CHAPTER 2

MORE FOR LESS, OPTIMALITY AND GENERALIZED KUHN TUCKER CONDITIONS

1. Introduction

Kuhn Tucker conditions (Kuhn and Tucker 1951) have been developed and used variously as consistency checks, as computational aids and as sources of optimal decision rules in many applications in which the relations concerned are continuous and differentiable. (For a wide range of applications see Zangwill 1969 or Bazaraa 1993.) My main purpose in this chapter is to develop new results analogous to the Kuhn Tucker conditions without assumptions either of continuity or of differentiability. A secondary purpose is to show how, whether or not the relevant conditions are continuous and differentiable, if the relevant systems are "conservative" in a sense to be defined in Section 7, then the corresponding Kuhn Tucker-like conditons will generally coincide with conditions for degeneracy and/or for alternate optima. The chapter concludes with economic examples which provide practical gift, exchange and learning related illustrations and applications of these phenomena.

2. Some preliminary results

The following Theorem appears as Theorem 3 in Section 2 of Chapter 1:

THEOREM 1
With $M >> h^+(s^+)$, $M >> h^-(s^-)$ and if an optimal solution exists for (I), then:

$$\begin{array}{llll} \text{Max } f(x) - Ms^+ - Ms^- = z \le z' = & \text{Max } f(x) - h^+(s^+) - h^-(s^-) & & \\ \text{st } g(x) + s^+ - s^- = b \quad \text{(I)} & \text{st } g(x) + s^+ - s^- = b & \text{(Ia)} \\ \quad x, s^+, s^- \ge 0 & \quad x, s^+, s^- \ge 0 & \end{array}$$

PROOF
Any optimal solution to (I) with all $s_i^+=0$, $s_{i0}^-=0$ is a feasible but not necessarily an optimal solution to (Ia). It follows that there may exist optimal solutions to (Ia) such that $z'>z$ or $z'=z$ with $s_i^+>0$ and/or $s_i^->0$ some s_i^+, s_i^-. (Notice that if variables s_i^+, s_i^- appear in each of constraints $i=1,2..m$ of (I) then there is *always* a feasible solution to that system.)

COROLLARY ***Less for More/Nothing***
There may be optimal solutions to (Ia) such that $s_i^+>0$ some i and $s_i^-=0$ all i so that $z'>z$ (resp $z'=z$). [Corresponding More for Less results arise via minimization cases for which $s_i^->0$ some i, $s_i^+=0$ all i and $(-)z'>(-)z$ (resp $z'=z$).]

REMARK
The less for more (nothing) and more for less results in this corollary do not require that either f(x) or g(x) be continuous or that constraint sets be connected. A fortiori, there is no requirement in Theorem 1 that f(x) or g(x) be differentiable.

In Chapter 1 I used a specialization of Theorem 1 and its corollary to extend and generalize results by Swarcz 1971 and Charnes and Klingman 1971 for the cost minimization related distribution problem and by Charnes et al 1987 for linear programming cases more generally. In that chapter I also generalized more for less results in Ryan 1980, 1998, Charnes et al 1980 to include individual and collective choice problems as well as gift and exchange related regulatory and bargaining problems. As a first step toward further generalizations of that approach to yield generalizations of the Kuhn Tucker conditions, now consider a specialization of Theorem 1:

THEOREM 1A
Assume that in Theorem 1: i) a feasible solution to (I) with $s^+=s^-=0$ exists and; ii) $h^+(s^+)=_{def}h^+s^+$, $h^-(s^-)=_{def}h^-s^-$ in (Ia) Then Theorem 1 is consistent with:

$$\begin{array}{llll} \text{Max } f(x) = z \le z' = & \text{Max } f(x) - h^+s^+ - h^-s^- & & \\ \text{st } g(x) = b \quad \text{(II)} & \text{st } g(x) + s^+ - s^- = b & \text{(IIa)} \\ \quad x \ge 0 & \quad x, s^+, s^- \ge 0 & \end{array}$$

PROOF
As for Theorem 1.

LEMMA 1

Let x^*,x^{**} be optimal evaluations of x respect-ively via (II) and (IIa). It follows that, unless these evaluations are identical, then $x_i^* \neq x_i^{**}$ some i. Equivalently $x_i^{**} =_{def} x_i^* + \Delta x_i$ with $\Delta x_i \neq 0$ at least one i. With this notation, at an optimum (II) and (IIa) become such that:

$$\begin{array}{ll} \text{Max } f(x) = f(x^*) \;=\; z \le z' = f(x^*+\Delta x) - h^+s^+ - h^-s^- & = \text{Max } f(x+\Delta x) - h^+s^+ - h^-s^- \\ \text{st } g(x) = b \quad \textbf{(III)} & \text{st } g(x+\Delta x) + s^+ - s^- = b \quad \textbf{(IIIa)} \\ x \ge 0 & x, s^+, s^- \ge 0 \end{array}$$

REMARK

Clearly (III) and (IIIa) are consistent with special cases of (I) and (Ia) of Theorem 1 for which optimally $s^+=s^-=0$ all i in (III) if h^+,h^- preemptively large.

Optimal solutions to (III) and (IIIa) can yield the wide variety of economic interpretations, which were derived from Theorem 3 in Chapter 1. Rather than focus on such interpretations, here I focus more narrowly on ways in which Lemma 1 can generate general classes of necessary and/or sufficient conditions for optimality for elements (III),(IIIa) and thence for (II),(IIa). These ideas are developed in the next Section.

3. Three classes of optimal solutions to (III), (IIIa)

CASE A:

Optima to (III),(IIIa) become consistent with z=z' via: **Ai)** $\Delta x_i=0$; **Aii)** $s_i^+=s_i^-=0$ all i in (IIIa).

CASE B:

Optima to (III),(IIIa) become equivalent but *not* consistent with z=z' via: **Bi)** at least one $\Delta x_i \neq 0$ and/or; **Bii)** $s_i^+>0$ and/or $s_i^->0$ some i with $s_i^+ \neq s_i^-$.

CASE C:

An optimum to (III) is suboptimal relative to (IIIa) with z<z' via: **Ci)** at least one $\Delta x_i \neq 0$ and/or; **Cii)** $s_i^+>0$ and/or $s_i>0$ some i, $s_i^+ \neq s_i^-$.

REMARKS

- For h_i^+,h_i^- sufficiently large and if $s_i^+=s_i^-=0$ is feasible, any optimal solution to (IIIa) will be such that $s_i^+=s_i^-=0$. If also $\Delta x_i=0$ all i, then Case A applies.
- Case B and Case C are of particular interest here because they potentially yield a variety of less for more interpretations, as in Theorem 2 below, and because refinements of them yield the discrete form of generalised Kuhn Tucker conditions which is developed in Section 4.

THEOREM 2 ***Less For More and More for Less Solutions to (III) and (IIIa)***

If $f(x^*)$, $f(x^*+\Delta x)$ are optimal respectively in (III) and (IIIa) and such that $f(x^*+\Delta x)-f(x^*)>0$, then, whereas **necessary** conditions for $f(x^*+\Delta x)-f(x^*)>0$ are $f(x^*+\Delta x)$ increasing in Δx_i at least one i, **sufficient** conditions are $h_i^+s_i^+, h_i^-s_i^- \ge 0$ all i, $h_i^+s_i^+, h_i^-s_i^- >0$, $s_i^+ \neq s_i^-$ at least one i with z=z', as in Case Bii).

PROOF

Necessity follows directly since $f(x^*+\Delta x)-f(x^*)>0$ implies $f(x^*+\Delta x)$ increasing in Δx_i at least one i. Sufficiency follows, too since, if optimally $h_i^+s_i^+, h_i^-s_i^- \ge 0$ all i and $h_i^+s_i^+, h_i^-s_i^- >0$, $s_i^+ \neq s_i^-$ at least one i (e.g. via $h_i^+,h_i^-<M$) and if z=z', as in Case Bii), these conditions are then sufficient to determine $f(x^*+\Delta x_i)-f(x^*)>0$ some i.

REMARKS

- Theorem 2 is potentially consistent with *less for more (*and, via transformations $f(x) =_{def} -f(\underline{x})$, *more for less)* optima for (IIIa) relative to (III) since these conditions are potentially consistent with $f(x^*+\Delta x)-f(x^*)>0$ together with $s_i^+>0$ *(*resp $s_i^->0$*)* some i in (IIIa).
- If $h_i^+s_i^+=h_i^-s_i^-=0$ all i then Theorem 2 is potentially consistent with *less for nothing* (resp *more for nothing*) optima for (IIIa) relative to (III) since it is potentially consistent with $f(x^*+\Delta x)-f(x^*)=0$ together with $h_i^+s_i^+=0$ all i and $s_i^+>0$ some i (resp $h_i^-s_i^-=0$ all i, $s_i^->0$ some i) in (IIIa).

ECONOMIC INTERPRETATIONS

It has already been noted that $f(x^*+\Delta x)-f(x^*)>0$ implies $f(x^*+\Delta x)$ increasing in Δx_i at least one i. Now note that Δx_i may be *negative*. In economic terms an individual may gain not only by gaining “goods” but by losing “bads”.

4. Further specializations and discrete Kuhn Tucker conditions

4.1 TWO EQUIVALENCES AND TWO SPECIAL CASES

LEMMA 2

If $h_i^+ =_{def} \lambda_i^+$, $h_i^- =_{def} \lambda_i^-$ with $\lambda =_{def} \{\lambda_i^+, -\lambda_i^-\}^T$ and if nonzero values for $s_i^+, s_i^- \geq 0$ are mutually exclusive so that $s_i^+ . s_i^- = 0$ then from (IIIa) *either* $s^+ = (b - g(x^* + \Delta x))$ *or* $s^- = (g(x^* + \Delta x) - b)$.

TWO EQUIVALENCES

Ei) Given Lemma 2, in Case B of Section 3 *equality* of the objectives of (III),(IIIa) becomes equivalent to:

$$\text{Max } f(x) = f(x^*) = z = z' = f(x^*+\Delta x) - \lambda^+ s^+ - \lambda^- s^- = \text{Max } f(x+\Delta x) - \lambda^+ s^+ - \lambda^- s^- \qquad (4.1)$$

So that:

$$f(x^*) = z = z' = f(x^*+\Delta x) - \lambda(b - g(x+\Delta x)) \qquad (4.1a)$$

Or:

$$f(x^*+\Delta x) - f(x^*) = \lambda(b - g(x^*+\Delta x)) \qquad (4.1b)$$

Eii) Given Lemma 2, in Case C of Section 3 *inequality* of the objectives of (III),(IIIa) becomes equivalent to:

$$\text{Max } f(x) = f(x^*) = z < z' = f(x^*+\Delta x) - \lambda^+ s^+ - \lambda^- s^- = \text{Max } f(x+\Delta x) - \lambda^+ s^+ - \lambda^- s^- \qquad (4.2)$$

So that

$$f(x^*) = z < z' = f(x^*+\Delta x) - \lambda(b - g(x+\Delta x)) \qquad (4.2a)$$

Or:

$$f(x^*+\Delta x) - f(x^*) > \lambda(b - g(x^*+\Delta x)) \qquad (4.2b)$$

TWO SPECIAL CASES

Si) Given Lemma 2, Case B of Section 3 and (4.1b) may apply in a disaggregated form such that, using Δx_i to denote variation only of the ith element at an optimum:

$$f(x^*+\Delta x_i) - f(x^*) = \lambda(b - g(x^*+\Delta x_i)) \quad \text{all relevant } i \qquad (4.1c)$$

Sii) Similarly, given Lemma 1, Case C of Section 3 and (4.2b) may apply in a disaggregated form such that, again using Δx_i to denote variation only of the ith element at an optimum:

$$f(x^*+\Delta x_i) - f(x^*) > \lambda(b - g(x^*+\Delta x_i)) \quad \text{all relevant } i \qquad (4.2c)$$

Conditions Si) and Sii) together yield a discrete form of Kuhn Tucker conditions via Theorem 3:

THEOREM 3 ***Disaggregated Discrete Kuhn Tucker Decision Rules***

Given Lemma 2 sufficient conditions for f(x*) in (III) *not* to be suboptimal relative to $f(x^*+\Delta x) - h^+ s^+ - h^- s^-$ in (IIIa) are:

$$x_i > 0, |\Delta x_i| \neq 0 \Rightarrow f(x^*+\Delta x_i) - f(x^*) = \lambda(b - g(x^*+\Delta x) \qquad (4.1c)^*$$

Potentially <u>select</u> $x_i > 0$, $|\Delta x_i| \neq 0$

and;

$$x_i > 0, |\Delta x_i| \neq 0 \Rightarrow f(x^*+\Delta x_i) - f(x^*) < \lambda(b - g(x^*+\Delta x_i)) \qquad (4.2c)^*$$

<u>Do not select</u> $x_i > 0, |\Delta x_i| \neq 0$

PROOF

Given Lemma 1, if $x_i > 0$ then $|\Delta x_i| \neq 0 \Rightarrow f(x^*+\Delta x_i) - h^+ s^+ - h^- s^- > f(x^*)$ and so $f(x^*+\Delta x_i) - h^+ s^+ - h^- s^- > f(x^*)$, then f(x*) in (III) is suboptimal relative to $f(x^*+\Delta x) - h^+ s^+ - h^- s^-$ in (IIIa). It follows that sufficient conditions for f(x*) in (III) to be *not* sub-optimal relative to $f(x^*+\Delta x) - h^+ s^+ - h^- s^-$ in (IIIa) are: i) $f(x^*+\Delta x_i) - f(x^*) = \lambda(b - g(x^*+\Delta x_i)$ if $x_i > 0$ and; ii) if $f(x^*+\Delta x_i) - f(x^*) < \lambda(b - g(x^*+\Delta x_i)$ then $x_i = 0$, $\Delta x_i = 0$.

REMARK

Theorem 3 in common with Theorems 1 and 2 requires no assumption of continuity, and *a fortiori* no assumption of differentiability, for f(x).

ECONOMIC APPLICATIONS

In economic contexts conditions (4.1c)* in Theorem 3 may have interpretations as optimal economic decision rules according to which: i) for $\Delta x_i > 0$ *more* of an activity i may be chosen if the marginal net reward $f(x^*+\Delta x_i) - f(x^*)$ to that activity is sufficient to recoup $\lambda(b - g(x^*+\Delta x_i)$, the opportunity costs of the resources employed in that expansion.

<u>**And;**</u> ii) for $\Delta x_i < 0$, *less* of an activity may be chosen as long as the marginal compensation $\lambda(b - g(x^*+\Delta x_i)$ for reducing that activity is sufficient to recoup the marginal net reward $f(x^*+\Delta x_i) - f(x^*)$ foregone in reducing that activity. Also, from (4.2c)* in Theorem 3, no activity would be chosen for which the marginal reward was less than the marginal

opportunity cost associated with achieving it. [I will return to these interpretations with a context of economic applications in Section 8.]

While sufficient, the conditions of Theorem 3 are not *necessary* for potentially mutually beneficial exchanges between individuals. Indeed mutually beneficial exchanges are consistent with circumstances where each individual not only prefers more of what they are getting (as in the examples just considered) *but also prefers less of what they are giving up*.

As this latter example implies, the conditions given in Theorem 3 may be such that simultaneously $f(x^*+\Delta x_i)-f(x^*) < \lambda(b-g(x^*+\Delta x_i))$ some i *and* $f(x^*+\Delta x_j)-f(x^*) > \lambda(b-g(x^*+\Delta x_j)$ some j, yet nevertheless an overall relationship may hold such that $f(x^*+\Delta x)-f(x^*)=\lambda(b-g(x^*+\Delta x))$, as in (4.1b). Theorem 4 is a formalization of this idea:

THEOREM 4 ***Aggregate Discrete Kuhn Tucker Conditions***

Necessary conditions for f(x*) in (III) *not* to be suboptimal relative to $f(x^*+\Delta x)-h^+s^+-h^-s^-$ in (IIIa) are:

$$f(x^*+\Delta x)-f(x^*)=h^+s^++h^-s^-\geq 0$$
$$x_i\neq 0, |\Delta x_i|\neq 0 \text{ some } \Delta x_i \qquad (4.1b)^*$$

PROOF

If $f(x^*+\Delta x)-h^+s^+-h^-s^->f(x^*)$ then f(x*) in (III) is suboptimal relative to $f(x^*+\Delta x)-h^+s^+-h^-s^-$ in (IIIa) and Case C of Section 3 applies. It follows that sufficient conditions for f(x*) in (III) to be *not* suboptimal relative to $f(x^*+\Delta x)-h^+s^+-h^-s^-$ are: $f(x^*+\Delta x)-f(x^*)=h^+s^++h^-s^-\geq 0$ if $x_i\neq 0$, $\Delta x_i\neq 0$ some i.

ECONOMIC INTERPRETATIONS

- The conditions of Theorem 4 are consistent with $f(x^*+\Delta x_i)>f(x^*)$ some i and thence with an economic interpretation according to which both parties to an exchange may *gain* according to their preference relations. They are also potentially open to an interpretation according to which one or more parties *lose* according to their preferences while others gain in such a way as to record an overall net gain. [In the former case an exchange related economic application of Theorem 4 may be consistent with a strict Pareto improvement according to which two individuals both gain in their own estimation from exchange.]
- Notice that, in distinction from Theorem 3, in general the conditions of Theorem 4 apply to potential net gains or losses from *bundles* of commodities.

Theorems 1-4 do not require assumptions of continuity, connectedness or differentiability. Nor do they require any particular structure for f(x) (or for g(x)). But clearly stronger results may be obtained with more restrictive conditions on f(x). Examples of more restrictive conditions are concavity and differentiability. These restrictions and their implications are the subjects of the next two Sections.

5. Concavity and optimality

DEFINITION 1 ***Concavity vs Strict Concavity***

A relation f(x) is *concave* if:

$$(1-\theta)f(x)+\theta f(x+\Delta x)\leq f(x+\theta\Delta x),\ 0<\theta<1 \qquad (5.1)$$

Or, equivalently, if:

$$\theta f(x+\Delta x)-\theta f(x)\leq f(x+\theta\Delta x)-f(x),\ 0<\theta<1 \qquad (5.2)$$

f(x) is *strictly concave* if the strict inequality holds.

DEFINITION 2 ***Maximal vs strictly maximal***

If $f(x+\theta\Delta x)-f(x)\geq 0,\quad 0<\theta<1 \qquad (5.3)$

Then $f(x+\theta\Delta x)$ is maximal relative to f(x) and;

If $f(x+\theta\Delta x)-f(x)>0,\quad 0<\theta<1 \qquad (5.4)$

Then $f(x+\theta\Delta x)$ is *strictly* maximal relative to f(x).

THEOREM 5 ***Concavity and Relative Maxima***

If f(x) is concave and if x*, $x^*+\Delta x$ are optimal solutions to (III) and (IIIa) and if Case B holds with $h^+s^++h^-s^-\geq 0$, then $f(x^*+\theta\Delta x)-f(x^*)\geq 0$, $0<\theta<1$ and $f(x^*+\theta\Delta x)$ is *maximal* relative to f(x*).

PROOF

$$\theta f(x^*+\Delta x)-\theta f(x^*)\leq f(x^*+\theta\Delta x)-f(x^*)$$
$$0<\theta<1 \text{ by concavity} \qquad (5.5)$$

Further:

Case B and $h^+s^++h^-s^-\geq 0$ =>

$$0\leq f(x^*+\Delta x)-f(x^*) \qquad (5.6)$$

So;

$$f(x^*)\leq f(x^*+\theta\Delta x)$$
$$0<\theta<1 \qquad (5.7)$$

And $f(x^*+\theta\Delta x)$ is maximal relative to f(x*).

REMARKS

- If f(x) is strictly concave then $f(x^*+\theta\Delta x)>f(x^*)$ so that $f(x^*+\theta\Delta x)$ is *strictly* maximal relative to f(x*)

- If $h^+s^++h^-s^-=0$ in Theorem 5 then the theorem holds a fortiori and $f(x^*+\theta\Delta x)$ is a maximum relative both to $f(x^*)$ and $f(x^*+\Delta x)$.

Another relative maximality result is:

THEOREM 6 ***Concavity and Sufficient Conditions for Maxima Relative to (III),(IIIa)***

If f(x) is concave and if $f(x^*+\theta\Delta x)$ is to be maximum relative to $f(x^*+\Delta x), f(x^*)$ in (III),(IIIa) then **sufficient** conditions are $f(x^*+\Delta x)-f(x^*)=0$ and z=z' in (III),(IIIa).

PROOF

$\theta f(x^*+\Delta x)-\theta f(x^*) \leq f(x^*+\theta\Delta x)- f(x^*)$, $0<\theta<1$ by concavity. If z=z' via (III),(IIIa) then $f(x^*+\Delta x)-f(x^*)= h^+s^++h^-s^-$. If also $f(x^*+\Delta x)-f(x^*)=0$ then $h^+s^++h^-s^-=0$ so that, for f(x) concave:

$$f(x^*)=f(x^*+\Delta x)\leq f(x^*+\theta\Delta x) \qquad (5.8)$$

And $f(x^*+\theta\Delta x)$ is maximal relative to f(x*), $f(x^*+\Delta x)$.

REMARKS

- If f(x) is strictly concave then $f(x^*+\theta\Delta x)>f(x^*)$ and $f(x^*+\theta\Delta x)$ is strictly maximal relative to f(x*)
- If $h^+s^++h^-s^-=0$ in Theorem 5 then Theorems 5 and 6 become equivalent.
- Under the conditions of Theorem 6 (and Theorem 5 if $h^+s^++h^-s^-=0$) if f(x) is strictly concave then $f(x^*+\theta\Delta x)$ is maximal relative *both* to $f(x^*+\Delta x)$ *and to* f(x*), even though $f(x^*+\Delta x)$,f(x*) are respect-ively *optimal* values of the maximands of (III) and (IIIa) .
- Although concavity is sufficient for a maximum of f(x) via Theorem 6, it is not necessary for optimality of (IIIa) relative to (III).
- More generally, if Case B applies and $f(x^*+\Delta x)$ and f(x*) are such that $f(x^*+\Delta x)-f(x^*)>0$, then optimally $h^+s^++h^-s^->0$. The latter conditions are *inconsistent* with the conditions for a relative maximum in Theorem 6.

6. Concavity and disaggregated Kuhn Tucker-like results for discrete cases

Condition $h^+s^++h^-s^-=0$ in Theorem 6 may apply in the disaggregated form $h_i^+s_i^++h_i^-s_i^-=0$. This leads in turn to disaggregated Kuhn Tucker-like Theorems 5A and 6A analogous to Theorems 5 and 6 for the discrete and concave case:

THEOREM 5A ***Complementary Slackness, Concavity and Sufficient Conditions for Relative Maxima***

If f(x) is concave, if x*, $x^*+\Delta x$ are optimal solutions to (III) and (IIIa) and if Case B holds together with the conditions of Lemma 2 (i.e. $\lambda_i^+s_i^+=0$, $\lambda_i^-s_i^-=0$, $s_i^+.s_i^-=0$ all i, where $s^+=(b-g(x^*+\Delta x))$ and $s^-=(g(x^*+\Delta x)-b)$, then $f(x^*+ \theta\Delta x)-f(x)\geq 0, 0<\theta<1$, and $f(x^*+\theta\Delta x)$ is maximal relative to f(x*) and $f(x^*+\Delta x)$.

PROOF

If $\lambda_i^+s_i^+=0$, $\lambda_i^-s_i^-=0$, $s_i^+.s_i^-=0$ all i where $s^+=(b-g(x^*+\Delta x))$ and $s^-=(g(x^* +\Delta x) -b)$ then $h^+s^++h^-s^-=0$ and the result follows both from Theorem 5 and from Theorem 6. [Recall that Theorems 5 and 6 are equivalent if $h^+s^++h^-s^-=0$.]

THEOREM 6A ***Complementary Slackness, Concavity and Disaggregated Sufficient Conditions for Maxima Relative to (III),(IIIa)***

If f(x) is concave and if $f(x^*+\theta\Delta x)$ is to be a maximum relative to $f(x^*+\Delta x)$, f(x*) in (III),(IIIa) then **sufficient** conditions are that z=z' and that the conditions of Lemma 2 apply (i.e. $\lambda_i^+s_i^+=0$, $\lambda_i^-s_i^-=0$, $s_i^+.s_i^-=0$ all i with $s^+=(b-g(x^*+\Delta x))$ and $s^-=(g(x^* +\Delta x) -b)$ in (III)).

PROOF

If $\lambda_i^+s_i^+=0$, $\lambda_i^-s_i^-=0$, $s_i^+.s_i^-=0$ all i where $s^+=(b-g(x^*+\Delta x))$ and $s^-=(g(x^* +\Delta x) -b)$ in (III), then $h^+s^++h^-s^-=0$ in (III) so that $f(x^*+\Delta x)-f(x^*)=0$ and the result follows from Theorem 5 and from Theorem 6. [Again recall that Theorems 5 and 6 are equivalent if $h^+s^++h^-s^-=0$.]

REMARK

In Theorems 5A and 6A s_i^+,s_i^- are *mutually exclusive.* Notice in this context that, if $s_i^+=s_i^-=0$ all i *and* $\Delta x_i=0$ all i, then Case A, which is mutually exclusive with Case B and so inconsistent with Theorem 5A, applies. (To be consistent with Case B, if $s_i^+=s_i^-=0$ all i then $\Delta x_i\neq 0$ at least one i.)

DEFINITION 3 ***Total :Partial Variation***

$df(x)= f(x^*+\Delta x)-f(x^*) =_{def}\Sigma[(f(x^*+\Delta x_i)-f(x^*))\Delta x_i]$

REMARK

It follows from Definition 3 that partial conditions $[(f(x^*+\Delta x_i)-f(x^*))\Delta x_i]=0$ all i are sufficient to determine total conditions $f(x^*+\Delta x)-f(x^*)=0$.

This Remark leads directly to a disaggregated variant of Theorem 6A:

THEOREM 6B *Disaggregated Discrete and Concave Kuhn Tucker*

If f(x) is strictly concave and if $f(x^*+\theta\Delta x)$ is to be a maximum relative to $f(x^*+\Delta x),f(x^*)$ in (III),(IIIa) then **sufficient** conditions are that $[(f(x^*+\Delta x_i)-f(x^*))\Delta x_i-\lambda(g(x^*+\Delta x_i)-g(x^*))\Delta x_i]=0$ all i and that the conditions of Lemma 2 apply (i.e. $\lambda_i^+s_i^+=0$, $\lambda_i^-s_i^-=0$, $s_i^+.s_i^-=0$ all i with $s_i^+=(b_i-g_i(x^*+\Delta x))$, $s^-=(g_i(x^*+\Delta x)-b_i)$ in (III)).

PROOF

By Definition 3:

$$\Sigma[(f(x^*+\Delta x_i)-f(x^*))\Delta x_i-\lambda(g(x^*+\Delta x_i)-g(x^*))\Delta x_i]=0 \text{ all i } \Rightarrow$$

$$(f(x^*+\Delta x)-f(x^*))-\lambda(g(x^*+\Delta x_i)-g(x^*))=0$$

Noting that $g_i(x^*)=b_i$ from (IIIa) it follows that z=z'. The result then follows from Theorem 6A.

REMARK

Again in Theorem 6B there are no assumptions of continuity or of differentiability for f(x) or for g(x).

DEFINITION 4 *Total Derivative*

If f(x) is continuous in the range x, $x+\Delta x$ with $\Delta x\neq 0$ then the *total derivative* for the continuous case is defined via:

$$\lim_{\Delta x\to 0} df(x)=\lim_{\Delta x\to 0} f(x+\Delta x)-f(x)$$

$$=_{def}\lim_{\Delta x_i\to 0}\Sigma_i[(f(x+\Delta x_i)-f(x)]dx_i] \qquad (6.1)$$

REMARK

Analogously to the discrete case of Definition 3 it follows from Definition 4 that partial conditions $\lim[(f(x^*+\Delta x_i)-f(x^*))dx_i]=0$ all i are sufficient to determine total conditions $\lim\ [f(x^*+\Delta x)-f(x^*)]=0$.

DEFINITION 5 *Partial Derivative*

If f(x) is continuous in the range x, $x+\Delta x$ with $\Delta x\neq 0$ then the *partial derivative* $\delta f(x)/\delta x_i$ is defined via:

$$\frac{\delta f(x)}{\delta x_i}=\lim_{\Delta x_i\to 0} f(x+\Delta x_i)-f(x) \qquad (6.2)$$

It follows from (6.1) and (6.2) that:

$$\lim_{\Delta x_i\to 0} df(x)=\lim_{\Delta x_i\to 0}\Sigma_i\frac{\delta f(x)}{\delta x_I}dx_i \qquad (6.3)$$

THEOREM 7 *Continuous Kuhn Tucker*

If f(x) is continuous and concave and if $f(x^*+\theta\Delta x)$ is to be maximal relative to $f(x^*+\Delta x),f(x^*)$ in (III),(IIIa) then **sufficient** conditions are: i) $\lim\ (f(x+\Delta x_i)-f(x))=\lim\ \lambda(g(x^*)-g(x^*+\Delta x_i))$ all i and; ii) the restriction that the conditions of Lemma 2 apply (i.e. $\lambda_i^+s_i^+=0$, $\lambda_i^-s_i^-=0$, $s_i^+.s_i^-=0$ all i with $s^+=(b-g(x^*+\Delta x))$, $s^-=(g(x^*+\Delta x)-b)$ in (III)):

PROOF

If $$\lim_{\Delta x_i\to 0} f(x+\Delta x_i)-f(x)=\lim_{\Delta x_i\to 0}\lambda(g(x^*)-g(x^*+\Delta x_i)) \text{ all i} \qquad (6.4)$$

Then from definition 4 and (6.1):

$$\lim_{\Delta x_i\to 0}\Sigma(f(x+\Delta x_i)-f(x))dx$$

$$-\lim_{\Delta x_i\to 0}\lambda\Sigma(g(x^*)-g(x^*+\Delta x_i))dx=0 \qquad (6.5)$$

This implies z=z' and the result follows from Theorem 6A

[Recall that, via Theorem 6A $f(x^*+\theta\Delta x)$ is maximal relative both to $f(x^*)$, $f(x^*+\Delta x)$. This in turn implies $\lim\ f(x^*+\theta\Delta x)-f(x^*)>0$ *and* $\lim\ f(x^*+\theta\Delta x)-f(x^*+\Delta x)>0$.]

THEOREM 7A *(Continuous Kuhn Tucker Decision Rules)*

Sufficient conditions for $f(x^*)$ in (III) *not* to be suboptimal relative to $f(x^*+\Delta x)-h^+s^+-h^-s^-$ in (IIIa) are:

- If $x_i>0$, $\lim_{\Delta x_i\to 0}\Delta x_i=0\Rightarrow$

$$\lim_{\Delta x_i\to 0} f(x^*+\Delta x_i)-f(x^*)=\lim_{\Delta x_i\to 0}\lambda(b-g(x^*+\Delta x_i) \qquad (6.6)$$

Potentially <u>select</u> $x_i>0,\lim\Delta x_i=0$.

- If $x_i>0$, $\lim_{\Delta x_i\to 0}\Delta x_i=0\Rightarrow$

$$\lim_{\Delta x_i\to 0} f(x^*+\Delta x_i)-f(x^*)>\lim_{\Delta x_i\to 0}\lambda(b-g(x^*+\Delta x_i)) \qquad (6.7)$$

<u>Do not select</u> $x_i>0$, $\lim\Delta x_i=0$ (i.e. select $x_i=0$, $\Delta x_i=_{def}0$).

Where (6.6) and (6.7) are subject to the restrictions of Lemma 2. That is: such that $h_i^+=_{def}\lambda_i^+,h_i^-=_{def}\lambda_i^-$ so that $\lambda=_{def}\{\lambda_i^+,-\lambda_i^-\}^T$ in (IIIa) and $s_i^+,s_i^-\geq 0$ are *mutually exclusive,* with $s^+=(b-g(x^*+\Delta x))$ or $s^-=(g(x^*+\Delta x)-b)$).

PROOF (*Similar to Theorem 3 for the discrete case*)

If $\lim f(x+\Delta x_i)-f(x)=\lim\lambda(g(x^*)-g(x^*+\Delta x_i))$ all i then $\lim f(x+\Delta x)-f(x)=\lim\lambda(g(x^*)-g(x^*+\Delta x))$ from (6.1), so that z=z' and the result follows from Theorem 6A (as in Theorem 7). But if $f(x+\Delta x_i)-f(x)>\lambda(g(x^*)-g(x^*+\Delta x_i))$ some i then $f(x+\Delta x)-f(x)>\lambda(g(x^*)-g(x^*+\Delta x))$ and f(x) is suboptimal relative to f(x) *unless* $x_i=0$, $\Delta x_i=_{def}0$ for such cases.

REMARK

Although sufficient, the conditions of Theorem 7A are not necessary. Overall conditions yielding equal optima for (III) and (IIIa) via $f(x^*+\Delta x)-f(x^*)=\lambda(g(x^*)-g(x^*+\Delta x))$ are consistent with some $f(x^*+\Delta x_i)-f(x^*)>\lambda(g(x^*)-g(x^*+\Delta x_i))$ and some $f(x^*+\Delta x_i)-f(x^*)<\lambda(g(x^*)-g(x^*+\Delta x_i))$

ECONOMIC APPLICATIONS

Economic applications here include examples analogous to those in relation to Theorems 3 and 4 for the discrete case.

7. Conservative systems and Kuhn Tucker results

DEFINITION 6 ***(Conservative system)***

System (IIIa) (reproduced from Section 2 below) is *conservative* iff $s_i^+=s_i^-=0$ all i.

$$\begin{array}{lcl} \text{Max } f(x)=f(x^*) & = z \le z' = f(x^*+\Delta x)-h^+s^+-h^-s^- = & \text{Max } f(x+\Delta x)-h^+s^+-h^-s^- \\ \text{st } g(x^*)=b \quad \textbf{(III)} & & \text{st } g(x^*+\Delta x)+s^+-s^-=b \quad \textbf{(IIIa)} \\ x^* \ge 0 & & x^*, s^+, s^- \ge 0 \end{array}$$

REMARK

The key idea here is that s_i^+, s_i^- can be interpreted as (framing) variables relating (III),(IIIa) to a wider system. In that context $s_i^+=s_i^-=0$ all i are necessary conditions for (III), (IIIa) to be conservative in the sense of self-contained.

LEMMA 1A

It follows from Definition 6 that, if (III) and (IIIa) are both conservative, Lemma 1 and thence Case A, Case B and Case C in Section 3 and <u>*all*</u> of the results stemming from them become restricted to cases for which $s_i^+=s_i^-=0$.

Via Lemma 1A Cases A, B and C become:

<u>**CASE A*:**</u>

(III),(IIIa) become identical with z=z' via $\Delta x_i=0$ all i in (IIIa).

<u>**CASE B***</u>:

(III),(IIIa) become equivalent but not identical with z=z' and $\Delta x_i \neq 0$ at least one i.

<u>**CASE C***</u>:

(III) is suboptimal relative to (IIIa) via z<z' and at least one $\Delta x_i \neq 0$.

These specializations in turn imply more restrictive applicability for Lemma 2:

LEMMA 2A

If $s_i^+=s_i^-=0$, Lemma 2 becomes ***degenerate.*** (Lemma 2 requires that *either* $s^+=(b-g(x^*+\Delta x))$ *or* $s^-=(g(x^*+\Delta x)-b$. But when $s_i^+=s_i^-=0$ these two classes of cases *inevitably* apply simultaneously.)

The conservative restriction in Definition 6 also implies:

THEOREM 8 ***Conservative Conditions, Alternative Optima and Opposite Sign***

If Case B* applies then: i) x*,x** are alternative optima in (III) and (IIIa) and; ii) from Definition 3, unless $f(x^*+\Delta x_i)-f(x^*)=0$ (resp $g(x^*+\Delta x_i)-g(x^*)=0$ all i, there must exist $(f(x^*+\Delta x_i)-f(x^*))\Delta x_i$, $(f(x^*+\Delta x_j)-f(x^*))\Delta x_j)$ (resp $g(x^*+\Delta x_i)-g(x^*)$, $g(x^*+\Delta x_j)-g(x^*)$) of opposite signs.

PROOF

If Case B* applies then: i) $f(x^*+\Delta x)-f(x^*)=0$ so that any optimum to x* (III) is an alternative optimum to (IIIa) and vice versa; ii) from Definition 3:

$$df(x)= f(x^*+\Delta x)-f(x^*) =_{def}\Sigma[(f(x^*+\Delta x_i)-f(x^*))\Delta x_i$$

But $f(x^*+\Delta x)-f(x^*)=0$ in Case B*. Similar arguments apply via Definition 3 and dg(x)=0 so that the result follows in both cases.

If (III) and (IIIa) are conservative in the sense of Definition 6, optima to (III) and (IIIa) must be consistent with Case B*, Lemmas 1A, 2A and Theorem 8. These results in turn restrict the applicability of Theorems 2-7 as summarized below.

[In what follows ***Degeneracy***=>degeneracy via Lemma 2A; ***AA*** =>alternative optima as in Theorem 8 and; **NSM** =>not strictly maximal.):

- Theorem 2 *(More for Less/Less for More).* ***AA****, i.e.* applies only via the more for nothing and less for nothing special cases.
- Theorem 3 *(Disaggregated Kuhn Tucker Conditions).* ***AA, Degeneracy***.
- Theorem 4 *(Aggregate Kuhn Tucker Conditions).* ***AA, Degeneracy*** and **NSM**.
- Theorem 5 *(Concavity and Relative Maxima).* ***AA.***
- Theorem 6 *(Concavity and Sufficient Conditions For Maxima Relative to (III),(IIIa)).* ***AA.*** (Recall that, if $h^+s^++h^-s^-=0$, Theorems 5 and 6 become equivalent so that, if (IIIa) is conservative, Theorems 5 and 6 become equivalent a fortiori.)
- Theorem 5A *(Complementary Slackness, Continuity, Concavity and Sufficient Conditions for Relative Maxima).* ***AA, Degeneracy.***
- Theorem 6A *(Complementary Slackness, Concavity and Disaggregated Sufficient Conditions for Maxima Relative to (III),(IIIa).* If (IIIa) ***Degeneracy,*** **NSM.**
- Theorem 6B *(Disaggregated Discrete and Concave Kuhn Tucker).* ***AA, Degeneracy.***
- Theorem 7 *(Continuous Kuhn Tucker)* ***AA, Degeneracy.***
- Theorem 7A *(Continuous Kuhn Tucker Decision Rules).* ***AA, Degeneracy.***

REMARKS

- Notice that, while the degenerate results in Theorems 5A, 6A and 6B follow via Theorem 5 or Theorem 6, results in Theorems 5 and 6 are *not* necessarily degenerate.
- The degenerate restrictions in Theorems 7 and 7A are consequences of the derivation of those results via Theorem 6A.
- From Theorem 8 in Case B*, x*, $(x+\Delta x)^*$ are alternative optima. Nevertheless, if f(x) are strictly concave (as they may be in Theorems 5-7 above), then there will be values such that $f(x^*+\theta\Delta x)>f(x^*)$ *and* $f(x^*+\theta\Delta x)> f(x+\Delta x)^*$ where $\Delta x \neq 0$, $0<\theta<1$.

Summarizing: under the conservative condition in Definition 6, Theorems 2-7 <u>all</u> become such that one or both of the ***AA*** and the ***degeneracy*** restriction applies. [In all cases the opposite sign property in Theorem 8 also applies.]

Before considering economic applications and interpretations of these results in the next section notice that Definition 6 might be strengthened via Definition 6A below to imply a stronger opposite sign property than that in Theorem 8. That in turn leads to two paradoxes as follows:

DEFINITION 6A *Conservative* Systems and Opposite Signs*

A system (IIIa) is conservative* iff $s_i^+=s_i^-=0$ all i <u>and</u> $\Delta x_i\neq 0$ any i implies the existence of at least one quantity $\Delta x_j\neq 0$ such that $\Delta x_i+\Sigma\Delta x_j=0$.

REMARK

This definition is not inconsistent with a condition $\Delta x_i+\Delta x_j=0$. That condition in turn is consistent with interpretations *either* as if these two quantities Δx_i, Δx_i are identical (in which case $\Delta x_i\equiv 0,\Delta x_j\equiv 0$), *or* as if the two quantities $\Delta x_i,\Delta x_j$ are equal and opposite in relative sign. More generally any conservative* system has an *opposite sign property* such that, if $\Delta x_i>0$ (resp $\Delta x_i<0$) at least one i, then there must exist at least one quantity $\Delta x_j<0$ (resp $\Delta x_j>0$) of equal and opposite magnitude to it.

THEOREM 9 *A Paradox of Conservatism**

For a case with two variables i=1,2, conservative* optima to (IIIa) will be such that, *unless* $f(x^*+\Delta x_i)-f(x^*)=0$, i=1,2 (resp $g(x^*+\Delta x_i)-g(x^*)=0$, i=1,2), then $f(x^*+\Delta x_i)-f(x^*)\neq 0$ i=1,2 (resp $g(x^*+\Delta x_i)-g(x^*)\neq 0$ i=1,2) must be equal and of *the same* sign and in that sense *not* conservative.

PROOF

A conservative* optimum to (IIIa) is consistent with Case B* and so such that:

$$df(x)=f(x^*+\Delta x)-f(x^*)=0=(f(x^*+\Delta x_1)-f(x^*))\Delta x_1 +(f(x^*+\Delta x_2)-f(x^*))\Delta x_2$$

In the two variable case a conservative* system is, by Definition 6A, also such that $\Delta x_1,\Delta x_2$ are equal and *opposite* in relative sign. With a similar argument via $dg(x)=0$ this gives the required result.

A similar paradox stems from the more general conservative case via Definition 6 as follows:

THEOREM 9A *A Paradox of Conservatism*

For a case with two variables i=1,2, conesrvative optima to (IIIa) will be such that, unless $f(x^*+\Delta x_i)-f(x^*))\Delta x_i=0$ (resp $(g(x^*+\Delta x_i)$ $-g(x^*))\Delta x_i)=0$ both i, then $f(x^*+\Delta x_i)-f(x^*))\Delta x_i\neq 0$ $i=1,2$ (resp $g(x^*+\Delta x_i)-g(x^*)\Delta x_i\neq 0$ $i=1,2$) must be *either*: i) such that $\Delta x_1,\Delta x_2$ opposite in relative sign and $f(x^*+\Delta x_1)-f(x^*)$, $f(x^*+\Delta x_2)-f(x^*)$ of *the same* sign and in that sense *not* conservative, or: ii) such that $\Delta x_1,\Delta x_2$ *the same* in relative sign with $f(x^*+\Delta x_i)-f(x^*))\Delta x_i\neq 0$ $i=1,2$ (resp $g(x^*+\Delta x_i)-g(x^*)\Delta x_i\neq 0$ $i=1,2$) *opposite* in relative sign and in *that* sense not conservative.

PROOF
The result under i) follows from an argument similar to that in Theorem 9. (In this context the fact that under conservative* conditions $\Delta x_1,\Delta x_2$ may be of unequal magnitude is not significant). The result under ii) follows because a conservative optimum to (IIIa) will be such that $f(x^*+\Delta x)-f(x^*)=0$ and $g(x^*+\Delta x)-g(x^*) = 0$.

REMARKS *(Conservatism and paradox)*

- One implication of Theorems 9 and 9A is that, if there is to be a gain $(f(x^*+\Delta x_i)-f(x^*))\Delta x_i>0$ at an optimum, there must necessarily be a loss via one or both payoffs $f(x^*+\Delta x_i)-f(x^*)$ or via one or both instruments Δx_i.
- Theorems 9 and 9A can be extended straight-forwardly so that a net *gain* $(f(x^*+\Delta x_i)-f(x^*))\Delta x_I>0$ (resp $g(x^*+\Delta x_i)-g(x^*)\Delta x_i>0$) implies a comple-mentary *loss* $\Sigma(f(x^*+\Delta x_j)-f(x^*))\Delta x_j$ $j\neq i$.
- Both Theorems 9 and 9A suggest interpretations in relation to processes of change to determine optimizing gains and losses relative to individuals or groups r and s. That in turn suggests economic applications of these results and potentially degenerate and alternative optimal and para-doxical implications.

9. An economic interpretation of conservative conditions,degeneracy and opposite signs

Consider an application of Theorem 1 taking the form of (IV),(IVa) below in which f(x) corresponds to an individual preference relation U(x) defined over consumption commodities x of which that individual has initial endowments b. With s_i^+,s_i^- as measures of interaction with a wider system, Theorem 1 implies that only exceptionally will this individual not prefer an optimum with positive interaction with a wider system (via $s_i^+>0,s_i^->0$ some i) to no interaction with that wider system (via $s_i^+=s_i^-=0$ all i). Theorem 1 implies further that an individual's propen-sity to interact with a wider system (via optima such that $s_i^+>0,s_i^->0$ some i in (IVa)) will increase if the effective effort required to do so, as measured by one or more of the parameters h_i^+,h_i^-, is sufficiently reduced. In these ways such an individual would at least weakly prefer *nonconservative* conditions with $s_i^+>0,s_i^->0$ some i in (IVa) to conser-vative conditions $s_i^+=s_i^-=0$ all i in (IV) - even if that led to unbalanced transfers (e.g. via unreciprocated gifts) to or from a relatively larger system.

$$\begin{array}{lcl} \text{Max } U(x)-Ms^+-Ms^- = U(x^*) & = z \le z' = & U(x^*+\Delta x)-h^+s^+-h^-s^- = \text{Max } U(x+\Delta x)-h^+s^+-h^-s^- \\ \text{st } g(x)+s^+-s^-=b \quad \text{(IV)} & & \text{st } g(x+\Delta x)+s^+-s^-=b \quad \text{(IVa)} \\ x,s^+,s^-\ge 0 & & x,s^+,s^-\ge 0 \end{array}$$

In these ways variables s_i^+,s_i^- have potential interpretations with reference not just to gains from gifts or exchanges of objects, but also to *learning* relative to otherwise potentially undiscovered elements of a wider system. Explicitly learning related extensions of (IV) and (IVa) will clarify this.

$$\begin{array}{lcl} \text{Max } U(x,x_0)-Ms_i^+-Ms_i-Ms_0^{--} & = z_1 \le z_2 & = \text{Max } U(x,x_0)-h^+s_i^+-h^-s_i^--h_0^-s_0^- \\ \text{st } g(x,x_0)+s_i^+-s_i^-=b \quad \text{(V)} & & \text{st } g(x,x_0)+s_i^+-s_i^-=b \quad \text{(Va)} \\ x_0-s_0=0 & & x_0-s_0^-=0 \\ x,x_0,s^+,s^-,s_0^-\ge 0 & & x,x_0,s^+,s^-,s_0^-\ge 0 \end{array}$$

With s_0^- in (V),(Va) interpreted as referring to relatively unknown inputs i=0 Theorem 1 implies that, other things equal, *chosen knowledge of* inputs i=0 via $s_0^- > 0$ will be at least weakly preferred to chosen ignorance of such inputs. [This is consistent with rational learning behaviour according to which additional knowledge would not be chosen unless it was understood to be preference enhancing.]

While individuals may potentially gain from gifts, exchanges and/or learning relative to an essentially impersonal wider system via variables s_i^+, s_i^-, s_0^-, they may also give, exchange and learn in a more personal way from other individuals. To see this consider another interpretation of (III),(IIIa) which has reference both to potentially Pareto enhancing gifts and exchanges between two individuals r=1,2 via variables x_{ri}^{12}, x_{ri}^{21} *and* with reference to potentially Pareto improving interactions with a wider environment via framing variables s_{rj0} and s_{r0}^- as follows:

$$
\begin{array}{ll}
\text{Max } U_1(y_{11}, y_{1j}, y_{1n}, y_{10}) - \Sigma h_{1j0} s_{1j0} - h_{10}^- s_{10}^- - \Sigma h_{1i}^{12} x_{1i}^{12} & + U_2(y_{21}, y_{2j}, y_{2n}, y_{20}) - \Sigma h_{2j0} s_{1j0} - h_{20}^- s_{10}^- - \Sigma h_{2i}^{21} x_{2i}^{21} \\
\quad \text{st } y_{1j} + s_{1j0} = g_{1j}(x_{1ij}, .. x_{10j}) & \\
\quad \Sigma x_{1ij} + x_{1i}^{12} - x_{2i}^{21} = x_{1i}^* & \\
\quad \Sigma x_{10j} + x_{10}^{12} - x_{20}^{21} = x_{10} & \qquad\qquad\qquad\qquad\qquad \text{(VIa)} \\
\quad x_{10} - s_{10}^- = 0 & \\
 & y_{2j} + s_{2j0} = g_{2j}(x_{2ij}, .. x_{20j}) \\
 & \Sigma x_{2ij} + x_{2i}^{21} - x_{1i}^{12} = x_{2i}^* \\
 & \Sigma x_{20j} + x_{20}^{21} - x_{20}^{12} = x_{20} \\
 & x_{20} - s_{20}^- = 0 \\
\quad U_1(\;) \geq U_1(\;)^* & U_2(\;) \geq U_2(\;)^*
\end{array}
$$

All variables nonnegative

First consider potentially Pareto improving gifts or exchanges between these individuals. If $U_1(\;)^*$ and $U_2(\;)^*$ correspond to optimal solutions to (VIa) in which h_{rj0}, h_{r0}^- are all arbitrarily large then a solution to (VIa) in which h_{rj0}, h_{r0}^- all arbitrarily large but h_{ri}^{12}, h_{ri}^{21} *not* all arbitrarily large potentially yields Pareto improving *economies of scope* via optimally positive values for one or more of the associated quantities x_{ri}^{12}, x_{ri}^{21}. In that case, if optimally $U_1(\;) > U_1(\;)^*$ and/or $U_1(\;) > U_1(\;)^*$, those inequalities will arise as if via chosen gift or exchange related connections between previously unconnnected units (e.g. individuals, regions, markets) r=1,2. [For more on such gift and exchange related cases see chapters 11 and 12 where I also relate systems analogous to (VIa) to new distinctions between *industrial contestability* and *market contestability* and in that way reconcile mutually inconsistent definitions of contestability in Baumol et al 1982, Shepherd 1984 and Cairns 1996.]

Next consider the variables s_{rj0}. These are analogues in (VIa) of framing variables s^+ in (I),(Ia) through (III),(IIIa), and correspondingly in Theorems 1 through 9. In the context of (VIa) s_{rj0} take on interpretations as *underconsumption* of commodities j by individuals r. With that interpretation a solution with h_{rjo} arbitrarily large would in effect *force* consumption of all that is produced (i.e. in effect force s_{rj0} to zero all r,j) as long as that outcome was feasible. Conversely, with h_{rjo} *not* arbitrarily large (e.g. zero), oppor-tunities to underconsume in this sense (e.g. via free disposal of any surplus) would be at least weakly Pareto preferred.

Finally consider s_{r0}^-. In the gift and exchange related context of (V) these quantities potentially yield a rich variety of meanings. Here I focus on meanings according to which via $s_{r0}^- > 0$ hitherto unavailable information concerning commodities i is correspondingly newly made known to individuals r. In that way individuals r=1,2 may learn in potentially Pareto improving ways from interactions additional to those potentially generating communications between those individuals via x_{ri}^{12}, x_{ri}^{21}, i≠0.

System (VIa) also captures the fact that learning via s_{r0}^- may have implications both for processes of production via x_{r0j} in the production relations $g_{rj}(x_{rij},..x_{r0j})$ and for new elements of communication via x_{r0}^{12},x_{r0}^{21}. In those ways relatively externally induced learning via s_{r0}^{-} may introduce productivity enhancing technological change as well as possibilities of producing entirely new kinds of commodities y_{r0}. In addition, via s_{r0}^{-} in (VIa), knowledge of new commodities, and so implicitly knowledge of the new processes with which they may be produced, may be conveyed to another individual by means of gifts or exchanges x_{ri}^{12},x_{ri}^{21} r=1,2. [I emphasize that exchanges of knowledge between individuals within (VIa) are also potentially Pareto enhancing in learning related ways. Indeed x_{ri}^{12},x_{ri}^{21} in (VIa) can be understood as analogues of (parts of) uncertainty and learning related variables s_i^{+} and s_i^{-}, in (Va).]

So far attention has been concentrated on implications of changing magnitudes of the framing parameters h_{ri}^{12},h_{ri}^{21} (resp h_{rj0},h_{r0}^{-}) for changes in x_{1i}^{12},x_{2i}^{21} (resp s_{rj0}, s_{r0}^{-}) and so for changes in the magnitudes of other variables in (VIa). But such changes have other implications. As one way of appreciating these consider (VIa) as a special case of Theorem 1A in Section 2 as follows:

THEOREM 10

If in (VII): i) a feasible solution with all $s_{rj0}=0$, all $s_{r0}^{-}=0$ and all $x_{1i}^{12}=0$, all $x_{2i}^{21}=0$ and $U_r(\ )\geq U_r(\ )^*$, r=1,2 exists then:

$$\begin{array}{c}\text{Max } U_1(y_{11},y_{1j},y_{1n},y_{10}) - \Sigma M s_{1j0} - M s_{10}^{-} - \Sigma M x_{1i}^{12} - \Sigma M x_{1i}^{21} + U_2(y_{21},y_{2j},y_{2n},y_{20}) - \Sigma M s_{1j0} - M s_{10}^{-} - \Sigma M x_{2i}^{12} - \Sigma M x_{2i}^{21} \\ \text{st constraints of (VIa)} \qquad\qquad \text{(VII)}\end{array}$$

$$= z \leq z' =$$

$$\begin{array}{c}\text{Max } U_1(y_{11},y_{1j},y_{1n},y_{10}) - \Sigma h_{1j0}s_{1j0} - h_{10}^{-}\, s_{10}^{-} - \Sigma h_{1i}^{12}x_{1i}^{12} - \Sigma h_{1i}^{21}x_{1i}^{21} + U_2(y_{21},y_{2j},y_{2n},y_{20}) - \Sigma h_{2j0}s_{1j0} - h_{20}^{-}\, s_{20}^{-} - \Sigma h_{2i}^{12}x_{2i}^{12} - \Sigma h_{2i}^{21}x_{2i}^{21} \\ \text{st constraints of (VIa)} \qquad\qquad \text{(VIIa)}\end{array}$$

PROOF

A feasible solution to (VII) exists and is feasible but not necessarily an optimal solution to (VIIa). for Theorem 1A.

The relation of Lemma 1B to Theorem 10 is analogous to the relation of Lemma 1 to Theorem 1A in Section 2:

Let $\{y^*,x^*\},\{y^{**},x^{**}\}$ be optimal evaluations of y via (VII) and (VIIa). It follows that, unless these evaluations are identical then $y_j^*\neq y_j^{**}$ and/or $x_i^*\neq x_i^{**}$ some i. Equivalently $y_i^{**}=_{def}y_i^*+\Delta y_i$, $x_i^{**}=_{def}x_i^*+\Delta x_i$. With this notation, at an optimum (VII) and (VIIa) become analogous to (III),(IIIa) and such that:

LEMMA 1B

$$\begin{array}{ll}\text{Max } f(y)=f(y^*) & = z \leq z' = \text{Max } f(y^*+\Delta y) - \Sigma h_{1i}^{12}x_{1i}^{12} - \Sigma h_{1i}^{21}x_{1i}^{21} - \Sigma h_{rj0}s_{rj0} - \Sigma h_{r0}^{-}\, s_{r0}^{-} \\ \text{st constraints of (VII)} \quad \textbf{(VIII)} & \qquad\qquad \text{st constraints of (VII)} \quad \textbf{(VIIIa)}\end{array}$$

Given the representation of (VII) and (VIIa) as in (VIII),(VIIIa) and the analogy between the latter systems and (III),(IIIa), all of the results in Sections 2 through 8 can be applied to (VIII),(VIIIa). In that way a corresponding variety of Kuhn Tucker conditions are generated for various forms of the relations f( ) and g( ) in those systems. Rather than develop all of these results with reference to the two person exchange related example in (VII),(VIIa), I note just four implications of those systems in that application:

- **Conservatism** From Theorem 6B and Definition 6 (for the discrete and concave case) and from Theorem 7 and definition 6A (for the continuous and concave case), if (VIIa) is conservative or conservative* in the following sense then an optimal solution is degenerate and so decomposable. Assume that in (VIIa): i) s_{rj0} has the interpretation of under-consumption of commodity j by individual r,; ii) s_{r0}^{-} has the interpretation of relatively external information concerning technology available to individual r and; iii) x_{ri}^{12},x_{ri}^{21} have interpretations as quantities of commodities i transferred (donated, bartered or traded) from individual 1 to individual 2 (resp 2 to 1). With those interpretations a conservative solution in the sense of Definition 6 implies that all of s_{rj0} and of s_{r0}^- would be zero. That is: no free disposal relative to a wider system and no technological information from a relatively external system. If also x_{ri}^{12} and x_{ri}^{12} were interpreted as corresponding to variables analogous to s_i^+,s_i^- in Theorems 6B and 7 so that all $x_{ri}^{12}=0$ and all $x_{ri}^{21}=0$ in (VIIa) under the conservative conditions of Definition 6, then there would be no positive exchanges between individuals 1 and 2. Under those conditions any conservative solution to (VIIa) would necessarily be not only *decomposable* with the two individuals consuming only their own endow-ments, but also *degenerate* because then all of the gift, exchange exchange or trade related variables x_{ri}^{12},x_{ri}^{21} would become redundant.

- **Conservatism*** Now consider a more narrowly conservative system in which the definition of a conservative system (Defin-ition 6 above) is applied to (VIIa) only through variables s_{rj0},s_{r0}^- i.e. for which only s_{rj0},s_{r0}^- are considered as relatively exterior to the system under consideration. Then solutions to (VIIa) consistent with Theorem 7 may be conservative* in the sense of Definition 6A, namely such that $x_{1i}^{12}-x_{2i}^{12}=0$ and $x_{1i}^{21}-x_{2i}^{21}=0$ for at least one $x_{ri}^{12}>0$ or $x_{ri}^{21}>0$. In context such a system corresponds to one in which one or more individuals may secure a Pareto improving gain by gift, barter or trade relative to the states associated with their initial preference evaluations $U_2(\ )^*,U_2(\ )^*$. [Such a solution can also illustrate how a conservative paradox consistent with the conditions of Theorem 9 or Theorem 9A may apply. Consider a single gift $x_{1i}^{12}>0$ from individual 1 to individual 2 so that $x_{1i}^{12}=x_{2i}^{12}>0$. Arguably the system as a whole is conservative in the sense that this gift is simply a transfer between two individuals within it. But, unless both individuals are indifferent to such a transfer, then $U_r(\ )>U_r(\ )^*$ for at least one individual r, with the consequence that the system described by (VIIa) will *not* in that sense be conservative.

- **Self contradiction** Any individual choosing to undertake a process leading to gifts, bartetr or trade with reference to their initial endowment of commodities *must* act as if purposively to have less if they want to guarantee more for another and, conversely, an individual *must* act as if to ensure that another has less if they want to guarantee more for self. [For more on this see Ryan 1992.] This in turn leads directly to a fourth implication;

- **Uncertainty** Even within a conservative system each of two individuals may prefer gifts or exchanges relative to each other and thus choose to subject themselves to conditions of less in order to get more in a manner consistent with conservative paradoxes and processes of purposive contradiction of kinds just considered. In general however such behaviours will not fully exploit the opportunities presented by the conditional nature of the conservatism of any system and the consequent uncertainty of the boundary of such a system. Specifically: *both* may prefer a relatively unknown alternative outside the system. As examples: one individual may prefer to give otherwise unobtainable commodities to another in contradiction of the conservation of their initial endowments as if wholly and only relative to themselves, or; two individuals may choose to prefer opportunities beyond their own collectively given initial boundaries. A key point here is that in order to define any system *as a system* individuals must be able to define it as not greater than that system. But a necessity for a constraint to preclude the possibility of "greater" outcomes beyond the boundaries of that consequently explicitly constrained system *itself explicitly* implies that such a "greater" system is possible. Further if both individuals potentially prefer such a "greater" outcome then both have a collective interest in generating the possibility of actualizing such an outcome. In such circumstances both individuals have an incentive to cooperate as if thereby collectively to *grow* the system which they have acted as if unanimously to constrain. With an economic context one way of doing this is by both acting as if to have financially less relative to themselves as if thereby unanimously to potentiate physic-ally more relative to themselves e.g. by savings and/or by reductions in taxes. Another way is to commit resources to exploring beyond the current boundaries of their system.

10. CONCLUSION

In this chapter I have focused on Kuhn Tucker conditions and generalizations of them with applications to more for less and more for nothing results, as the developments in Chapter I have already demonstrated. Kuhn Tucker conditions and associated generalizations stemming from (VIIa) as from other models in this paper potentially yield interpretations both with reference to prices and with reference to taxes/subsidies and with reference to learning. These ideas are all developed in subsequent chapters within specifically economic contexts of inherently self contradictory and incompletely informed processes of giving, bartering and trading trading.

In the latter part of the chapter I have focused on learning related applications in contexts of conservative restrictions to obtain degeneracy and alternate optima conditions as consequences of such classes of more for less or more for nothing solutions. In concluding the paper I emphasize two learning related ideas. First: if a system is not conservative in the sense used in this chapter, then there is room for innovation from sources relatively *exterior* to that nonconservative system. Secondly: even if a system is conservative overall there can nevertheless be scope for processes of learning and innovation from sources relatively *interior* to that conservative system.

It follows that, especially if preferences are defined with reference to processes of exchange of commodities, then individuals can potentially learn/innovate relative to each other. In particular a quantity relatively inside one part of a system may transmit itself/be transmitted to become that quantity relatively outside another part of that system, leading to interpretations with reference to teaching and to learning and associated interpretations corresponding to relative ignorance and uncertainty.

Finally, the focus in this chapter on Kuhn Tucker conditions in turn focuses on interpretations of multipliers λ as *potentials*. While these are not essential for the description and analysis of gift and/or exchange related cases these, too, are information bearing. In that connection the opposite sign results of the concluding part of Section 8 are particularly suggestive since opposite signs potentially relate directly to processes of purposive contradiction, in turn corresponding to signalling and dually related processes of interaction between individuals. These will be key ideas in subsequent chapters.

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