CHAPTER 9

INTERVENING DUALITY BASED ANALYSES OF NONCONSTANT SUM BIMATRIX GAMES

1. Introduction

I introduced the intervening duality idea in Ryan 1995 with a context of two person constant sum coin tossing games. After briefly reviewing that paper in Section 2, I extend it to the Prisoner's Dilemma and Cournot, Stackelberg and collusion classes of non-constant sum duopoly games in Sections 3 and 4. Then, in Section 5, I use strategic equivalence to extend these respectively constant sum and symmetric nonconstant sum character-izations and analyses to wider classes of duopoly related nonconstant sum bimatrix games.

In every case, and in distinction from more standard treatments, such as recent work by Cooper et al 1996, Hoekstra 1995 or Daniel 1994 for the prisoner's dilemma, the focus on intervening duality in this paper provides a means of explicitly modelling cooperative or noncooperative inter-actions between players. The intervening duality idea augments and enriches existing results and generates new ones by explicitly modelling essentially subjective processes of interaction between players and games by considering each player as if each choosing to make themselves dual to one of the duals of an intervening bimatrix game.

While the treatment in this paper concentrates on pure strategy cases the way is open for mixed strategy extensions in future work.

2. Intervening duality

The two person matching pennies game is well known (see Shubik 1982, Wang 1988). But nevertheless it has been interpreted in two quite distinct ways: as a game against nature and; as a game between two distinct and dually interrelating persons. In either case, with contingent payoffs as in Table 1, and assuming for simplicity that maximin/minimax behaviours are appropriate, the problems for the two players are implicitly assumed to be associated with interrelating dual programs as in (I),(I)':

	H	**T**
H	1	-1
T	-1	1

Table 1

$$\begin{array}{lll} \text{Max} & \rho - Mp^{+} - Mp^{-} & \\ \text{st} & \sum_{j\varepsilon J} \pi_{jk}\, p_j \geq \rho & \\ & \sum_{j\varepsilon J} p + p^{+} - p^{-} = 1 & \textbf{(I)} \\ & -M \leq \rho \leq M & \\ & p, p^{+}, p^{-} \geq 0 & \end{array} \qquad \begin{array}{lll} \text{Min} & \mu + Mq^{+} + Mq^{-} & \\ \text{st} & \sum_{k\varepsilon K} \pi_{jk} q_k \leq \mu & \\ & \sum_{k\varepsilon K} q + q^{+} - q^{-} = 1 & \textbf{(I)'} \\ & -M \leq \mu \leq M & \\ & q, q^{+}, q^{-} \geq 0 & \end{array}$$

As in Ryan 1995, (I),(I)' explicitly introduce preemptive *frames* via arbitrarily large weights M associated with any deviations p^+,p^-,q^+,q^-. In this if such solutions are feasible, optima will be restricted to subsets jεJ and kεK. (For ease of exposition I will assume throughout the next three sections that solutions jεJ,kεK within the relevant preemptive frames are not only feasible but optimal.)

With payoffs as in Table 1, optimal solutions to (I),(I)' yield $\rho=\mu=0$ with $p_1=p_2=1/2$ and $q_1=q_2=1/2$. This "fair" equiprobable outcome may seem unsurprising. But it arises from the nature of the contingent payoffs in Table 1, and not from any prior information or beliefs concerning contingent probabilities. If indeed Player 1 had such prior knowledge or beliefs, (I),(I)' would properly be modified to take this into account via a *constrained game* specification such as (II),(II)' below which introduces prior probabilities q_k^* and associated dual variables R_k as follows. (Constrained games were first introduced by Charnes 1953, see also Owen 1982 and Ryan 1994, where I considered strict and weak probability examples too).
In (II) the quantities R_k may take on a variety of interpretations at an optimum, including

interpretations as evaluators of prior information and/or interpretations in relation to relative bias, as well as in relation to prior/posterior distinctions. Notice that in each of these connections nature is essentially *benevolent* insofar as it potentially rewards prior information in this way. (By the principle of optimality (II)', being at least as tightly constrained as (I)', yields at least as high an overall solution.)

Now reconsider the two different matching pennies related interpretations referred to above. In the first nature has a distinct problem (problem (I)' or (II)'), whereas, if (I),(I)' or (II),(II)' are interpreted as a game between two persons, each of problem represents a person and there is no distinct problem for a player as nature. This observation immediately prompts the intervening duality idea. According to it the two person matching pennies interpretation is specified by setting one individual as dual to the coin and the other as dual to the dual of the (thereby intervening) coin. If, further, each individual imputes prior beliefs to such intervening duals via additional constraints relating to prior beliefs p'^{*}_{j} and q'^{*}_{k} respectively, the overall specification extends to include (III) and (III)':

$$\begin{array}{ll}
\text{Max} & \rho + \Sigma R_k q'^{*}_k - Mp^{+} - Mp^{-} \\
\text{st} & 1p_1 - 1p_2 - R_1 \geq \rho \\
& -1p_1 + 1p_2 - R_2 \geq \rho \\
& p_1 + p_2 + p^{+} - p^{-} = 1 \\
& -M^2 \leq R_k \leq M^2 \\
& -M \leq \rho \leq M \\
& p_j, p^{+}, p^{-} \geq 0
\end{array} \qquad \textbf{(II)}$$

$$\begin{array}{ll}
\text{Min} & \mu' + Mq^{+\prime} + Mq^{-\prime} + M^2 \Sigma(q_k^{+\prime} + q_k^{-\prime}) \\
\text{st} & 1q_1' - 1q_2' \leq \mu' \\
& -1q_1' + 1q_2' \leq \mu' \\
& q_1' + q_2' + q'^{+} - q'^{-} = 1 \\
& q_k' + q_k^{+\prime} - q_k^{-\prime} = q_k'^{*} \\
& -M \leq \mu' \leq M \\
& q_k', q_k^{+\prime}, q_k^{-\prime} \geq 0
\end{array} \qquad \textbf{(II)'}$$

$$\begin{array}{ll}
\text{Max} & \mu + \Sigma S_j p'^{*}_j - Mq^{+} - Mq^{-} \\
\text{st} & 1q_1 - 1q_2 - S_1 \geq \mu \\
& -1q_1 + 1q_2 - S_2 \geq \mu \\
& q_1 + q_2 + q^{+} - q^{-} = 1 \\
& -M^2 \leq S_j \leq M^2 \\
& -M \leq \mu \leq M \\
& q, q^{+}, q^{-} \geq 0
\end{array} \qquad \textbf{(III)}$$

$$\begin{array}{ll}
\text{Min} & \rho' - Mp^{+\prime} - Mp^{-\prime} - M^2 \Sigma(p_j^{+\prime} + p_j^{-\prime}) \\
\text{st} & 1p_1' - 1p_2' \leq \rho' \\
& -1p_1' + 1p_2' \leq \rho' \\
& p_1' + p_2' + p^{+\prime} - p^{-\prime} = 1 \\
& p_j' + p_j^{+\prime} - p_j^{-\prime} = p_k'^{*} \\
& -M \leq \rho \leq M \\
& p_j', p_j^{+}, p_j^{-} \geq 0
\end{array} \qquad \textbf{(III)'}$$

By construction (II),(II)' and (III),(III)' each potent-ally constitute an independently dually optimal pair of linear programs. But, when viewed in the more general intervening duality context, the system (II),(II)',(III),(III)' includes other possibilities.

First (II)',(III)' are potentially dually optimal to each other. Specifically, for the coin tossing case with $p_1=p_2=1/2$ and $q_1=q_2=1/2$, the additional constraints in (II)',(III)' are as if redundant and these two programmes constitute mutually dual solutions. Second, (II),(III) are potentially dual to each other. Indeed they are consistent not just with optimal solutions $p_1=p_2=1/2$, $R_1=R_2=0$, $\rho=0=\mu$, $q_1=q_2=1/2$, $S_1=S_2=0$, but with $p_1=p_2=1/2$, $R_1=1, R_2= -1$, $\rho=0=\mu$, $q_1=q_2=1/2$, $S_1= -1, S_2=1$. Together these various potential dualities provide the basis for rationalization of a "fair" coin tossing game as follows:

> Two players each independently and freely accept: i) an agreed payoff matrix (as in Table 1); ii)an agreed frame (via preemptive weights to restrict contingent payoffs to sets $j\varepsilon J$, $k\varepsilon K$ respectively); and iii) agreed prior probabilities (agreement that the coin in question is unbiased relative to the system). From this specification it follows that (II)',(III)' are conditionally dual with that condition being in effect that, both relative to the system and relative to players 1 and 2, $R_1=R_2=S_1=S_2=0$.
>
> Given this specification, via potential dualities of (II),(II)' and (III),(III)', either or both of the players may act as if potentially to generate contingent gains relative to such an ostensibly "fair" prior specification. Without loss of generality player 1 may perceive net advantage in backing heads, and player 2 in backing tails with relatively private

evaluations as if respectively $R_1=1, R_2= -1$, $S_1= -1, S_2=1$.

Notice that a sequence of such interrelationships and solutions would be consistent with each class of interpretations in relation to the variables R_k,S_j referred to above. E.g. for player 1 $R_1=1,R_2= -1$ have interpretations both in relation to prior information or beliefs q_k^* concerning probabilities of the two potential outcomes relative to the system and concerning relatively prior/posterior interpretations. A third class of interpretations in relation to relative *bias* becomes clearer if it is noted that, as specified, systems (II)',(III)' are degenerate (see Charnes 1951), and that partial regularizations to resolve such degeneracies are potentially consistent with interpretations in relation to relatively favourable or unfavourable bias.

For example: the degeneracy of (II)' for values $q_1=1/2,q_2=1/2$ is removed if these right hand sides are perturbed to give $q_1=1/2+\in,q_2=1/2-\in$. In that case the objective of the relatively dual programme (II) potentially yields a relative gain, or, equiv-alently, a measure of relative *bias* of $(R_1-R_2)\in$ relative to the first player. Similar developments and interpretations apply, via S_1,S_2, to (III),(III)' and player 2.

The specification (II),(II)',(III),(III)' also yields contingent duality between (II),(III). In context it is consistent with a coin tossing story as follows:

Via (II),(III) and $R_1=1,R_2=-1,S_1 = -1,S_2=1$, two individuals discover that they have *opposite* prior beliefs concerning heads and tails probabilities q_k^*,p_j^*. They perceive that, via corresponding quantities R_k,S_j, there is an opportunity for each to gain relative to the relatively neutral intervening specification (II)',(III)'.

For more on coin related intervening duality specifications again see Ryan 1995. Here I extend the intervening duality idea to non constant sum cases, first with symmetric payoffs, as in the Prisoners Dilemma, and then by considering more general classes of non symmetric nonconstant sum cases.

3. Intervening duality and the prisoners dilemma

The Prisoner's Dilemma game is well known (see Cooper 1996, Daniel and Arce 1994, Hoekstra 1995, Moyer and McGuigan 1993). It can be summarized briefly as: A crime has been committed. Two suspects have been apprehended and must choose whether or not to confess to it with the knowledge that their contingent sentences (in years) would be as in Table 2 (taken from McGuigan and Moyer p477):

		SUSPECT 1	
		Confess	Dont Confess
SUSPECT 2	Confess	6,6	0,15
	Dont Confess	15,0	1,1

Table 2

Clearly this game is not constant sum. Clearly, too, there are incentives to cooperate. Finally, when considered as a non repeated game, the prisoner's dilemma game is naturally analyzed solely with reference to pure strategy (either confess or dont confess) outcomes and in general players' decisions will depend on how, if at all, information is made available to each concerning the other's decision. (I will consider mixed strategies in Section 5.) Now distinguish interactive from noninteractive and cooperative from noncooperative cases and consider links to the intervening duality idea in stages as follows:

***STAGE 1 As if noncooperative and non-interactive cases*.** Assume that Suspect 1 seeks to minimize his/her maximum potential sentence in response to the explicit imputation of probabilities $q_k'^*$ to the second suspect's confess/dont confess strategies in (IV), (IV)' below. If :

i)$q_1'=1,q_2'=0$, $\mu'=0$ in (IV)' this is consistent with an optimal solution to (IV) with $p_1=0,p_2=1$, $\rho=0,R_1=0,R_2=-1$;

ii)$q_1'=0,q_2'=1,\mu'=1$ in (IV)' this is consistent with an optimal solution to (IV) with $p_1=0,p_2=1$, $\rho=1,R_1=0,R_2=0$.

$$
\begin{aligned}
\text{Min}\ \ & \rho + \Sigma R_j q_j'^{*} - Mp^{+} - Mp^{-} \\
\text{st}\ \ & 6p_1 + 0p_2 - R_1 \le \rho \\
& 15p_1 + 1p_2 - R_2 \le \rho \qquad \textbf{(IV)} \\
& p_1 + p_2 + p^{+} - p^{-} = 1 \\
& -M^2 \le R_k \le M^2 \\
& -M \le \rho \le M \\
& p_j, p_j^{+}, p_j^{-} \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Max}\ \ & \mu' - Mq^{+'} - Mq^{-'} - M^2 \Sigma(q_j^{+'} + q_j^{-'}) \\
\text{st}\ \ & 6q_1' + 15q_2' \ge \mu' \\
& 0q_1' + 1q_2' \ge \mu' \qquad \textbf{(IV)'} \\
& q_1' + q_2' + q^{+'} - q^{-'} = 1 \\
& q_k' + q_k^{+'} - q_k^{-'} = q_k'^{*} \\
& -M \le \mu' \le M \\
& q_k', q^{+'}, q^{-'}, q_k^{+'}, q_k^{-'} \ge 0
\end{aligned}
$$

Solutions i,ii) reflect the strategically dominant character of dont confess over confess strategies for player 1. (In i)R_2=-1 is a also a measure of contingent value of prior dont confess information relative to Player 2.) Analogous specifications (V),(V)' for player 2 lead to analogous conclusions as follows:

$$
\begin{aligned}
\text{Min}\ \ & \mu + \Sigma S_j p_j'^{*} - Mq^{+} - Mq^{-} \\
\text{st}\ \ & 6q_1 + 0q_2 - S_1 \le \mu \\
& 15q_1 + 1q_2 - S_2 \le \mu \qquad \textbf{(V)} \\
& q_1 + q_2 + q^{+} - q^{-} = 1 \\
& -M^2 \le S_j \le M^2 \\
& -M \le \mu \le M \\
& q_k, q_k^{+}, q_k^{-} \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Max}\ \ & \rho' - Mp^{+'} - Mp^{-'} - M^2 \Sigma(p_j^{+'} + p_j^{-'}) \\
\text{st}\ \ & 6p_1' + 15p_2' \ge \rho' \\
& 0p_1' + 1p_2' \ge \rho' \qquad \textbf{(V)'} \\
& p_1' + p_2' + p^{+'} - p^{-'} = 1 \\
& p_j' + p_j^{+'} - p_j^{-'} = p_k'^{*} \\
& -M \le \rho' \le M \\
& p_j', p^{+'}, p^{-'}, p_j^{+'}, p_j^{-'} \ge 0
\end{aligned}
$$

If:

i) $p_1'=1, p_2'=0, \rho'=0$ in (V) ' this is consistent with an optimal solution to (V) with $q_1=1, q_2=0$, $\mu=0$, $S_1=0, S_2=-1$; ii)$p_1'=0$, $p_2'=1$, $\rho'=1$ in (V)' this is consistent with an optimal solution to (V) with $q_1=1, q_2=0$, $\mu=1, S_1=0$, $S_2=0$.

Again these solutions reflect the strategically dominant character of dont confess over confess strategies.

STAGE 2. Interactive cases. In fact (IV),(IV)' and (V),(V)' will interact if only because strategic decisions by one player will have implications for the payoffs of the other. An optimal solution to (V)' corresponds both to a prediction on the part of player 2 of choices of nature relative to player 2 and, by complem-entary slackness, of strategies and payoffs for player 1. If correct the latter prediction would yield conclusions consistent with optimal solutions to (IV). Similar considerations apply to (IV)' and player 1 vis a vis player 2.

There are three classes of interactive equilibria /disequilibria which correspond to the three different kinds of cells (1,1),{(0,15),(15,0)} and (6,6) in Table 2 as follows:

2A. Cooperative interactive equilibria. If $p_1'=p_1=0, p_2'=p_2=1$ the unique optimum for (V)' with $\rho'=1$ is perfectly predictive of a cooperatively interactive optimum with $\rho=1$, $R_1=0$, $R_2=0$ in (IV). This optimum is in turn perfectly predictive of an optimum (IV)' its dual, for which $\mu'=1, q_1'=0, q_2'=1$. But, with these values and conditions perfectly predict-ively $q_1'=q_1=1$ $,q_1'=q_2=0$, the correspond-ingly unique optimum to (IV)' with $\mu'=1$ is perfectly predictive of (V) with $\mu=1, S_1=0, S_2=0$ and thence of (V)' its dual. That in turn is perfectly predictive of the initial optimum to (IV). This sequence is not only essentially cooperative in the sense that it maximizes contingently individual and thence collective rewards, (with $R_1=0, R_2=0, S_1=0$, $S_2=0$), it also yields overall equilibria in the sense that each player's actions and reactions are as if perfectly predictive of the other's.

2B. Type 1 Non cooperative interactive dis-equilibria. If $p_1'=p_1=1, p_2'=p_2=0$ the unique optimum to (V)' with ρ‘=0, is perfectly predict-ive of a noncooperatively interactively optimum to

(IV) with $\rho=0, R_1=0, R_2=0$. This in turn is perfectly predictive of an interactive optimum to (IV)', its dual, with $\mu'=0$, $q_1'=1$, $q_2'=0$. But, with these values, and conditions $q_1'=q_1=1$, $q_2'=q_2=0$, this is predictive of a noninteractively optimum to (V) with $\mu=0$, $S_1=0, S_2= -15$ (if $q_2{*}=1$) and thence of (V)' its dual with $p_1'=0, p_2'=1$ and $\rho'=0$.
But these latter predictions ($p_1'=0, p_2'=1$) for player 2 relative to player 2 are *inconsistent with* the initial predictions ($p_1'=1, p_2'=0$) by player 1 relative to player 2 and, by contrast with 2A and Type 1 equilibria, this sequence corresponds to an overall *diseqilibrium* corresponding to cell (0,15) in Table 2. [A similar argument starts with $q_1'=q_1=1, q_2'=q_2=0$ and an optimum to (IV)' and leads to an overall disequil-ibrium corresponding to cell (15,0) in Table 2.]

2C Type 2 Noncooperative interactive dis-equilibria. This is a class of cases for which players seek to predict outcomes by reference to the *other's* contingent payoff matrices, ie to predict each others' decisions as if via an intervening duality framework as follows:

$$\begin{array}{ll} \text{Min} & \rho + \Sigma R_j q_j'^{*} - Mp^{+} - Mp^{-} \\ \text{st} & 6p_1 + 15p_2 - R_1 \le \rho \\ & 0p_1 + 1p_2 - R_2 \le \rho \qquad \textbf{(IV)*} \\ & p_1 + p_2 + p^{+} - p^{-} = 1 \\ & -M^2 \le R_k \le M^2 \\ & -M \le \rho \le M \\ & p, p^{+}, p^{-} \ge 0 \end{array}$$

$$\begin{array}{ll} \text{Max} & \mu' - Mq^{+\prime} - Mq^{-\prime} - M^2\Sigma(q_j^{+\prime} + q_j^{-\prime}) \\ \text{st} & 6q_1' + 0q_2' \ge \mu' \\ & 15q_1' + 1q_2' \ge \mu' \qquad \textbf{(IV)*'} \\ & q_1' + q_2' + q^{+\prime} - q^{-\prime} = 1 \\ & q_k' + q_k^{+\prime} - q_k^{-\prime} = q_k^{*} \\ & -M \le \mu' \le M \\ & q_k', q^{+\prime}, q^{-\prime}, q_k^{+\prime}, q_k^{-\prime} \ge 0 \end{array}$$

$$\begin{array}{ll} \text{Min} & \mu + \Sigma S_j p_j'^{*} - Mq^{+} - Mq^{-} \\ \text{st} & 6q_1 + 15q_2 - S_1 \le \mu \\ & 0q_1 + 1q_2 - S_2 \le \mu \qquad \textbf{(V)*} \\ & q_1 + q_2 + q^{+} - q^{-} = 1 \\ & -M^2 \le S_j \le M^2 \\ & -M \le \mu \le M \\ & q, q^{+}, q^{-} \ge 0 \end{array}$$

$$\begin{array}{ll} \text{Max} & \rho' - Mp^{+\prime} - Mp^{-\prime} - M^2\Sigma(p_j^{+\prime} + p_j^{-\prime}) \\ \text{st} & 6p_1' + 0p_2' \ge \rho' \\ & 15p_1' + 1p_2' \ge \rho' \qquad \textbf{(V)*'} \\ & p_1' + p_2' + p^{+\prime} - p^{-\prime} = 1 \\ & p_j' + p_j^{+\prime} - p_j^{-\prime} = p_k'^{*} \\ & -M \le \rho' \le M \\ & p_j', p^{+\prime}, p^{-\prime}, p_j^{+\prime}, p_j^{-\prime} \ge 0 \end{array}$$

In this case, if $p_1'=p_1=1, p_2'=p_2=0$, an optimum to (V)*' with $\rho'=6$, is perfectly predictive of a non-cooperatively interactive optimum to (IV)* with $\rho=6, R_1=0, R_2=0$. This in turn perfectly predictive of an optimum to (IV)*', its dual, with $\mu'=6$, $q_1'=1, q_2'=0$. But, with these values, and conditions perfectly predictively $q_1'=q_1=1, q_2'= q_2=0$, this is perfectly predictive of (V)* with $\mu=6, S_1=0$, $S_2=0$ and thence of (V)*', its dual, with $p_1{\cdot}'=1, p_2'=0$ and $\rho'=6$. But predictions $p_1'=1 p_2'=0$ for player 2 relative to player 2 are consistent with the initial predictions ($p_1=1, p_2=0$) by player 1 relative to player 2. In that sense they correspond to an overall equilibrium. (A similar argument would start with $q_1'=q_1=1, q_2'=q_2=0$ and an optimal solution to (IV)*' and go to the same as if perfectly self predictive equilibrium solution [Confess,Confess].

Summarizing: With these intervening duality contexts in each of cases 2A and 2C, the associated processes are as if perfectly consistent with as if perfectly self predictive *cycles*, while for 2B, the associated processes are consistent with as if non self predictive cycles.

The cooperative solution 2A corresponds to as if mutually altruistic prediction. collective action. Case 2B is associated with as if independently/ noncooperatively selfish prediction and action. Case 2C is consistent with interactively non-cooperative/ selfish prediction and action.

4. Cournot/Stackelberg/collusion cases

In the previous section I focussed on the Prisoner's Dilemma. I now consider more general specifications before focussing on intervening duality processes and interpretations for symmetric Cournot/Stackelberg related cases. For concreteness, and without loss of generality, consider a game with payoffs as in Table 3:

		PLAYER 1	
		P1	P2
PLAYER 2	Q1	π_{111},π_{211}	π_{112},π_{221}
	Q2	π_{121},π_{212}	π_{122},π_{222}

Table 3

Table 3 comprehends constant sum and symmetric nonconstant sum cases, including the coin tossing case of Section 2 and the Prisoner's Dilemma cases of section 3, as types of special cases for which, respectively: i) $\pi_{ajk}+\pi_{bjk}$=constant; ii)$\pi_{ajk}=\pi_{bjk}$, j=k, $\pi_{ajk}=\pi_{bkj}$, j≠k.

Another class of symmetric cases correspond to Cournot/Stackelberg cases for which the strategies would be variously Stackelberg Leader and Stackelberg (Cournot) Follower for each of the two players. With corresponding notation Table 3 takes the form shown in Table 4 with $\pi_{111}=\pi_{121}>\pi_{112},\pi_{122}$ and $\pi_{211}=\pi_{221}>\pi_{212},\pi_{222}$.

Now consider a two stage approach to processes of solution for this class of games:

		PLAYER 1	
		Leader	Follower
PLAYER 2	Leader	π_{1LL},π_{2LL}	π_{1FL},π_{2FL}
	Follower	π_{1LF},π_{2LF}	π_{1FF},π_{2FF}

Table 4

STAGE 1. As if noncooperative and non-interactive cases. In the PD case individuals were assumed to pursue minimax strategies, in this class of cases, with reference to the expected duration of their sentences. In profit oriented settings independently maximin strategies seem appropriate. Accord-ingly consider a representation with a leader seeking to maximize his/her minimum potential profit via (VI) or (VII) in response to the imputation of probabilities $q_k'^{*}$ (resp $p_j'^{*}$) to the other player's (follower's) follow/dont follow strategies in (VI)',(VII)'. In that way an intervening dual specification is generated as follows (compare (IV),(IV)',(V),(V)'):

$$\begin{aligned}
\text{Max}\quad & \rho + \Sigma R_j q_j'^{*} - Mp^{+} - Mp^{-} \\
\text{st}\quad & \pi_{111}p_1 + \pi_{112}p_2 - R_1 \geq \rho \\
& \pi_{121}p_1 + \pi_{122}p_2 - R_2 \geq \rho \qquad \textbf{(VI)} \\
& p_1 + p_2 + p^{+} - p^{-} = 1 \\
& -M^2 \leq R_k \leq M^2 \\
& -M \leq \rho \leq M \\
& p_j, p^{+}, p^{-} \geq 0
\end{aligned}$$

$$\begin{aligned}
\text{Min}\quad & \mu' - Mq'^{+} - Mq^{-\prime} - M^2\Sigma(q_j^{+\prime} + q_j^{-\prime}) \\
\text{st}\quad & \pi_{111}q_1' + \pi_{121}q_2' \leq \mu' \\
& \pi_{112}q_1' + \pi_{122}q_2' \leq \mu' \qquad \textbf{(VI)'} \\
& q_1' + q_2' + q^{+\prime} - q^{-\prime} = 1 \\
& q_k' + q_k^{+\prime} - q_k^{-\prime} = q_k'^{*} \\
& -M \leq \mu' \leq M \\
& q_k', q^{+\prime}, q^{-\prime}, q_k^{+\prime}, q_k^{-\prime} \geq 0
\end{aligned}$$

$$\begin{aligned}
\text{Max}\quad & \mu + \Sigma S_j p_j'^{*} - Mq^{+} - Mq^{-} \\
\text{st}\quad & \pi_{211}q_1 + \pi_{212}q_2 - S_1 \geq \mu \\
& \pi_{221}q_1 + \pi_{222}q_2 - S_2 \geq \mu \qquad \textbf{(VII)} \\
& q_1 + q_2 + q^{+} - q^{-} = 1 \\
& -M^2 \leq R_j \leq M^2 \\
& -M \leq \mu \leq M \\
& q_k, q_k^{+}, q_k^{-} \geq 0
\end{aligned}$$

$$\begin{aligned}
\text{Min}\quad & \rho' - Mp'^{+} - Mp^{-\prime} - M^2\Sigma(p_j^{+\prime} + p_j^{-\prime}) \\
\text{st}\quad & \pi_{211}p_1' + \pi_{221}p_2' \leq \rho' \\
& \pi_{212}p_1' + \pi_{222}p_2' \leq \rho' \qquad \textbf{(VII)'} \\
& p_1' + p_2' + p^{+\prime} - p^{-\prime} = 1 \\
& p_j' + p_j^{+\prime} - p_j^{-\prime} = p_k'^{*} \\
& -M \leq \rho' \leq M \\
& p_j', p^{+\prime}, p^{-\prime}, p_j', p_j^{+\prime}, p_j^{-\prime} \geq 0
\end{aligned}$$

With reference to (VI),(VI)' and Cournot /Stackelberg interpretations for elements of payoffs in Tables 3 and 4:

i) $q_1'=1, q_2'=0, \mu'=\pi_{111}$ in (VI)', which is consis-tent with an optimum to (VI) with $p_1=1$, $p_2=0, \rho=\pi_{111}$, $R_1=(\pi_{121}-\pi_{111}), R_2=0$;.

ii) $q_1'=0, q_2'=1, \mu'=\pi_{121}$ in (VI)' and this is consistent with an optimum to (VI) with $p_1=1$, $p_2=0, \rho=\pi_{121}, R_1=0, R_2=0$.

Similarly, with reference to (VII),(VII)'

i) $p_1'=1, p_2'=0, \rho'=\pi_{211}$ in (VII)' which is consistent with an optimum to (VII) with $q_1=1, q_2=0, \mu=\pi_{211}$, $S_1=(\pi_{221}-\pi_{211}), S_2=0$;

ii) $p_1'=0, p_2'=1, \rho'=\pi_{221}$ 1 in (VII)' which is consistent with an optimum to (VII) with $q_1=1$, $q_2=0, \mu=\pi_{221}, S_1=0$, $S_2=0$..

Notice that: a) these solutions stem essentially from dominance considerations; b) with $\pi_{121}=\pi_{111}$ and $\pi_{221}=\pi_{211,}$ solutions i) and ii) in each case constitute alternative optima and; c) with these values these solutions for the two players are collectively infeasible.

STAGE 2.Interactive cases As for the Prisoner's Dilemma, in fact (VI),(VI)' and (VII),(VII)' will interact, at least implicitly. There are three classes of interactive equilibria disequilibria corresponding to the three different kinds of cells LL,LF (or FL) and FF. Now consider these using terminology analogous to those for PD cases:

2A. Type 1 Noncooperative interactive disequilibria LL. If $p_1'=p_1=1, p_2'=p_2=0$ the optimal solution to (VII)' with $\rho'=\pi_{211}$ is as if perfectly predictive of a non-cooperatively interactive optimum with $\rho=\pi_{111}, R_1=(\pi_{211}-\pi_{111})$, $R_2=(\pi_{211}-\pi_{121})$, (if $q_1'^*=1, q_2'^*=0$) in (VI). In turn this is perfectly predictive of an optimum to (VI)' its dual, for which $\mu'=\pi_{111}$, $q_1'=1, q_2'=0$. Moreover, with these values and conditions as if $q_1=q_1'=1, q_2=q_2'=0$, the corresponding optimum to (VI)' with $\mu'=\pi_{111}$ is as if predictive of (VII) with $\mu=\pi_{211}, S_1=(\pi_{111}-\pi_{211})$, $S_2=(\pi_{111}-\pi_{221})$, (if $p_1'^*=1$, $p_2'^*=0$), and thence, if $p_1'=p_1=1, p_2'=p_2=0$ to an optimum for (VII)', with $\rho'=\pi_{211}$.

This class of cases is not only essentially noncooperative in the sense that each individual acts as if to maximize rewards relative to self but is inherently collectively infeasible and unstable. Here the LL solution sequence corresponds to cyclic *switches* between the two Stackelberg LF and FL cases.

2B. Type 2 Noncooperative interactive equilibria LF,FL. Here each player seeks to predict outcomes for him/herself by reference to the *other* player's contingent payoff matrices, i.e. to predict each others' decisions as if via an intervening maximin/minimax duality framework as follows:

$$\begin{array}{ll}
\text{Max} & \rho + \Sigma R_j q_j'^* - Mp^+ - Mp^- \\
\text{st} & \pi_{211}p_1 + \pi_{221}p_2 - R_1 \geq \rho \\
& \pi_{212}p_1 + \pi_{222}p_2 - R_2 \geq \rho \qquad \textbf{(VI)*} \\
& p_1 + p_2 + p^+ - p^- = 1 \\
& -M^2 \leq R_k \leq M^2 \\
& -M \leq \rho \leq M \\
& p_j, p^+, p^- \geq 0
\end{array}$$

$$\begin{array}{ll}
\text{Min} & \mu' - Mq^{+\prime} - Mq^{-\prime} - M^2\Sigma(q_j^{+\prime} + q_j^{-\prime}) \\
\text{st} & \pi_{211}q_1' + \pi_{212}q_2' \leq \mu' \\
& \pi_{221}q_1' + \pi_{222}q_2' \leq \mu' \qquad \textbf{(VI)*'} \\
& q_1' + q_2' + q^{+\prime} - q^{-\prime} = 1 \\
& q_k' + q_k^{+\prime} - q_k^{-\prime} = q_k'^* \\
& -M \leq \mu' \leq M \\
& q_k', q^{+\prime}, q^{-\prime}, q_k^{+\prime}, q_k^{-\prime} \geq 0
\end{array}$$

$$\begin{array}{ll}
\text{Max} & \mu + \Sigma S_j p_j'^* - Mq^+ - Mq^- \\
\text{st} & \pi_{111}q_1 + \pi_{121}q_2 - S_1 \geq \mu \\
& \pi_{112}q_1 + \pi_{122}q_2 - S_2 \geq \mu \qquad \textbf{(VII)*} \\
& q_1 + q_2 + q^+ - q^- = 1 \\
& -M^2 \leq R_j \leq M^2 \\
& -M \leq \mu \leq M \\
& q, q^+, q^- \geq 0
\end{array}$$

$$\begin{array}{ll}
\text{Min} & \rho' - Mp^{+\prime} - Mp^{-\prime} - M^2\Sigma(p_j^{+\prime} + p_j^{-\prime}) \\
\text{st} & \pi_{111}p_1' + \pi_{112}p_2' \leq \rho' \\
& \pi_{121}p_1' + \pi_{122}p_2' \leq \rho' \qquad \textbf{(VII)*'} \\
& p_1' + p_2' + p^{+\prime} - p^{-\prime} = 1 \\
& p_j' + p_j^{+\prime} - p_j^{-\prime} = p_k'^* \\
& -M \leq \rho' \leq M \\
& p_j, p^{+\prime}, p^{-\prime}, p_j^{+\prime}, p_j^{-\prime} \geq 0
\end{array}$$

Consider the Stackelberg leadership case FL: If $p_1'=0, p_2'=1$, a solution to (VII)*' with $\rho'=\pi_{112}$ ($=\pi_{122}$) is as if interactively dually predictive of a solution to (VI)* with $p_1=0, p_2=1$ and $\rho=\pi_{112}., R_1=(\pi_{221}-\pi_{112})$, $R_2=0$ with conditions as if $q_1'^*=1$, $q_2'^*=0$. That in turn is consistent with leadership for player 2 relative to player 1 via the individually dual solution $\mu'=\pi_{221}$ for (VI)*' with $q_1'^*=1$, $q_2'^*=0$. But, with conditions as if interactively perfectly predictively $q_1=q_1'=1$, $q_2=q_2'=0$, the optimum to (VII)* then becomes as if $\mu=\pi_{221}$ with $S_1=0$, $S_2=(\pi_{112}-\pi_{221})$ and conditions as if $p_1'^*=0$, $p_2'^*=1$ and so as if *cyclically* predictive of an optimum to (VII)* '.

Similar considerations and processes will generate the other Stackelberg leadership case LF. In both cases solutions will constitute equilibria in the sense that the corresponding sequences of actions and responses correspond to elements of as if perfectly self predictive *cycles*. But both cases are also arguably inherently **unstable**. They imply positive **regret** for the following player ($R_1=(\pi_{221}-\pi_{112})$ in the FL case).

2C. Type 2 Noncooperative interactive equilibria FF. Still considering cases in which each player seeks to predict outcomes by reference to the other player's contingent payoff matrices as in (VI)*,(VI)*',(VII)*,(VII)*', now consider Cournot-like follower-follower predictions.

Starting with player 2, if $p_1'=p_1=0,p_2'=p_2=1$ the optimum to (VII)*' with $\rho'=\pi_{112}$ ($=\pi_{122}$), is perfectly predictive of a noncooperatively interactive solution to (VI)* with $\rho=\pi_{112}$, $R_1=0$, $R_2=(\pi_{222}-\pi_{112})$ with $q_2'^*=1$. This in turn is perfectly predictive of an optimum to (VI)*', its dual, with $\mu'=\pi_{222}$, $q_1'=0$, $q_2'=1$. But, with these values, and conditions as if predictively $q_1'=q_1=0,q_2'=q_2=1$, this is interactively predictive of (VII)* with $\mu=\pi_{122}$, $S_1=0,S_2=(\pi_{222}-\pi_{122})$ with $p_2'^*=1$, and thence individually predictive of (VII)*' its dual with $p_1'=0,p_2'=1$ and $\rho'=\pi_{112}$. But these latter predictions ($p_1'=0,p_2'=1$) for player 2 relative to player 2 are consistent with the initial predictions ($p_1=0,p_2=1$) by player 1 relative to player 2 and in that sense correspond to an overall equilibrium solution. (A similar argument would start with $q_1'=q_1=0,q_2'=q_2=1$ and an optimal solution to (VI)*' and go to the same as if perfectly self predictive equilibrium solution FF.)

Summarizing: In a manner similar to that for prisoners dilemma cases, with these intervening duality contexts for cases 2B and 2C associated processes are consistent with perfectly self predictive and feasible *cycles* while, for cases 2A, the Stackelberg disequilibrium case, the associated processes are consistent with as if perfectly self predictive *switches* between leaders.

Also in a manner similar to that of PD cases, case 2B is associated with as if *independently*/ non-cooperatively *selfish* prediction and action, and case 2C is consistent with *interactively* noncooperative/*selfish* prediction and action.

2D Cooperative interactive equilibria CC. Notice finally that a class of potentially collusive cases emerges, analogous to PD cases 2A, if the Stackelberg leadership payoffs are considered as corresponding to two distinct and respectively collusion and non collusion oriented types via perturbations $\in_1,\in_2>0$ to break ties as follows: $\pi_{111}=\pi_{121}+\in_1$, $\pi_{211}=\pi_{221}+\in_2$. Clearly, with $\in_1,\in_1>0$, the corresponding dominance of such collusion over Stackelberg payoffs will invalidate the predictions of Stackelberg sequences 2B by perturbing both of them to the collusive case.

5.Strategic equivalence and mixed strategy extensions

Evidently the preceding results both with reference to the Prisoner's Dilemma Cournot/Stackelberg/ Collusive behaviours hinge on dominance considerations and thence on pure strategies. But a species of strategic equivalence is nevertheless inherent in the intervening duality idea in the sense that the measures R_k,S_j potentiate strategic equivalence of contingent payoffs relative to others.
More general classes of strategic equivalence are implicit in intervening duality systems obtained by using parameters $\theta_1,\theta_2,\theta_3,\theta_4$ to modify (VI),(VI)', (VII),(VII)' to give the more generally defined nonconstant sum and asymmetric cases shown as $(VI)_s,(VI)_s',(VII)_s',(VII)_s'$ below.

Some special cases. If $\theta_1=\theta_2=\theta_3=\theta_4=1$, $(VI)_s,(VI)_s'$, $(VII)_s,(VII)_s'$ become equivalent to (VI),(VI)', (VII),(VII)'. Another class of special cases with $\pi_{1jk}+\pi_{2jk11}$=const, $\theta_1=\theta_2=1$ in $(VI)_s,(VI)_s'$, $\theta_3,\theta_4\neq1$ in $(VII)_s,(VII)_s'$ and $S_1=R_1+\Delta$, $S_2=R_2+\Delta$, corresponds to strategic equivalence of the kind familiar in standard two person constant sum games. A third class of special cases is suggested by the matching pennies examples of Section 2. If $\pi_{1jk}+\pi_{2jk}=0$, $\theta_2=\theta_3=1$ in $(VI)_s',(VII)_s'$ and, relative to those systems, feasible solutions exist to $(VI)_s,(VII)_s$ with $S_1=R_1=0$, $S_2=R_2=0$, in those senses individuals may act as if to agree relative to the system. But conditions with $S_1,R_1,S_2,R_2\neq0$ are also feasible and by that means individuals may see opportunities for gain in *disagreeing* relative to the system and/or relative to themselves.

In each of these three classes of cases associated intervening duality arguments are not restricted to pure strategies: they can work, too, for mixed strategies, as would these related strategic equivalence extensions. Further, conditions for mixed strategies as well as for strategic equivalence are not just related to the quantities R_k, S_j, but interrelated through these quantities interpreted as "slack", via the associated principles and processes of complementary slackness which have been used throughout the preceding duality and intervening duality arguments.

$$\begin{aligned} \text{Max} \quad & \rho + \Sigma R_j q_j'^{*} - Mp^{+} - Mp^{-} \\ \text{st} \quad & \theta_1(\pi_{111}p_1 + \pi_{112}p_2) - R_1 \geq \rho \\ & \theta_1(\pi_{121}p_1 + \pi_{122}p_2) - R_2 \geq \rho \qquad \textbf{(VI)}_{\mathbf{s}} \\ & p_1 + p_2 + p^{+} - p^{-} = 1 \\ & -M^2 \leq R_k \leq M^2 \\ & -M \leq \rho \leq M \\ & p_j, p^{+}, p^{-} \geq 0 \end{aligned}$$

$$\begin{aligned} \text{Min} \quad & \mu' - Mq^{+\prime} - Mq^{-\prime} - M^2 \Sigma(q_j^{+\prime} + q_j^{-\prime}) \\ \text{st} \quad & \theta_2(\pi_{111}q_1' + \pi_{121}q_2') \leq \mu' \\ & \theta_2(\pi_{112}q_1' + \pi_{122}q_2') \leq \mu' \qquad \textbf{(VI)}_{\mathbf{s}}' \\ & q_1' + q_2' + q^{+\prime} - q^{-\prime} = 1 \\ & q_k' + q_k^{+\prime} - q_k^{-\prime} = q_k'^{*} \\ & -M \leq \mu' \leq M \\ & q_k', q^{+\prime}, q^{-\prime}, q_k^{+\prime}, q_k^{-\prime} \geq 0 \end{aligned}$$

$$\begin{aligned} \text{Max} \quad & \mu + \Sigma S_j p_j'^{*} - Mq^{+} - Mq^{-} \\ \text{st} \quad & \theta_3(\pi_{211}q_1 + \pi_{212}q_2) - S_1 \geq \mu \\ & \theta_3(\pi_{221}q_1 + \pi_{222}q_2) - S_2 \geq \mu \qquad \textbf{(VII)}_{\mathbf{s}} \\ & q_1 + q_2 + q^{+} - q^{-} = 1 \\ & -M^2 \leq R_j \leq M^2 \\ & -M \leq \mu \leq M \\ & q_k, q, q^{+}, q^{-} \geq 0 \end{aligned}$$

$$\begin{aligned} \text{Min} \quad & \rho' - Mp^{+\prime} - Mp^{-\prime} - M^2 \Sigma(p_j^{+\prime} + p_j^{-\prime}) \\ \text{st} \quad & \theta_4(\pi_{211}p_1' + \pi_{221}p_2') \leq \rho' \\ & \theta_4(\pi_{212}p_1' + \pi_{222}p_2') \leq \rho' \qquad \textbf{(VII)}_{\mathbf{s}}' \\ & p_1' + p_2' + p^{+\prime} - p^{-\prime} = 1 \\ & p_j + p_j^{+\prime} - p_j^{-\prime -} = p_k^{*} \\ & -M \leq \rho' \leq M \\ & p', p'^{+}, p'^{-} \geq 0 \end{aligned}$$

Apart from extensions stemming from strategic equivalence and mixed strategies a more subtle way of extending and generalizing earlier arguments is by focussing on potential degeneracy and consequent issues and processes pertaining to regularization. An example of degeneracy and partial regularization was considered briefly in Section 2 with a context of prior probabilities for coin tossing cases. But degeneracy and partial or complete regularization can be illuminating in other cases too. As an example reconsider the Type 2 Noncooperative Interactive Equilibrium case (the Cournot case) which referred to the intervening duality system (VI)*,(VI)*', (VII)*,(VII)*' and concluded Section 4. That argument began with an assumption that in (VII)*' conditions obtained as if both $p_2'=1$ and $\Sigma p_k'=1$. That clearly implies degeneracy at that constrained optimum. Now consider a partially regularized system (VII)*' with $p_2'=1-\varepsilon$. The optimal solution to (VII)*' then becomes predictive of a uniquely "ε mixed" strategy $\rho'=\pi_{212}\varepsilon + \pi_{222}(1-\varepsilon)$. This has a "trembling hand" character in the sense that this ε noncooperatively interactive solution to (VII)*' not only implies conditions as if such a solution has non zero probability $p_1>0$ relative to (VI)*' and individual 1, but as if, by complementary slackness, $q_1=1$. In that sense a collusive solution could become certain relative to (VII)*' and individual 2.

6. Conclusion

I have focussed on representations and processes of solution for various classes of constant and non-constant sum bimatrix games, including PD and Cournot/Stackelberg/Collusion conjecture related duopoly games, using a constrained game approach in conjunction with intervening duality principles and processes.

In all cases I have implicitly assumed conditions of complete information concerning contingent payoffs. The possibility of extensions to incomplete information specifications and associated learning related developments is obvious. Another direction for extensions is via nonpreemptively frame related specifications and associated developments and interpretations such as I have pursued elsewhere with reference to Allais-like switching behaviours stemming from changes in the magnitudes of relatively exterior frame determining penalties and rewards. (See Ryan 1996.)

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