\geq 1$
λ_{rS}		$\Sigma 1/S\varphi_{irS} +\Sigma(S-1)/S\varphi_{irS}{}^{+}\leq\psi_{rS} -\Sigma\psi_{r+1S+1} \quad S\geq 2$
λ_{r1}		$\Sigma\psi_{r2}\leq -\varphi_{ir1} \quad (VII)$
		$i\varepsilon r2$
$\lambda_{rS}{}^{+}$		$e_{rs}{}^{+}\leq\psi_{rS} \qquad\qquad S\geq 2$
		$x_i ,x_{rS}{}^{+},x_{rS}{}^{-}, x_{irS}{}^{++}, x_{irS}{}^{+-}\geq 0$

A solution consistent with $\lambda_{ir1}>0$ all irl is feasible in (VII)'. And, by complementary slackness,

$$\lambda_{ir1}>0 => x_i + \varphi_{ir1} + x_{ir1}{}^{+}-x_{ir1}{}^{-} =V_{r1} \qquad S\geq 1 \qquad (5.5)$$

Also, from (5.2), $\lambda_{ir1}>0 =>\lambda_{r1} >0$ so that, by complementary slackness:

$$\Sigma\psi_{r2}= -\varphi_{ir1} \qquad i\varepsilon r2 \qquad (5.6)$$

Next, if optimally $\lambda_{rS}{}^{+}=0$ $rS\subseteq N=H-1$ then $\lambda_{rS}>0$ for all permutations $rS\subseteq N=H-1$. By complementary slackness, sufficient conditions for continuation /nondefection of coalition members rS are $e_{rs}{}^{+}<\psi_{rS}$

$$\Rightarrow \lambda_{rS}^+ = 0 \quad \Rightarrow \lambda_{rS}>0 \Rightarrow \Sigma 1/S\varphi_{irS} + \Sigma(S-1)/S\varphi_{irS}^+$$
$$= \psi_{rS} - \Sigma\psi_{r+1S+1} \quad S=2,..H-1 \tag{5.7}$$

Sufficient conditions for *non*continuation/ defection of potential coalition members rN+1, N≤ H-1 are:

$$\lambda_{rN+1}^+ > 0 \Rightarrow \quad e_{rN+1}^+ = \psi_{rN+1} \tag{5.8}$$

Now $\lambda_{rS} > 0$ implies $\lambda_{irS}>0$, $\lambda_{irS}^+ >0$ S≥1 from (5.2),(5.3) and, by complementary slackness:

$$\lambda_{irS}>0 \Rightarrow \quad -\varphi_{irS} = x_i - V_{rS} + x_{irS}^+ - x_{irS}^- \tag{5.9}$$
$$\lambda_{irS}^+ >0 \Rightarrow \quad -\varphi_{irS}^+ = x_i - V_{rS} + x_{irS}^{++} - x_{irS}^{+-} \tag{5.10}$$

In (5.9) (resp (5.10)) $-\varphi_{irS}$, $-\varphi_{irS}^+$ are opportunity costs to players respectively of *not entering* a coalition rS and of *not continuing* in a coalition r-iS-i to form a coalition rS. Equivalently, the right hand sides of (5.9), (5.10) are *contributive gains* (or losses) attributable to a player i entering (resp continuing in) a coalition rS, where such gains or losses equate to the difference between the immediate payoff V_{rS} to that coalition and the present imputation x_i to that player, *less* net process contingent redistributions x_{irS}^+ (resp x_{irS}^{++}) to others, and *plus* net process contingent redistributions x_{irS}^+ (resp x_{irS}^{+-}) from others.

If optimally (IV)' is also such that $\lambda_{i1}=1$, then, by complementary slackness:

$$\lambda_{i1}>0 \Rightarrow \quad -\varphi_{ir1}= x_i - V_{r1} + x_{ir1}^+ - x_{ir1}^- \quad S=1 \tag{5.11}$$

An interpretation of (5.11) is that $-\varphi_{ir1}$ is the opportunity cost to player i of not joining a game associated with an imputation x_i. Equivalently φ_{ir1} is the *relative gain* to player i from entering a game leading to an imputation x_i. Next: follow Shapley in defining the *immediate gain* to an incoming member i to form a coalition rS, S≥2 as:

$$\varphi_{irS} =_{def} V_{rS} - V_{r-iS-i} \quad S≥2 \tag{5.12}$$

Immediate gains $\varphi_{irS}=V_{rS}-V_{r-iS-i}$ in (5.12) for incoming players i imply $\varphi_{irS}^+=0$ for continuing players and from (5.7) $\varphi_{irS}^+=0$ implies:
$$\Sigma 1/S\varphi_{irS} + \Sigma\psi_{r+1S+1} = \psi_{rS} \quad S=2,..N-1 \tag{5.13}$$

In (5.13) ψ_{rS} is a measure of *anticipated gain* to

the addition of a member to an existing coalitions rS to form coalition r+1S+1. This anticipated gain is made up of two parts: the expected immediate gain $\Sigma 1/S\varphi_{irS}$ and the future gain $\Sigma\psi_{r+1S+1}$ to the formation in that way of coalition S.

Now, if the boundary conditions are such that all coalitions S⊆N form, then $\lambda_{rS}>0$ all rS⊆N so (5.7) holds for all ψ_{rS} in every coalition sequence and the summation in the left hand term in (5.13) is over the number of ways of choosing a coalition of cardinality S from N players so that (5.7) and thence (5.13) gives:

$$\frac{(S-1)!(N-S)!(V_{rS}-V_{r-iS-i})}{N!} + \Sigma\psi_{r+1S+1} = \psi_{rS}$$
$$S=2,..N-1 \tag{5.14}$$

Recalling that at an optimum $x_i + \varphi_{ir1} + x_{ir1}^+ - x_{ir1}^- = V_{r1}$ for S≥1, from (5.5), and $\Sigma\psi_{r2}= -\varphi_{ir1}$, i$\varepsilon$r2, from (5.6), and assuming that $x_{ir1}^+ = x_{ir1}^- = 0$ and using (5.14) recursively gives:

$$\frac{\Sigma(S-1)!(N-S)!(V_{rS}-V_{r-iS-i})}{N!} + \Sigma\psi_{N+1N+1} = \Sigma\psi_{r2} = x_i - V_{i1}$$
$$S=2,..N-1 \tag{5.15}$$

If $\psi_{N+1N+1}=V_{i1}=0$ then (5.15) yields the standard form of the Shapley Value.

Together the preceding developments establish the following result:

THEOREM 4

The following conditions are sufficient to generate the N player Shapley value from the dual pair of linear programs (VII), (VII)':
i) all players iεN enter the game, so that $\lambda_{ir1}>0$ all iεNN;
ii) at an optimum a grand coalition of cardinality N forms so that $\lambda_{NN}=1$;
iii) coalitions rS⊆NN form and non defection conditions obtain such that $\lambda_{rS}>0$, $\lambda_{rS}^+=0$ all rS⊆NN;
iv) positive contingent probabilities of entering and continuing in coalitions so that $\lambda_{irS}>0$, $\lambda_{irS}^+>0$ all iεrS⊂NN;
v) for each player i the opportunity cost of entering and net transfers x_{ir1}^+ and x_{ir1}^- are also zero so that $x_{ir1}^+ = x_{ir1}^- = V_{i1}=0$, ;
vi) for each potential coalition of cardinality N+1 pot-entially inclusive of the grand coalition N, $\psi_{N+1N+1}=0$.

PROOF

Conditions i)-iv) are necessary and sufficient for condition (5.14) to follow from (VII) by complementary slackness, and conditions v) and vi) are sufficient for (5.14) to yield (5.15) and the standard Shapley value result.

COROLLARY

If any of conditions i)-vi) of Theorem 4 does *not* hold then optimal solutions to (VII), (VII)' will *not* yield the standard Shapley values.

THREE REMARKS

- Through terms ψ_{N+1N+1} there is a role in in (5.15) for (N+1)st players. With the assumption that $\psi_{N+1N+1}=0$, which reduces (5.15) to the standard Shapley values, this role might be interpreted as that of a "dummy player". Notice here that the specialization of (5.15) to Shapley values also requires $V_{i1}=0$. In that way the role of the "dummy player" becomes isomorphic with that of those determining the reservation payoffs - opportunity costs of playing the game - for players i. (This idea can be developed in relation to repeated games with the condition for stopping one game being preconditions for starting another. For more on this, as well as on associated interpretations of $\lambda_{irS}{}^+,\lambda_{irS}{}^-$ as probabilities of *leadership* and *followership,* see Ryan 1998a.)

- Secondly: If optimally $\lambda_{rS}{}^+>0$ for one or more coalitions $rS\subseteq N$, $\lambda_{rS}{}^+>0$ then, by complementary slackness, $\psi_{rS}=e_{rS}{}^+$ and the analogue of (5.15) will take on an interpretation as a class of *modified* Shapley values incorporating the fact that one or more coalition sequences will then optimally terminate prior to formation of a relatively grand coalition NN.

- The context of (VII),(VII)' and Theorem 4 emphasizes that the quantities ψ_{rS}, and through them, Shapley values, are explicitly frame dependent. In particular Shapley values are contingent on the values of parameters in (VII),(VII)' being those appropriate for optimal starting, stopping and continuation criteria for Shapley-like coalition building behaviours. For those values optimal solutions to (VII),(VII)' determined the properties of marginal contributions and Shapley efficient imputations simultaneously with appropriate probabilities of formation of Shapley-like coalition building sequences. By contrast Shapley's own work omits any consideration of framing, as does the alternative approach by means of "potentials", which here correspond to the quantities ψ_{rS} in (VII)', by Hart and Mas Collel 1988. A fortiori neither of those approaches considers the appropriateness, or not, of

restrictions on frame related transition probabilities and/or of non zero magnitudes for frame related opportunity costs.

6. Framing, the Shapley value and optimal starting, continuation and stopping criteria

In Section 5 via (5.15) Shapley values have been derived with explicit reference to a more inclusive frame. It has already been noted that, if $\psi_{N+1N+1}=V_{i1}=0$ all $i\varepsilon NN$, these values in (5.15) correspond to the standard form of the Shapley Value. But the explicit framing conditions associated with the derivation of (5.15) relate more generally to coalition building process and associated starting, continuation and stopping criteria. Specifically: implications of (VII),(VII)' are that: i) possibly under duress (via $c_{irS}{}^+$, $c_{irS}{}^-$) a player will not **start** a coalition building sequence if the net opportunity cost $-\varphi_{ir1}$ of doing so is less than the contributive gain from doing so. And; ii) the opportunity cost of starting a coalition building sequence is at least equal to that if that player remained a singleton. These interpretations follow from a rearrangement of the first and fourth constraints of (VI), viz:

$$\lambda_{ir1}\geq 0 - \qquad \varphi_{ir1} \leq x_i - V_{r1} + x_{ir1}{}^+ - x_{ir1}{}^- \qquad (6.1)$$
$$\lambda_{r2}\geq 0 \qquad -\varphi_{ir1} \geq \Sigma\psi_{r2} \qquad (6.2)$$
$$\lambda_{i2}{}^+\geq 0 \qquad e_{i2}{}^+ \leq \psi_{i2} \qquad (6.3)$$

Next; iii) **continue** only as long as optimally $\lambda_{rS}{}^+=0$ S≥2 and; iv) a coalition building sequence will **stop** with a relatively grand coalition NN when optimally $e_{rN+1}{}^+=\psi_{rN+1}$ so that $\lambda_{rN+1}{}^+>0$ and $\lambda_{irN+1}=0$. That is: as soon as the predicted gain by augmenting a coalition rN is less than the alternative of not augmenting it. [Note that that predicted gain ψ_{rN+1} here refers to a potential *beyond* the immediate frame.]

Parenthetically, with this context of optimal starting, continuation and stopping criteria the quantities ψ_{rS} are consistent with Shapley's interpretation of Shapley values as *promises* in the quote at the beginning of Section 5 viz: "..the amount V(N) - exactly enough to meet all promises" as well as with interpretations as measures of prospective gain. [Throughout these Shapley related developments φ_{irS} has been independent of the identity of the incoming player. In this way the derivation of (5.15) also subsumes Shapley's assumption that the game in

question is *abstract*: it is assumed that payoffs to coalitions depend on their final composition rather than on their mode of composition.]

7. Framing , the Shapley value and emptiness or nonemptiness of the core

Clearly the Shapley value is not always in the core since a Shapley Value exists for any characteristic function game and not all characteristic function games have non empty cores. Nevertheless there are cases in which Shapley values generated as optimal solutions to (VII), (VII)' will be simultaneously consistent with nonemptiness of the core to the corresponding game.

Consider a coalition rS. For the entering player $\lambda_{irs}>0$ and for the other players $\lambda_{irs}^{+}>0$, so that, by complementary slackness, conditions (5.9) and (5.10) apply. But with $\varphi_{irS}=_{def}(V_{rS}-V_{r-iS-i})$, $\varphi_{irS}^{+}=_{def}0$ those conditions give:

$$\Sigma x_i = V_{rS}+\varphi_{irS} -x_{irS}^{+} + x_{irS}^{-} + \Sigma x_{irS}^{++} - \Sigma x_{irS}^{-+}$$
$$i\varepsilon rS \qquad (7.1)$$

A necessary condition for nonemptiness of the core to that game is then that:

$$\varphi_{irS} -x_{irS}^{+} + x_{irS}^{-} + \Sigma x_{irS}^{++} - \Sigma x_{irS}^{-+}\geq0 \qquad i\varepsilon rS$$
$$(7.2)$$

Or:

$$-\varphi_{irS} \leq (-x_{irS}^{+} + x_{irS}^{-} + \Sigma x_{irS}^{++} - \Sigma x_{irS}^{-+}) \qquad i\varepsilon rS$$
$$(7.3)$$

Condition (7.3) is a core related entry condition: A player i will optimally enter a game if the opportunity cost $-\varphi_{irS}$ of doing so is less than or equal to the net sum of redistributions consequent on doing so.

Clearly, given superadditivity and contingent rules that any ex ante gain to an entering member of a coalition would be redistributed equally among any coalition thus formed, Shapley imputations are consistent with (7.3) and so consistent with *nonemptiness* of the core. Conversely, in the absence of such contingent redistribution criteria, the Shapley value may be consistent with *emptiness* of the core.

More generally, since the Shapley solution can be determined via (VII), and (VII) is not necessarily inconsistent with emptiness of the core of the associated game, the Shapley solution is not necessarily inconsistent with emptiness of the core of that game. But (VII) was specified as a relatively more tightly (Shapley) constrained specialization of the more generally core related specification (III). And, by a general principle of optimality a relatively more constrained maximization problem will lead to an equal or a relatively reduced optimum:

THEOREM 5 *(Shapley values and the core)*
   Assuming that a feasible solution exists to (VII)' and that $e_{rS}^{+}\geq0$ then:

$$\underset{rS\subseteq H}{Max}\ \Sigma\lambda_{irS}V_{rS} +\Sigma e_{rS}^{+}\lambda_{rS}^{+} \qquad \geq \qquad \underset{rS\subseteq H}{Max}\ \Sigma\lambda_{rS}V_{rS}$$

st $\Sigma\lambda_{irS} +\Sigma\lambda_{irS}^{+}=1$ $i\varepsilon rS\subseteq H$ st $\Sigma\lambda_{rS}\leq1$ $rS\subseteq H$
   $-c_{irS}^{+-}\leq\lambda_{irS}^{+}\leq c_{irS}^{++}$
   $-c_{irS}^{-}\leq\lambda_{irS}\leq c_{irS}^{+}$ (VII)' $-\Sigma c_{irS}^{-}\leq\lambda_{rS}\leq\Sigma c_{irS}^{+}$
(VIa)'
   $\lambda_{irS} =1/S\lambda_{rS}$ $S\geq1$ $i\varepsilon rS$ $i\varepsilon rS$
   $\lambda_{irS}^{+} = (S-1)/S\lambda_{rS}$ $S\geq1$
   $\lambda_{rS} +\lambda_{rS}^{+} = \Sigma\lambda_{r-iS-i}$ $S\geq2$

That is: Shapley Value related restrictions as in (VII)' cannot increase, and may reduce, the possibility that the Shapley Value is in the core of an N person characteristic function game.

PROOF

Any feasible solution to (VII)' is a feasible solution to (VIa)', but not conversely. So any optimal solution to (VII)' is a feasible but not necessarily an optimal solution to (VIa)'. In particular, with H=N and $\Sigma c_{irN}^{-}=M^{-}$, $\Sigma c_{irN}^{+}=M^{+}$,$\lambda_{NN}=1$,$\lambda_{rS}=0$ all $rS\subset N$, an optimal solution to (Va) is potentially equivalent to an optimal solution to (II)' and so consistent with non emptiness of the core of an N person game. It follows that an optimal solution consistent with nonemptiness of the core of an N person characteristic function game via (VIa)' may be *inconsistent* with nonemptiness of the core via the Shapley value constrained extension (VII)'.

It follows from Theorem 3 that the specialized constraints on (VIa)' to determine Shapley solutions as if via (VII)' will, if anything, *reduce* the possibility that the associated game has a non empty core.

8. Conclusion

In this chapter I first focused on interrelations between framing and emptiness or nonemptines of the core. I then turned to frame related restrictions and the elicitation of Shapley values and associated conditions according to which the Shapley is, or is not, in the core using an explicitly framed optimization approach.

Clearly the results in this chapter could be extended to other classes of solution by considering more specific orderings on the parameters of (VI), (VI)', as I have shown in Ryan 1998a for the case of the nucleolus. They could be extended, too, by considering other types of probabilistic restrictions on (VI),(VI)' instead of and/or in addition to the Shapley Value related constraints which here have specialized (VI), (VI)' to the Shapley value related systems (VII),(VII)'. Another direction for extensions, which I have already explored in a preliminary way in Ryan 1998a, is a more detailed consideration of the extensive form of the N person game which is implicit in the optimal starting, stopping and continuation rules associated with the coalition building motivation for the Shapley Value which has been reported here. In particular, a point which has not been emphasised but which has been implicit throughout the chapter, is that an explicitly framed approach makes it possible to investigate interrelationships between a solution criterion (e.g. core and/or Shapley Value), the number of players electing to play a game given that solution criterion, and a particular parameterization of relevant boundary conditions.

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