CHAPTER 8

CONSTRAINED GAMES, INTERVENING DUALITY AND EXPERIMENTER-EXPERIMENT INTERACTIONS

1.Introduction

In this chapter I consider experimental inter-actions, including those relating to choices of experimental frames, within an explicitly optimizing structure using a constrained gaming approach.

The structure of the paper is as follows: The next Section provides a brief introduction to constant sum games and constrained games (see Charnes 1953, Owen 1982, Ryan 1994) primarily to introduce relevant notation. In Section 3 I go on to model a die casting example as a constrained game with emphasis on optimizing criteria, not just for the choice of strategies within a given frame, but for the choice of that frame vis a vis relatively exterior alternatives.

In Section 4 I extend work on intervening duality in Ryan 1995 to model explicit distinctions between experimenter, experiment and subject for a class of die casting examples. In that section I develop a more general analysis which includes coin tossing as well as die casting examples as special cases. Using this approach it becomes evident, *inter alia*, how individuals might rationally differ in the sense of conditions of Allais' Paradox (see Machina 1993) relative to implications of different outcomes, even under unanimously agreed experimental conditions.

In Section 5 I turn to more subtle kinds of framing issues by focusing on packing related extensions of the models in Section 4. As one kind of application I show how these packing related intervening duality representations can provide new kinds of rationalisations of packing related phenomena which have puzzled Tversky and Kahneman 1983, Tversky and Kahneman 1986, Johnson et al 1993, Starmer and Sugden 1993, and others, in a variety of different experimental conditions.

Finally in Sections 6 and 7 I use strategic equivalence arguments to extend the earlier intervening duality based analyses to include *tracer games* and more generally nonconstant sum experimental applications and examples.

2.Dual representations of constant sum and constrained games

In a constant sum game contingent payoffs to two players with strategies $j \varepsilon J$ and $k \varepsilon K$ can be represented as π_{kj} and $\Pi-\pi_{kj}$. Because in these cases gains to one are equivalent to losses to the other, if both players adopt the apparently pessimistic objective of maximizing the minimum attainable expected return to self, optimal solutions $\rho^*=\Sigma\pi_{kj}p_j^*$, $-\mu^*=\Sigma(\Pi-\pi_{kj})q_k^*$ can be found via:

$$\text{Maximize } \rho$$
$$\text{st} \quad \Sigma\pi_{kj}p_j \geq \rho \qquad (I)$$
$$\Sigma_j p_j = 1$$
$$p_j \geq 0$$

$$\text{Maximize } -\mu$$
$$\text{st} \quad \Sigma_k(\Pi-\pi_{kj})q_{kj} \geq -\mu \qquad (I)'$$
$$\Sigma_k q_k = 1$$
$$q_k \geq 0$$

More generally strategically equivalent solutions p_j^*, $j\varepsilon J$, q_k^*, $k\varepsilon K$ to these problems can be found as optimal solutions to a dual pair of linear programmes, where ranges of the strategies $j\varepsilon J$ and $k\varepsilon K$ are each preemptively set equal to "1" as follows:

$$\text{Maximize } \rho- Mp^+-Mp^-$$
$$\text{st} \quad \Sigma\pi_{kj}p_j \geq \rho \qquad (I)$$
$$\Sigma_j p_j + p^+-p^-=1$$
$$p_j, p^+, p^- \geq 0$$

Maximize $-\mu + Mq^+ + Mq^-$

st $\sum_k (\Pi - \pi_{kj})q_{kj} \geq -\mu$ (I)'

$\sum_k q_k + q^+ - q^- = 1$

$q_k, q^+, q^- \geq 0$

Notice immediately that the weights M relate explicitly to framing issues and processes. For example, in a die casting context, these preemptive weights have natural interpretations as one means of framing a procedure according to which an individual agrees to a relatively restricted specification kϵK (heads-tails) vis a vis a relatively less restricted set of thereby relatively exterior alternatives.

However, in many constant sum games, including market sharing games, as well as coin tossing and die casting games, players are likely to have prior information or beliefs concerning probabilities of particular outcomes or collections of outcomes. If this information can be expressed in a linear form, one way of including it in an extremal specification is by adding further conditions, as in (III), and associating dual variables R_r with those additional constraints to obtain a *constrained game* as follows:

Maximize $\rho - Mp^+ - Mp^-$

st $\sum \pi_{kj}p_j \geq \rho$ (III)

$\sum_j p_j + p^+ - p^- = 1$

$\sum d_{rj}p_j \leq d_r$

$j\epsilon J_r$

$p_j, p^+, p^- \geq 0$

Minimize $\mu + \sum R_r d_r + Mq^+ + Mq^-$

st $\sum_k \pi_{kj}q_k - \sum R_r d_{rj} \leq \mu$ μ (III)'

$\sum_k q_k + q^+ - q^- = 1$

$R_r, q_k, q^+, q^- \geq 0$

In (III)' the quantity R_r is a measure of gain to information implicit in the rth additional constraint in (III). (Incidentally, as I noted in Ryan 1994, the maximin-minimax formulation, which is often seen as unduly pessimistic and for that reason unreasonable, has the very attractive *dominance* property with respect to additional information of this kind. If an initially "worst", i.e. maximin, set of potential outcomes is improved via relatively relaxations of constraints all outcomes are improved via such relaxations of those constraints.)

The specification (III),(III)' covers a wide variety of possibilities including correspondences to Fishburn-like relations between preference orderings and dually related weak or strict rankings on probabilities (Fishburn 1964) and/or to elements of labour and production capacity related constraints in a production scheduling context. (For more on each of these types of cases see Ryan 1994.) Here attention will be confined to apparently simpler examples in which additional constraints and associated dual variables refer to initial specifications of various types of experimental conditions.

3. ALLAIS' PARADOX, FRAMING AND DIE CASTING GAMES

Consider two alternative sets of payoffs to tosses of a die:

	1	2	3	4	5	6
Alternative 1	500	600	700	800	900	1000
Alternative 2	600	700	800	900	1000	500

TABLE 1

Assuming that the die is fair, the expected payoff is 750 for each alternative so that an individual whose behaviour is consistent with expected subjective utility axioms, which include a compound lottery axiom (e.g. von Neumann Morgenstern expected utility axioms), would be indifferent between them. However experimenters, including Howard 1992 using this example, have found individuals who are not. For such individuals expected return related preference criteria and empirical choice criteria appear inconsistent, not just with each other, but with the prior indifference predictions of the analyst concerned. In that sense their behaviour may appear paradoxical in ways which can be seen as a variant of those generating Allais' paradox (see Machina 1993, p24).

Before considering intervening duality extensions, which can model and resolve such apparent Allais-like inconsistencies between subject and experimenter, first consider a more restricted constrained gaming specification.

Assume that an experimenter plays a die casting game against nature with payoffs as in Table 1 by imputing: i) equal prior probabilities of one sixth

to outcomes of tosses of the die; ii) a minimax expected payoff criterion relative to an experimenter and, correspondingly; iii) a maximin objective relative to nature.

Player 1 (Experimenter)

Min $\rho + M*\Sigma_k(p_k^+ + p_k^-) + M(p^+ + p^-)$

$600p_1 + 700p_2 + 800p_3 + 900p_4 + 1000p_5 + 500p_6 \leq \rho$
$500p_1 + 600p_2 + 700p_3 + 800p_4 + 900p_5 + 1000p_6 \leq \rho$

$p_k + p_k^+ - p_k^- = 1/6$

$\sum_{k=1}^{6} p_k + p^+ - p^- = 1$ \qquad (IV)

$-M \leq \rho \leq M$

$\rho, p_k, p^+, p^-, p_k^+, p_k^-, \geq 0$

The association of weights $M*$ with potential deviations from the six die related outcomes in (IV) and the two alternatives in (IV)' is consistent with lexicographic preferences by the analyst for a prior decision to frame a game with only two possible alternatives j=1,2 relative to a wider system - and then, by associating preemptive weights M with deviations from ex ante evaluations $p_k^*=1/6$, to play it as if believing outcomes k=1,...6 equiprobable relative to a relatively abstract self. With this preemptively restrictively framed specification the uniquely optimal solution for the experimenter is, via (IV):

$\rho = 750, p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1/6.$

But optimality is consistent with a variety of outcomes for the experiment via (IV)':

i) $\mu' = 750, q_1' = 1, q_2' = 0, R_1' = -150, R_2' = -50,$
$R_3' = 50, R_4' = 150, R_5' = 250, R_6' = -250$
ii) $\mu' = 750, q_1' = 0, q_2' = 1, R_1' = -250, R_2' = -150, R_3' = -50, R_4' = 50, R_5' = 150, R_6' = 250$
iii) $\mu' = 750, q_1' = 1/2, q_2' = 1/2, R_1' = -200, \quad R_2' = -100,$
$R_3' = 0, R_4' = 100, R_5' = 200, R_6' = 0$
iv) Any other linear combination of i) and ii).

Of these cases iii) corresponds to as if indifference between all or nothing bets on i) and ii) and is consistent with the revealed indifference which would be predicted for an experiment -

Given this specification optimal solutions follow from the dual pair of programmes (IV),(IV)', where kϵK in (IV) is the set of six potential outcomes to a die toss over the two potential alternatives jϵJ in programme (IV)':

Nature (Experiment)

Max $\mu' - M(q'^+ + q'^-) + \Sigma_k 1/6 R_k$

$600q_1' + 500q_2' - R_1' \geq \mu$
$700q_1' + 600q_2' - R_2' \geq \mu$
$800q_1' + 700q_2' - R_3' \geq \mu$
$900q_1' + 800q_2' - R_4' \geq \mu$
$1000q_1' + 900q_2' - R_5' \geq \mu$
$500q_1' + 1000q_2' - R_6' \geq \mu$

$\sum_{j=1}^{2} q_j' + q'^+ - q'^- = 1$ \qquad (IV)'

$-M \leq \mu' \leq M$

$-M* \leq R_k' \leq M*$

$\mu', q_j', q'^+, q'^-, q_j'^+, q_j'^- \geq 0$

and an experimenter - by standard expected utility axioms. Such indifference between outcomes can be interpreted as corresponding, via the measures R_k', to effective absence of explicit prior probability information or, more subtly, to as if indifference to prior information on the part of the experimenter concerning potential probabilities q_1', q_2' of occurrence of the alternative lotteries. (This would be consistent too with Harsanyi or Bayes Laplace prior ignorance criteria.)

Now consider not just potentially explicit distinctions between experimenter and experiment, as in (IV),(IV)', but conditions according to which experimenter and subject may interact differently to each other via differing interactions relative to an intervening die and the alternatives in Table 1. To do this model a second individual as if potentially incorporating agreement to fairness of a die relative to the system via (V)', and, through the dually related system (V), predicting relative gains or losses to strict preferences between between two Allais-like fair die related alternatives relative to self as follows:

Player 2(Subject)

Max $\mu + \Sigma R_k * 1/6 - M(q^+ + q^-)$

$600q_1 + 500q_2 - R_1 \geq \mu$
$700q_1 + 600q_2 - R_2 \geq \mu$
$800q_1 + 700q_2 - R_3 \geq \mu$
$900q_1 + 800q_2 - R_4 \geq \mu$
$1000q_1 + 900q_2 - R_5 \geq \mu$
$500q_1 + 1000q_2 - R_6 \geq \mu$

$$\sum_{j=1}^{2} q_j + q^+ - q^- = 1 \qquad (V)$$

$-M \leq \rho \leq M$
$-M^* \leq R_k \leq M^*$
$\mu, q_j, q^+, q^-, q_j^+, q_j \geq 0$

Die (Nature/Experiment)

Min $\rho' + M^* \Sigma(p_k'^+ + p_k'^-) + M(p'^+ + p'^-)$

$600p_1' + 700p_2' + 800p_3' + 900p_4' + 1000p_5' + 500p_6' \leq \rho'$
$500 p_1' + 600p_2' + 700p_3' + 800p_4' + 900p_5' + 1000p_6' \leq \rho'$

$$p_k' + p_k'^+ - p_k'^- = 1/6$$

$$\sum_{k=1}^{6} p_k' + p'^+ - p_k'^- = 1 \qquad (V)'$$

$-M \leq \rho' \leq M$

$\rho', p_k', p'^+, p'^-, p_k'^+, p_k'^- \geq 0$

Again weights M associated with potential deviations from overall probabilities "1" are consistent with lexicographic preferences and a prior decision, in this case by the subject, to frame a game with only six die related outcomes - and then, by associating preemptively large weights M* with deviations from the values $p_k'^*=1/6$, to play it as if only with these probabilities relative to the system. In this case an equiprobable solution to the experimental specification (V)' is uniquely optimal giving:

$\rho = 750$, $p_1' = p_2' = p_3' = p_4' = p_5' = p_6' = 1/6$

But, analogously to the relationship of (IV)' to (IV), there is a variety of correspondingly optimal solutions for the subject and that experiment via the dual programme (V). These include:

i) $\mu=750$, $q_1=1$, $q_2=0$, $R_1=-150$, $R_2=-50$, $R_3=50$, $R_4=150$, $R_5=250$, $R_6=-250$
ii) $\mu =750$, $q_1=0$, $q_2=1$, $R_1=-250$, $R_2=-150$, $R_3=-50$, $R_4=50$, $R_5=150$, $R_6=250$
iii) $\mu =750$, $q_1=1/2$, $q_2=1/2$, $R_1=-200$, $R_2=-100$, $R_3=0$, $R_4=100$, $R_5=200$, $R_6=0$
iv) Any other linear combination of i) and ii).

As for the experimenter relative to (IV)', the third kind of optimum corresponds to as if indifference beween all or nothing bets on i) and ii) and appears consistent with an analyst's prediction of a subject's behaviour in Allais' story. That is, alternative iii) corresponds to revealed indifference by a subject and so potentially validates an analyst's prediction of standard expected utility maximizing behaviour by a subject.

But cases i) and ii) include other possibilities according to which, via $q_1=1, q_2=0$, or $q_1=0, q_2=1$, a subject may act as if strictly to prefer one alternative over the other, even though their expected values are each 750, in ways potentially contradicting standard expected utility axioms in general, and potentially consistent with conditions of Allais' paradox in particular. I consider these and other possibilities in more detail in the context of a more general intervening duality structure in the next Section.

4.Intervening duality and experiment: experimenter interactions

Analogously to the incorporation of constraints concerning the experimenter's and subject's prior assumptions or beliefs concerning the fairness or otherwise of the die in (IV) and (IV)', if an experimenter or a subject had prior beliefs (or a personal bias) in favour of one or other of the two alternative outcomes, properly that should be taken into account by appropriately modifying (IV)', and so its dual, (IV). Accordingly, for the experimenter append constraints $q_j' + q_j'^+ - q_j'^- = q_j'^*$ with associated dual variables S_j to (IV)' to generate the modified system (VI)' and its dual (VI). Similarly, for the subject append $q_j + q_j^+ - q_j^- = q_j^*$ with associated dual variables S_j' to (V) to generate the modified system (VII) and its dual (VII)'. In this way a more general intervening duality specification (VI),(VI)', (VII),(VII)' is generated in which (VI) represents an experimenter's problem, (VI)',(VII)' represent an intervening experiment and (VII) represents a subject's problem as follows:

Player 1 (Experimenter)

$$\text{Min } \rho + M*\Sigma(p_k^+ + p_k^-) + M(p^+ + p^-) + \Sigma S_j q_j'^*$$
$$\Sigma\pi_{kj}p_k - S_j \leq \rho$$
$$\Sigma p_k + p^+ - p^- = 1 \qquad (VI)$$
$$p_k + p_k^+ - p_k^- = p_k^*$$
$$-M \leq \rho \leq M$$
$$-M^* \leq S_j \leq M^*$$
$$\rho, p_k, p^+, p^-, p_k^+, p_k^-, \geq 0$$

Nature (Experiment)

$$\text{Max } \mu' - M*\Sigma(q_j'^+ + q_j'^-) - M(q'^+ + q'^-) + \Sigma p_k^* R_k'$$
$$\Sigma\pi_{kj}q_k' - R_k' \geq \mu$$
$$\Sigma q_j' + q'^+ - q'^- = 1 \qquad (VI)'$$
$$q_j' + q_j'^+ - q_j'^- = q_j'^*$$
$$-M \leq \mu' \leq M$$
$$-M^* \leq R_k' \leq M^*$$
$$\mu', q_j', q'^+, q', q_j'^+, q_j' \geq 0$$

Player 2 (Subject)

$$\text{Max } \mu + \Sigma R_k^* p_k'^* - M(q^+ + q^-) - M\Sigma(q_j^+ + q_j^-)$$
$$\Sigma\pi_{kj}q_k - R_k \geq \mu$$
$$q_j + q_j^+ - q_j^- = q_j^*$$
$$\Sigma q_j + q^+ - q^- = 1 \qquad (VII)$$
$$-M \leq \mu \leq M$$
$$-M^* \leq R_k \leq M^*$$
$$\mu, q_j, q^+, q^-, q_j^+, q_j^- \geq 0$$

Die (Nature/Experiment)

$$\text{Min } \rho' + M*\Sigma(p_k'^+ + p_k'^-) + M(p'^+ + p'^-) + \Sigma S_j' q_j^*$$
$$\Sigma\pi_{kj}p_k' - S_j' \leq \rho'$$
$$p_k' + p_k'^+ - p_k'^- = p_k'^*$$
$$\Sigma p_k' + p'^+ - p_k'^- = 1 \qquad (VII)'$$
$$-M \leq \rho' \leq M$$
$$-M^* \leq S_j' \leq M^*$$
$$\rho', p_k', p'^+, p'^-, p_k'^+, p_k'^- \geq 0$$

In this more general framework an experimenter can be seen as dual to his/her experiment via (VI),(VI)' and a subject as dual to that experiment via (VII),(VII)' and the intervening experiment as if dual to itself via (VI)',(VII)'.

While solutions associated with the intervening dual structure (IV),(IV)', (V),(V)', remain optimal in (VI),(VI)', (VII),(VII)', further solutions become attainable via relative penalties or rewards S_j in (VI) and S_j' in (VII)' to the explicit incorporation of elements of prior information $q_j'^*$ and q_j^* respectively in (VI)' and (VII). Formally, interpretations associated with the system (IV)',(VI) are included via conditions as if $S_j=0$ in (VI) and $q_j'^*=0$ in (VI)'. Similarly, interpretations associated with (V)',(VI) are included via conditions as if $q_j^*=0$ in (VII) and $S_j'=0$ in (VII)'.

Conditions with $S_j=0$ (resp $S_j'=0$) are as if the experimenter (resp subject) were indifferent, not just to further information concerning probabilities of alternative payoffs *per se*, but to implications of potentials as if via additional constraints on probabilities $q_j'^*, q_j^*$ in (VI)', (VII) for the validation (or otherwise) of an initially chosen model.

More subtly, if $p_k^*=p_k'^*$, $q_j'^*=q_j^*$ and $R_k'=R_k$, $S_j=S_j'$ all k,j then experimenter, experiment and subject will all be in agreement, both relative to their own dual and relative to each other. Under those circumstances (VI) becomes as if identical to (VII)' and (VII) as if identical to (VI)'. Elements of (VI) initially relative to an experimenter are as if perfectly predictive of elements of an experimental specification (VII)' relative to a subject, and the dually induced response (VII) is as if perfectly predictive of an experimentally induced response (VII)' relative to the experimenter.

But the specification (VI),(VI)', (VII),(VII)' includes more general classes of sequence. They can also be solved for cases under which either or both $S_j, S_j' \neq 0$ all j.

To illustrate and clarify this further, consider a sequence of interpretations with (VI),(VI)' (VI),(VI)' interpreted in relation to stages of an Allais-like experimental specification and execution as follows:

Experimenter

- An experimenter as if via (VI) and $S_j=0$ makes themselves dual to a system (VI)' with conditions as if $q_j'^*$ arbitrary. In that way the experimenter becomes as if dual to a pair of mutually exclusive die related experiments Alternative 1 vs Alternative 2 with $q_j'^*$ arbitrary. Then;
- The experimenter transmits to the subject an experimental specification to the effect that the subject must act as if consistently with

(VII)' in such a way that it is potentially dual to (VI) with q_j^* arbitrary.

Subject

- The subject given the experimenter's specification, may select any particular values q_j^* in (VII) as long as an optimal solution tp (VII) consistent with these values is in turn potentially consistent with an optimal solution to (VI)

Possible Experimental Outcomes

- The experimenter's Allais-like hypothesis of apparent indifference is confirmed by the subject's choice of values $q_1^*=1/2$, $q_2^*=1/2$ in (VII). In that case, for values of π_{kj} as in Table 1, the optimal solution to (VII) is consistent with a prediction $S_j'=0$ in (VII)', which is in turn consistent with an experimenter's actions as if to select $q_j'^*$ arbitrarily in (VI)', and respond via $S_j=0$ in (VI) accordingly. Or;
- The experimenter's Allais-like hypothesis of indifference is apparently refuted by the subject's choice of values *either* $q_1^*=1$, $q_2^*=0$ *or* with $q_1^*=1$, $q_2^*=0$ in (VII). In either case, for values of π_{kj} as in Table 1, the optimal solution to (VII) is consistent with a prediction $S_1'=0$, $\Sigma\pi_{k2}p_k'\geq S_2'\geq 0$, (resp $\Sigma\pi_{k1}p_k' \geq S_1'\geq 0$, $S_2'=0$) in (VII)'. Those probabilities and potential payoffs in turn are *inconsistent* with an experimenter's actions as if to select $q_j'^*=1/2$ in (VI)' and respond via $S_j=0$ in (VI) accordingly.

In the latter cases apparent refutations of an experimenter's prior hypotheses $q_j'^*=1/2$ with the associated possibilities that $S_j'\neq S_j$, may also correspond to cases in which, via results $S_2'>0$, (resp $S_1'>0$), the subject acts as if having strict preferences for one experimental alternative over the other. This is a great strength of this intervening duality approach to the representation and analysis of experimenter-experiment interractions. It explicitly recognizes that outcomes may lie within the frame specified for the experiment and yet constitute refutations of an experimenter's initial hypothesis - in this case of strict indifference between the two alternatives.

In this wider framework experimenter and subject are seen as making explicitly optimizing choices within constraints potentially set *inter alia* by each other - and not just by "nature". Here a subject is not just choosing alternatives within a given frame for the magnitudes and ranges of probabilities of outcomes but, via R_k, S_j', marginal potentials on those magnitudes and ranges. In this context refutations of Allais-like hypotheses of precisely the kind found in Howard 1992 may potentially arise via a subject's choice $q_1^*=1$, $q_2^*=0$ with $S_2'>0$ (or $q_1^*=1$, $q_2^*=0$ with $S_1'>0$). As just demonstrated these could be consistent with optimizing solutions to (VII), (VII)' for an experimental subject, but inconsistent with arbitrary $q_j'^*$ together with $S_j=0$ in optimising solutions to (VI), (VI)' for the experimenter.

While open to interpretation as if potentially generating contradictions of initial hypotheses in this way, evidently such an intervening duality specification of experimenter-experiment interactions does not *inevitably* generate such contradictions and falsification of an experimenter's hypothesis relative to a subject. Indeed it is open to the subject to select outcomes, for example via $q_1=1/2$, $q_2=1/2$ and $S_2'=S_1'=0$, as if wholly consistent with such an initial hypothesis.

In any case such an intervening duality specification is consistent with duality based *learning* by the subject from the experimenter and by the experimenter from the subject. Notice in that context that in these examples experimenter and subject not only act *sequentially* but may act as if respectively to perfectly predict and thereby to validate each other's actions and reactions.

Cases of as if perfect prediction with $q_j^*=q_j'^*$ and $S_j'=S_j=0$ can generate games with conditions corresponding to unanimity. More richly such an intervening dual specification can be seen as incorporating potentially *validating* principles and processes according to which each of the two individuals (or groups) may learn about the other's preferences and beliefs within given frames, whether or not they accord with their prior hypotheses.

Emphasizing that potential differences of preferences and beliefs may be important, as I have already noted in general outcomes may not be consistent with experimenter's initial choice. Indeed if an experiment were designed in such a

way as only to admit conclusions entirely consistent with the experimenter's prior hypotheses concerning its potential outcomes why conduct it? More constructively, an intervening duality framework as above might be intended simply to elicit a subject's reponse prior to further experiments with the same intervening dual structure but with revised parameters founded on learning from such a response.

These examples illustrate the more general point that in no case need experimenter and subject be in agreement in any sense beyond their agreement, here as if via weights M,M*, to preconditions framing a particular intervening dual-type game. Indeed, in less tightly framed experimental conditions, experimenter and subject may retain not just potentially different, but possibly oppositely oriented, prior probabilities and/or valuations on information at the margin.

Consider here a variant of these examples in which a first individual acts, as did the "experimenter" above, as if hypothesizing $p_k^*=1/6$ in (VI) and selecting q'^* arbitrarily in (VI)' and being as if indifferent between the alternatives $j=1,2$ by setting $S_1=S_2=0$ in (VI) . Given correspondingly prior probabilities $p_k'^*=1/6$ in (VII)' relative to the system the subject responds with a prediction $q_1=1, q_2=0$ with $S_1'= S_2'=0$ in (VII)'. While values $q_1=1, q_2=0$ are formally consistent with the experimenter's specification of arbitrary $q_j'^*$ in (VI)', they may in fact be at variance with the experimenter's true beliefs concerning q'^*.

But these are the preconditions for a potentially mutually advantageous die casting variant of the coin tossing game with which I introduced intervening duality idea in Ryan 1995. I will return to this example in Section 6 in connection with *tracer games* which might correspond, among other things, to ways of modelling learning related preliminaries to coin tossing or die casting games founded on oppositions of individual's beliefs concerning intervening "fair" coins or dies. Before that I reinforce the potential of the intervening duality approach to the modelling of experimenter-experimenter inter-actions by using it to model packing related

experiments and to explain packing related phenomena in the next Section.

5. Packing and unpacking extensions of intervening duality

Intervening duality structures can be used as new ways of investigating packing related experiments - e.g. packing probabilities of drawing particular colours of balls into probabilities of more inclusive subsets of colours of balls. Such experiments can be accommodated by expanding the intervening dual structure (VI), (VI)', (VII), (VII)' to include packing related possibilities as in (VIII), (VIII)', (IX), (IX)'.

An example here is a relatively packed high-low version of a die casting game. With $k=1,2...6$, $k_1=\{1,2,3\}$ and $k_2=\{4,5,6\}$, if $j=1,2$ such a relatively packed game is potentially isomorphic with - and in that sense strategically equivalent to - a coin tossing game.

If $M^{**}>>M^*$ a relatively packed version of (IX)', (IX), with $p_{kv}'^+=p_{kv}'^-=0$ all $k_v \subset K$ will be as if preemptively prescribed (lexicographically preferred) by the experimenter to the relatively unpacked version. Conversely, if $M^*>>M^{**}$ the relatively unpacked version, with $p_k'^+=p_k'^-=0$ will be as if lexicographically preferred. That is, an experimenter can act as if purposively to induce a *switch* between relatively packed and unpacked versions, eg of a diecasting experiment, by means of a switch in the relative weights attaching to the two alternative *framing* conditions $p_k'^+=p_k'^-=0$ or $p_{kv}'^+=p_{kv}'^-=0$ relative to the subject.

Going further, if the subject accepts the experimenter's prioritization of relatively packed vs relatively unpacked specifications they could be induced to act as if to *switch* preferences as if by accepting such reversals of weights M*, M**. But, as in the simpler unpacked die casting cases of the previous section, even if a subject accepts the ranges of outcomes and contingent probabilities as if preemptively specified by an experimenter in this way, they may nevertheless differ through their choices of S_j', R_k and V_v for the subject vs S_j, R_k' and V_v' for the experimenter.

Player 1 (Experimenter)

$$\text{Min } \rho + M^*\Sigma(p_k^+ + p_k^-) + M(p^+ + p^-) + \Sigma S_j q_j'^*$$
$$+M^{**}\Sigma(p_{kv}^+ + p_{kv}^-)$$
$$\Sigma\pi_{kj}p_k - S_j \le \rho$$
$$\Sigma p_k + p^+ - p^- = 1 \qquad k\varepsilon K \qquad \text{(VIII)}$$
$$p_k + p_k^+ - p_k^- = p_k^*$$
$$\Sigma p_k + p_{kv}^+ - p_{kv}^- = p_{kv}^* \qquad k\varepsilon k_v \subset K$$
$$-M \le \rho \le M$$
$$-M^* \le S_j \le M^*$$

$$\rho, p_k, p^+, p^-, p_k^+, p_k^-, \ge 0$$

Nature (Experiment)

$$\text{Max } \mu' - M^*\Sigma(q_j'^+ + q_j'^-) - M(q'^+ + q'^-) + \Sigma p_k^* R_k'$$
$$+\Sigma p_{kv}^* V_v'$$
$$\Sigma\pi_{kj}q_k' - R_k' - V_v' \ge \mu' \qquad k\varepsilon K_v$$
$$\Sigma q_j' + q'^+ - q'^- = 1 \qquad\qquad \text{(VIII)}'$$
$$q_j' + q_j'^+ - q_j'^- = q_j'^*$$

$$-M \le \mu' \le M$$
$$-M^* \le R_k' \le M^*$$
$$-M^{**} \le V_v' \le M^{**}$$
$$\mu', q_j', q'^+, q'^-, q_j'^+, q_j' \ge 0$$

Player 2(Subject)

$$\text{Max } \mu - M^*(q^+ + q^-) - M^*\Sigma(q_j^+ + q_j^-)$$
$$+\Sigma p_k^* R_k + \Sigma p_{kv}^* V_v$$
$$\Sigma\pi_{kj}q_k - R_k - V_v \ge \mu \qquad k\varepsilon K_v$$

$$q_j + q_j^+ - q_j^- = q_j^*$$
$$\Sigma q_j + q^+ - q^- = 1 \qquad\qquad \text{(IX)}$$
$$-M^* \le \mu \le M^*$$
$$-M \le R_k \le M$$
$$-M^{**} \le V_v \le M^{**}$$
$$\mu, q_j, q^+, q^-, q_j^+, q_j^- \ge 0$$

Die (Nature/Experiment)

$$\text{Min } \rho' + M\Sigma(p_k'^+ + p_k'^-) + M^*(p'^+ + p'^-) + \Sigma S_j' q_j^*$$
$$+M^{**}\Sigma(p_{kv}'^+ + p_{kv}'^-)$$
$$\Sigma\pi_{kj}p_k' - S_j' \le \rho'$$
$$p_k' + p_k'^+ - p_k'^- = p_k'^*$$
$$\Sigma p_{kv}' + p_{kv}'^+ - p_{kv}'^- = p_{kv}'^* \qquad k \subset k_v \subset K$$
$$\Sigma p_k' + p'^+ - p_k'^- = 1 \qquad\qquad \text{(IX)}'$$
$$-M^* \le \rho' \le M^*$$
$$-M^* \le S_j' \le M^*$$

$$\rho', p_k', p'^+, p'^-, p_k'^+, p_k'^- \ge 0$$

Specifically, if the experimenter selects $S_1 = S_2 = 0$ and $V_1' = V_2' = 0$ in the augmented intervening duality system (VIII),(VIII)',(IX),(IX)' and if the subject responds with $S_1' = S_2' = 0$ and $V_1 = V_2 = 0$, not only is the experimenter as if indifferent between relatively packed and relatively unpacked specifications; so is the subject. But, if $S_1 = S_2 = 0$ and $V_1' = V_2' = 0$ for the experimenter and $\Sigma p_{kv} V_v = 0$ with $V_1, V_2 \ne 0$ for the subject, not only is the subject *not* as if indifferent to elements of relatively packed vs unpacked specifications but, to that extent, would not be indifferent to a change in relatively packed vs relatively unpacked specifications by a relatively external experimenter. (Again this emphasises interpretations of quantities M*, M** as not just frame related but potentially as measures on a relatively external experimenter.)

Put another way: if packed and unpacked specifications are as if wholly consistent via conditions as if $V_1' = V_2' = 0$ (resp $V_1 = V_2 = 0$), an experimenter (resp subject) would be as if indifferent to information concerning the magnitudes of packing related probabilities $q_j'^*$ (resp q_j^*). If dual variables are evaluated as zero it is as if the corresponding constraints were omitted from the relatively dual problem. Although that is one possibility, it is not the only possibility. Indeed

the relation via measures V_v' and V_v between an initial specification for experimenter and subject and a relatively more tightly framed high-low specification can be used to investigate and explain others' observations, (see Tversky and Kahneman 1983, Tversky and Kahneman 1986, Johnson et al 1993, Starmer and Sugden 1993), that subjects' perceptions of probabilities relative to relatively packed outcomes will generally change relative to the relatively unpacked alternative and in particular will be weakly superadditive, if relatively favourable, and weakly subadditive if relatively unfavourable.

First, from the principle of optimality, since a relatively packed specification of (VI)' or of (VII) is in that sense a relatively more constrained system, it will give at least as large an optimum as the relatively packed alternative. In the present context, too, the fact that the quantities M, M*, M** are measures on the experiment setter by the subject is also significant. In general a change from a preemptively unpacked to a preemptively packed specification by the experimenter must not only induce a response in the subject, but one with at least as high an evaluation for the subject's problem.

To see how such changes as if induced *inter alia* via V_v', V_v may be consistent with findings by Tversky and Kahnemann and Starmer and Sugden, notice that the incremental contribution to the objective of (VIII)' due to the inclusion of the packing constraint in (VIII) is $\Sigma p_{kv} V_v'$. If $V_v'=0$ all v in (VIII)' (resp $V_v=0$ all v in (IX)) relatively unpacked and packed systems (VI)' and (VIII)' (resp (VII) and (IX)) are as if wholly equivalent. Less restrictively, such pairs of systems might appear equivalent via evaluations variously as if $V_1'=V_2'=0$ for the experimenter and $\Sigma p_{kv} V_v=0$ for the subject, as above. In that case the marginal evaluators $V_v \neq 0$ for the subject may nevertheless take on interpretations of a relatively preference for high outcomes over low outcomes in a relatively packed high-low specification of a die casting game. This is precisely the type of finding obtained by Tversky and Kahnemann 1983 and Starmer and Sugden 1993.

More technically, packing implies linear dependence for the constraints of (VIII) and of (IX)'. That in turn implies *degeneracy* of optimal solutions to these systems. Perturbations ε_v, ε_v' could be introduced in (VIII) (resp (IX)') to remove such degeneracy but then the objectives of the corresponding duals would be augmented by $\Sigma V_v' \varepsilon_v'$ (resp $\Sigma V_v \varepsilon_v$).

For example in the high-low die casting case perturbations ε_1, ε_2, $\varepsilon_2 \neq \varepsilon_2$ (resp ε_1', ε_2', $\varepsilon_2' \neq \varepsilon_2'$) could be associated with the packing constraints in (VIII) (resp (IX)') to remove such degeneracy. The corresponding duals would then be augmented respectively by $V_1' \varepsilon_1' + V_2' \varepsilon_2'$ and $V_1 \varepsilon_1 + V_2 \varepsilon_2$. Now if $V_1'=V_2'=0$ for the experimenter and $V_1+V_2=0$ $V_1, V_2 \neq 0$ for the subject (as above) then, while the experimenter is open to interpretation as if via $V_1'=V_2'=0$ indifferent to relatively packed vs unpacked specifications, the subject, via $V_1+V_2=0$, $V_1, V_2 \neq 0$ is open to interpretation as having a net preference for relatively packed over relatively unpacked alternatives, if $V_1 \varepsilon_1 + V_2 \varepsilon_2 > 0$, (or vice versa if $V_1 \varepsilon_1 + V_2 \varepsilon_2 < 0$).

For a rational individual for this case, and via appropriate perturbations of (VIII) and (IX)' in general, outcomes of the latter type would correspond to a strengthened preference for relatively more favourable and relatively packed events and a weakened preference for relatively less favourable and packed events. This is a general result and comprehends packing related experiments of kinds considered by Tversky and Kahneman and Starmer and Sugden as special cases. [These and other cases can be interpreted in terms of experiment related *bias*. For example, if a die is perceived by an experimental subject as such that $\varepsilon_1 > \varepsilon_2$ for relatively high over relatively low outcomes, even if, via conditions $V_1 + V_2 = 0$, that subject is otherwise neutral to information concerning outcomes of a high-low die toss, he/she may nevertheless express a strict preference for one outcome over the other stemming solely from that perceived bias.]

More generally a subject may reason that bias, either toward elements of ε_v or toward elements of V_v, is implicit in a relatively packed over a relatively unpacked specification - why otherwise would an experimenter expend time and resource in requiring both him/herself and the subject to consider it? In that context any experiment stemming from a null hypothesis of rationally unbiased outcomes, i.e. zero net informational gain outcomes, could be perceived as itself essentially *irrational,* since any experiment to test it would require nonzero inputs of effort by experimenter and subject to achieve.

6. Strategic equivalence and generalized nonconstant sum games

Consider problems (X),(X)',(XI),(XI)' according to which payoffs in (VIII),(VIII)' ,(IX),(IX)' modified to $(\theta_s \pi_{kj}^{\ s} + c_s)$.

If $\pi_{kj}^{\ s} = \pi_{kj}$, $c_s = c$, $\theta_s = \theta$ all s and $(p_k^*, S_j = 0)$, $(q_j'^*, R_k', V_v' = 0)$, $(q_j^*, R_k, V_v = 0)$, $(p_k'^*, S_j = 0)$, are consistent with optimality of (VIII),(VIII)', (IX),(IX)' those same values are potentially consistent with optimality for (X),(X)', (XI),(XI)'.

With an important qualification the only substantive change would then be that, if optimal values to the initial systems are ρ, μ', ρ', μ, optimal values for the transformed systems are $\underline{\rho} = \Sigma(\theta \pi_{kj} + c) p_k = \underline{\rho}' = \Sigma(\theta \pi_{kj} + c) p_k'$ and $\underline{\mu}' = \Sigma(\theta \pi_{kj} + c) q_j' = \underline{\mu} = \Sigma(\theta \pi_{kj} + c) q_j'$. In those circumstances (X), (X)',(XI),(XI)' correspond to an intervening duality analogue of the standard representation of strategic equivalence for unconstrained game cases. The important qualification is that this correspondence to the standard representation of strategic equivalence holds only if *M, M*, M** are sufficiently large*. If not all M, M*,M** sufficiently large, increasing/reducing payoffs π_{kj}

via parameters θ and c may induce players to expand (resp shrink) the chosen set of payoffs. That is: to switch to or from a relatively external work or leisure related alternatives. [This is pursued with reference to frame related resolutions of Allais' paradoxes in Chapter 10.]

More generally, if optimally $\Sigma p_k=1$, $\Sigma q_j'=1$ and $\Sigma q_j=1$, $\Sigma p_k'=1$, the intervening duality system (X),(X)',(XI),(XI)' is potentially strategically equivalent to (VIII),(VIII)', (IX),(IX)', even if S_j,

S_j', V_j,V_j' are not all equal to zero in (IX),(XI)', if the quantities S_j, S_j', R_k, R_k' and/or V_j,V_j' are amended accordingly. For example, if R_k is optimal in (VIII) and $V_v=0$, a strategically equivalent optimum for (XI) is attainable if $\underline{R_k}=_{def}R_k+(1-\theta)\Sigma\pi_{kj}q_j$ -c in that system, again with the important qualification that the relevant experimental frame does not change as a consequence of such transformations relative to relatively exterior alternatives.

Player 1 (Experimenter)

Min ρ + M*Σp_k^++ p_k^-) + M(p^++ p^-) + $\Sigma S_j q_j'$*

$\qquad$ +M**$\Sigma(p_{kv}^+$+ $p_{kv}^-)$

$\quad$ st $\Sigma(\theta_1\pi_{kj}^1 +c_1)p_k - S_j \leq \rho$

$\qquad \Sigma p_k + p^+ - p^- = 1 \qquad k\varepsilon K \qquad$ (X)

$\qquad p_k + p_k^+ - p_k^- = p_k$*

$\qquad \Sigma p_k + p_{kv}^+ - p_{kv}^- = p_{kv}$* $\quad k\varepsilon k_v \subset K$

$\qquad -M \leq \rho \leq M$

$\qquad -M^* \leq S_j \leq M^*$

$\quad \rho, p_k ,p^+, p^-,p_k^+,p_k^-, \geq 0$

Nature (Experiment)

Max μ' - M*$\Sigma(q_j'^+ + q_j'^-)$ -M($q'^++q'^-$) +$\Sigma p_k^* R_k'$+$\Sigma p_{kv}^* V_v'$

$\quad$ st $\Sigma(\theta_2\pi_{kj}^2 +c_2)q_k'$ - R_k' -$V_v' \geq \mu'$ $\quad k\varepsilon K_v$

$\qquad \Sigma q_j' + q'^+- q'^- = 1 \qquad$ (X)'

$\qquad q_j' + q_j'^+- q_j'^- = q_j'$*

$\qquad -M \leq \mu' \leq M$

$\qquad -M^* \leq R_k' \leq M^*$

$\qquad -M^{**} \leq V_v' \leq M^{**}$

$\qquad \mu', q_j',q'^+,q'^-,q_j'^+,q_j' \geq 0$

Player 2(Subject)

Max μ - M*$\Sigma(q_j^++q_j^-)$

$\quad$ - M$\Sigma(q^++q^-)$+$\Sigma R_k^* p_k'$*+$\Sigma p_{kv}^* V_v$

$\quad$ st $\Sigma(\theta_3\pi_{kj}^3 +c_3)q_j - R_k$ -$V_v \geq \mu$ $\quad k\varepsilon K_v$

$\qquad q_j + q_j^+ - q_j^- = q_j$*

$\qquad \Sigma q_j + q^+ - q^- = 1 \qquad$ (XI)

$\qquad -M \leq \mu \leq M$

$\qquad -M^* \leq R_k \leq M^*$

$\qquad -M^{**} \leq V_v \leq M^{**}$

$\qquad \mu,q_j,q^+,q^-,q_j^+,q_j^- \geq 0$

Die (Nature/Experiment)

Min ρ' + M*$\Sigma(p_k'^+ + p_k'^-)$+ M$\Sigma(p'^+ + p'^-)$+ $\Sigma S_j' q_j$*

$\qquad$ + M**$\Sigma p_{kv}'^++ p_{kv}'^-)$

$\quad$ st $\Sigma(\theta_4\pi_{kj}^4 +c_4)p_k'$- $S_j' \leq \rho'$

$\qquad p_k'+p_k'^+-p_k'^- =p_k'$*

$\qquad \Sigma p_k'+p_{kv}'^+-p_{kv}'^- =p_{kv}'$* $\quad k\subset k_v \subset K$

$\qquad \Sigma p_k' +p'^+-p_k'^- =1 \qquad$ (XI)'

$\qquad -M \leq \rho' \leq M$

$\qquad -M^* \leq S_j' \leq M^*$

$\quad \rho', p_k' ,p'^+, p'^-,p_k'^+,p_k'^- \geq 0,$

This again emphasises the information and frame related role of the measures S_j, S_j', R_k, R_k' and/or V_j,V_j'. Even if the (linear) transformation of contingent payoffs within a particular frame is maintained, transforming parameters θ and c nevertheless change the inducement to play this particular game vis a vis alternatives. Another species of conditional strategic equivalence would consider strategic implications of linear transformations of the framing weights M, M*,M** - in that way potentially effecting a relatively *exterior* transformation of the present preemptively framed analysis to a relatively nonpreemptively framed extension of it.

So far I have considered strategic equivalence as if identical linear transformations θ,c of identical payoffs π_{kj} might generate (X),(X)',(XI),(XI)' from (VIII),(VIII)', (IX),(IX). Now reconsider (X),(X)',(XI),(XI)' and notice that conditions as if optimally $q_j'=q_j$* all j in (XI)' are as if perfectly predictive not just of relatively exterior *information* q_j* in the relatively dual system (X), but, by complementary slackness, as if perfectly predictive of the payoff related subset of binding constraints in that relatively dual system *regardless of the actual evaluations* $(\theta_1\pi_{kj}^1 +c_1)$ *of such contingent payoffs*. If, further, the game is as if preemptively framed so that in (X), at an optimum, $p^+=p^-=0$ and $p_k^+=p_k^-=0$ all k (and/or $p_{kv}^+=p_{kv}^-=0$ all v), then an optimal solution to (X) can be found - via appropriate

transformations of variables S_j - with $\theta_1 \neq 0$, $c_1 \neq 0$ *arbitrary*. Conversely, if optimally $p_k = p_k^*$ and/or $p_{kv} = p_{kv}^*$ all k,v in $(X)'$ and the game is as if preemptively framed so that, in $(X)'$, at an optimum, $q^+ = q^- = 0$ and $q_j^+ = q_j^- = 0$ all j then the relatively exterior information is such that the optimally binding payoff related constraints in $(X)'$ are fixed and optimal solutions can be found - via appropriate transformations of variables R_k' -with $\theta_2 \neq 0$, $c_2 \neq 0$ arbitrary in *that* system. Similar considerations apply to $(XI),(XI)'$.

It follows that, with important frame and complementary slackness related qualifications, $(X),(X)'$ (resp $(XI),(XI)'$) may be both strategically equivalent to the correspondingly untransformed systems $(VIII),(VIII)'$ (resp $(IX),(IX)'$) and informationally dual to each other, even if those earlier systems are *independently* transformed via conditions such that $\pi_{kj} => \theta_s \pi_{kj}^s + c_s$.

If $\theta^s = \theta$, $c^s = c$ all s, the latter expressions yield problem independent strategically equivalent specializations $\pi_{kj}' = \theta \pi_{kj} + c$ of the standard form. But these *are* indeed specializations. The more general transformations $\pi_{kj}^s = \theta^s \pi_{kj} + c^s$ comp-rehends a much more general class of strategically equivalent cases, including non-constant sum as well as constant sum cases, given: i) *frame equivalence* i.e. as if preemptive agreement via magnitudes M, M^*, M^{**} to given frames and; ii) *information equivalence* i.e. as if perfect information via $q_j'^*, p_k^*, p_{kv}^* \quad q_j'^*, p_k^*, p_{kv}^*, \quad p_k^*, p_{kv}^*, q_j^*$, concerning probabilities with which a relatively dual player will adopt an optimal set, or subset, of strategies.

Under these conditions, and as long as the frame of the experiment remains invariant to such transformations, the relevant parts of the overall problem yield strategically equivalent solutions, *whether or not* payoffs $\pi_{kj}^s = \theta_s \pi_{kj}^s + c_s$ to relatively dual elements of games variously between experimenter and experiment, experimenter and subject and subject and experiment, which constitute parts of an overall intervening duality structure, *are constant sum*.

7. Strategic equivalence and tracer games

In this section strategic equivalence is used to focus on a class of experimenter-experiment interactions that might be seen as taking place with reference to a lower order of payoffs than those in the game proper. In this way tracer games can be used to elicit information concerning an opponent's potential strategies and thence as means of formulating optimal strategies - including a no play strategy - for a game with a larger order of payoffs. (The term tracer was chosen because of the analogy with ranging shots using light calibre tracer or lasers prior to deciding whether, and when, to fire heavier calibre weapons, for example in tank-antitank warfare or in radar related antisubmarine operations.)

Consider tracer analogues of $(X),(X)',(XI),(XI)'$ interpreting them with reference to a relatively packed high-low die related specification with $\pi_{kj}^s = \theta_s \pi_{kj}$ all s, $j=1,2$, $k=1, 2..6$, $v=h,l$ and $0 < \theta^s << 1$, $c^s = 0$, $s = 1,2,3,4$.

For simplicity assume M, M*, M** arbitrarily large so that frames restricting outcomes $j \epsilon \{1,2\}$, $k \epsilon \{1,2,..6\}$, $v \epsilon \{h,l\}$ are as if preemptively acceptable to both players. (If these weights were not preemptively large "no play" solutions $q_j', q_j = 0$ all j and/or $p_k, p_k' = 0$ all k might be selected by one or both players and that information transmitted accordingly.)

For simplicity, too, attention will be focused on relatively unpacked cases by assuming that both player and experimenter will select $V_v' = V_v = 0$ and in that sense be as if indifferent between relatively packed and relatively unpacked specifications.

With these assumptions players accepting high-low die related frames via M,M*,M** may act with reference to a tracer game with payoffs $\theta^s \pi_{kj}$, $\theta^s << 1$ as if perfectly to predict strategically equivalent behaviours relative to a potentially higher ordered game with payoffs π_{kj} as follows :

First Player/Experimenter

- As if via (X) with $p_k^* = 1/6$, $p_{kh}^* = p_{kl}^* = 1/2$ and $S_1 = 0$, $S_2 = 0$, makes themselves dual to a system $(X)'$ with conditions as if $q_1'^* = q_2'^* = 1/2$. In that way the first player/experimenter becomes as if potentially dual to a pair of mutually exclusive high-low die related outcomes such as Alternative 1 vs Alternative 2 in Table 1. Then;
- The first player/experimenter transmits to the subject an experimental specification to the effect that the second player/subject, if accepting the frame of the game via M, M* in $(XI)'$, must set $p_k'^* = 1/6$, $p_{kh}'^* = p_{kl}'^* = 1/2$ in $(XI)'$ and act via (XI) as if that system is potentially dual to it by selecting $q_j^* > 0$ some j accordingly.

Subject

• Given the experimenter's specification the subject may select any values q_j^* in (XI) as long as an optimal solution to (XI) consistent with these values is potentially dual to an optimal solution to (XI)'.

Possible Experimental Outcomes Include

• Given $q_j = q_j' = 1/2$ and $p_k' = p_k = 1/6$ the subject selects $S_j' = 0$, in agreement also with the experimenter's choice $S_j = 0$. Or;
• Given $q_j' = 1/2$ and $p_k' = p_k = 1/6$ the subject selects $q_1 = 1, q_2 = 0$ with $S_1' = 0$, $\Sigma\pi_{k2}p_k' \geq S_2' \geq 0$, (or $q_1 = 0$, $q_2 = 1$ with $\Sigma\pi_{k1}p_k' \geq S_1' \geq 0$, $S_2' = 0$). This is in disagree-ment with the experimenter on probabilities q_j, (resp q_j') and on contingent valuations S_j', S_j.

With the first type of outcome both players would be indifferent as to whether or not to (continue) to play either via the tracer game or its scaled up analogue. With the second one or both players may be induced to (continue) to prefer the tracer game to an initial alternative and then to prefer play via a scaled up analogue as if generated via that tracer game to (continued) play of the tracer game. In more detail:

Indifference related outcomes with agreed probabilities and magnitudes S_j, S_j'

• Through selections in a tracer game with payoffs $\theta^s\pi_{kj}$ two players find themselves in agreement on relative probabilities $q_j = q_j' = 1/2$, $p_k' = p_k = 1/6$ and on relative payoffs via $S_j = S_j' = 0$.

In this case both would discover not just that they agreed with each other but that that agreement was as if informationally identical to their initial agreements variously (via (X),(XI)) with themselves and (via (X)',(XI)') with systems dually related to themselves.

In these circumstances (X) is as if perfectly predictive of (XI)' and (XI) as if perfectly predictive of (X)', so both players would discover themselves to be *indifferent* between play against an intervening system - e.g. an intervening high-low die toss - and play directly against each other, even if the payoffs were transformed by $1/\theta^s$ to equate to those of the game proper. Under these circumstances each would discover themselves to prefer *any* alternative activity (e.g. alternative work or leisure related activity) that may be or become available with a positive expected payoff beyond the initial frame to (continued) play of the high-low

die casting game or of any scaled up analogue. (This is the promised extension of the case considered on pp.11-13.)

Preference Related Outcomes without agreed probabilities/magnitudes S_j, S_j'

In these cases players may not only disagree with each other but with themselves. They might undertake a tracer game specifically to discover another's perceived probabilities and/or payoffs and thence their willingness (or otherwise) to play the game proper. Here I consider only this subclass of tracer games and potentially scaled up analogues in two stages as follows:

• **Stage 1**: Through selections in a tracer game with payoffs $\theta^s\pi_{kj}$ two players find themselves in agreement on (high-low die casting) probabilities $q_j' = 1/2$ and $p_k' = 1/6$ relative to a system via (X)',(XI)', as above. The first player acts as if to choose $p_k = p_k' = 1/6$ and $S_1 = 0$, $S_2 = 0$ in (X) (again as above). The second selects $q_1 = 1, q_2 = 0$ in (XI) with $S_1' = 0$, $\Sigma\theta_3\pi_{k2}p_k' \geq S_2' > 0$ in (XI)', (or $q_1 = 0$, $q_2 = 1$ with $\Sigma\theta_3\pi_{k1}p_k' \geq S_1' > 0$, $S_2' = 0$). But now, even though both players are respectively in agreement with, in the sense of dual to, elements of (X)',(XI)', they *disagree* with each other on probabilities q_j, (resp q_j') and on contingent valuations S_j', S_j. (If the experiment finished here this conflict between experimenter and subject would be equivalent to a tracer game analogue of the Allais-like outcome case considered on p11 above.) But now continue;

• **Stage 2.** Instead of simply recording outcomes under Stage 1, one or both players may *learn*. In particular, since $S_j' \neq S_j$ and $q_j \neq q_j'$ some j, the dual pairs (X),(XI)' and (XI),(X)') are informationally different. There may be opportunities to exploit these differences in further plays of learning- modified tracer games and/or of scaled up strategically equivalent analogues of them. Consider three subcases:

2a: The first player may choose to *learn from/*agree with the second via $q_1' = 1, q_2' = 0$ in (X)' and correspondingly *learn/*agree $S_1 = 0$, $\Sigma\theta_1\pi_{k2}p_k \geq S_2 > 0$ with $p_k = p_k'$ in (X). In that way the first player would become in agreement with the second so that measures of overall net advantage are $\Sigma S_j q_j'^* = 0$, $\Sigma p_k * R_k = 0$, $\Sigma p_k' * R_k = 0$, $\Sigma S_j' q_j^* = 0$ with no *apparent* incentive for either player to continue to play tracer games (X),(X)', (XI),(XI)';
2b: the first player may *learn from/*agree with the second via $q_1' = 1, q_2' = 0$ in (X)' bit retain $S_1 = S_2 = 0$. Even though apparently disagreeing on relative preferences between outcomes j again $\Sigma S_j q_j'^* = 0$, $\Sigma p_k * R_k = 0$, $\Sigma p_k' * R_k = 0$, $\Sigma S_j' q_j^* = 0$ with no *apparent* incentive to continue to play tracer games (X),(X)', (XI),(XI);
2c: the first player in response to *learning from the second may choose to disagree with them* by retaining

p_k=1/6 and setting q_1'=q_2=1, q_2'=q_1=0 in (X)' and correspondingly S_1=0, $\Sigma\theta_1\pi_{k2}p_k\geq S_2$>0 in (X) so that again so that measures of overall net advantage are $\Sigma S_j q_j'^*$=0, $\Sigma p_k^*R_k'$=0, $\Sigma p_k'^*R_k$=0, $\Sigma S_j'q_j^*$=0 with no *apparent* incentive for either player to continue to play tracer games (X),(X)', (XI),(XI)';

In each of subcases 2a,2b,2c I have emphasized *apparent* because in each case, if contingent payoffs $\theta_s\pi_{kj}$ are augmented by corresponding amounts c_s>0, although outcomes of each of these augmented payoff cases would be strategically equivalent to its unaugmented correspond, the measures $\Sigma S_j q_j'^*$, $\Sigma p_k^*R_k'$, $\Sigma p_k'^*R_k$, $\Sigma S_j'q_j^*$ of potential gain to continued play of the tracer game would in all cases become strictly *positive*. *A fortiori,* if contingent payoffs were augmented by amounts $(1/\theta_s-1)\pi_{kj}+c_s$ with θ_s,<<1, π_{kj}>0, c_s>1 outcomes of each *scaled up* analogue of these tracer games would not only be strategically equivalent to, but preferable to. the (unaug-mented) tracer correspond which preceded it *as if because* measures $\Sigma S_j q_j'^*$, $\Sigma p_k^*R_k'$, $\Sigma p_k'^*R_k$, $\Sigma S_j'q_j^*$ of potential gain to playing the game proper would in each case be strictly positive. This emphasizes interpretations of S_j, R_k', R_k, S_j' as measures of marginal gain, if positive - or marginal loss/regret if negative - to learning about/possession of corresponding increments of relatively advantageous (resp disadvantageous information at that margin.

Even under the highly restrictive assumptions that players will continue to preemptively select relatively restricted strategy sets with j=1,2, k=1,..6, v=h,l and V_v=V_v'=0 even when payoffs π_{jk} are increased or reduced by magnitudes θ_s and/or $+c_s$ there is a variety of solutions here. That variety would be considerably enhanced if cases were also considered for other ranges of the indices j,k,v, for which V_v,V_v'≠0 and/or for which analogues of the the magnitudes M,M*,M** were not assumed to be arbitrarily large.

But clearly this learning related variety could be extended in other ways too, including ways in which the second player learns of the first player's revised behaviours under subcases 2a,2b,2c and revises his/her behaviour in turn to generate further learning related intervening duality specifications and results.

More generally I emphasise again that if R_k, R_k' ,V_v,V_v' and S_j, S_j' are varied, for example in response to learning, even if a game was initially

constant sum, in general it would not remain so. Clearly therefore a considerably wider range of individually or mutually advantageous cases is attainable by means of the intervening duality structure (X),(X)',(XI),(XI)' than has been considered in detail here. But the subcases considered here are sufficient to demonstrate how, one or both players may generate cumulative gains for themselves in a number of learning related and sequentially interrelated ways. First individuals may gain in the context of tracer games by offering larger opportunity sets to each other via the prospect of playing at all (through an offer of otherwise unknown/unattainable interpersonal experimental frame) Next - and still in the context of a tracer game - individuals may gain/learn - as in the experimenter-subject cases considered here - by inducing another to provide the opportunity of a mutually preferred specification via information concerning their potential strategies and payoffs given that prior frame. Then, using that prior information and learning individuals, may discover a mutual advantage in preferring play in a strategically equivalent game with a higher order of payoffs.

8. Conclusions and extensions

The principal focus in this paper has been on constrained game formulations of apparently tightly defined classes relatively unpacked and then relatively packed die casting experiments and their associated intervening duality structures. A major conclusion is that even in explicitly fair die casting cases, where it might be thought there is no room for mutually advantageous differences of opinion and observation, an intervening duality approach can be usefully employed to model and exploit such potential differences.

From this perspective these die related examples can be regarded as a first stage on the way to barter and trade related extensions of kinds anticipated in the conclusion of Ryan 1995. Looking forward to the next stage, whereas in this paper the focus has been primarily upon an experimenter represented as if indifferent to the outcome of an intervening experiment, in more narrowly economic extensions and applications each of two interrelating individuals would generally be seeking to gain from any exchange. In such cases, as in two person coin tossing or die casting games, in fact neither would be indifferent to the framing and specification of elements of an intervening dual. In contrast to the emphasis here on an as if neutral experimenter

emphasis in that context will be on cases where both parties perceive advantage to differences relative to an intervening and as if agreed state (e.g. commodity/ transaction) and both seek to gain from relatively oppositely inclined exchange related perceptions.

Finally, in this chapter attention has been confined to preemptively framed cases of experimenter-experiment interaction in an intervening duality framework. With a different context Chapter 10 will used focus on a subject's choice of experimental outcomes in a non-preemptively framed analysis but without the intervening duality framework. A more comprehensive analysis would extend and generalize results in both chapters through an analysis of experimenter-experiment interactions in a non-preemptively framed intervening duality framework.

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