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Theory and Methodology

The distribution problem, the more for less (nothing) paradox and economies of scale and scope

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Abstract

In this paper, I develop a goal programming approach to the representation and resolution of the more for less and more for nothing paradoxes in the distribution problem. In doing so I establish new ways of deriving more for less and more for nothing results in relation variously to competitive and non-competitive market structures. Within these contexts I also introduce new and generally applicable definitions of economies of scale and scope and illustrate them by means of extended numerical examples.

Keywords: Mathematical programming; Distribution problem; More for less; Scale; Scope

1. Introduction

In Ryan (1980) and Charnes et al. (1980), we showed that if an optimal solution to the distribution problem exhibits the more for less or more for nothing paradox, any subsequent solution which fully exploits those conditions is necessarily degenerate and decomposable. It is the main purpose of this paper to use both the theorems and examples to show how those more for less and more for nothing results and associated economic interpretations can be extended to include economies of scale and of scope.

The paper is organised as follows. The next section presents generalized more for less (nothing) theorems and specializations of them to the distribution problem. Then, in Sections 3 and 4, distribution structures are related to degeneracy and decomposability with contexts variously of spatially competitive and non-competitive markets. Next, in Section 5, I introduce new goal programme related definitions of economies of scope and scale. These will stem respectively from opportunities to reduce costs by connecting previously unconnected production plants, and from opportunities to reduce costs in response to increasing supply and demand at a single production plant within a set of already connected plants. Finally, in Sections 6 and 7, I turn to examples using these definitions, the first with reference to a homogeneous commodity and

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interregional exchange, and the second with reference to a bank reorganization with associated implications for redundancy and retraining costs in a heterogeneous labour market.

2. General more for less (nothing) results

Theorem 1. *If M is arbitrarily large and if a feasible solution exists for programme (I) then*:

$$\begin{array}{llcll} \max & \sum f(x_j) - M\sum x_i^+ - M\sum x_i^- & = z \leqslant z' = & \max & \sum f(x_j) - \sum c_i^+ x_i^+ - \sum c_i^- x_i^- \\ s.t. & \sum a_{ij}x_j + x_i^+ - x_i^- = b_i, \qquad \text{(I)} & & s.t. & \sum a_{ij}x_j + x_i^+ - x_i^- = b_i, \qquad \text{(Ia)} \\ & x_j, x_i^+, x_i^- \geqslant 0. & & & x_j, x_i^+, x_i^- \geqslant 0. \end{array}$$

Proof. Any feasible solution to (I) is a feasible solution to (Ia) and conversely. But an optimal solution to (I) is a feasible but not necessarily an optimal solution to (Ia). It follows that there may exist optimal solutions to (Ia) such that $z' > z$ or $z' = z$ with $x_i^+, x_i^- > 0$ some $x_i^+,\ x_i^-$. □

For example it is not always optimal for a profit maximizing farmer to choose crop production plans so that they exactly exhaust all of his/her resources of land, machinery and time. More generally there may be *more for less* (MFL) or *more for nothing* (MFN) cases for which optimally $x_i^- > 0$ some i with $x_i^+ = 0$ all i implies $z < z'$ (resp. $z = z'$).

One class of special cases are those in which both $f(x_j)$ are linear.

Theorem 1*. *If a feasible solution exists for programme (I)* then*:

$$\begin{array}{llcll} \max & \sum f_j x_j - M\sum x_i^+ - M\sum x_i^- & = z \leqslant z' = & \max & \sum f_j x_j - \sum c_i^+ x_i^+ - \sum c_i^- x_i^- \\ s.t. & \sum a_{ij}x_j + x_i^+ - x_i^- = b_i, \qquad \text{(I)}^* & & s.t. & \sum a_{ij}x_j + x_i^+ - x_i^- = b_i, \qquad \text{(Ia)}^* \\ & x_j, x_i^+, x_i^- \geqslant 0. & & & x_j, x_i^+, x_i^- \geqslant 0. \end{array}$$

Proof. As for Theorem 1. □

Analogous to Theorem 1 are a class of minimization cases as follows.

Theorem 2. *If a feasible solution exists for programme (II) then*:

$$\begin{array}{llcll} \min & \sum c(x_j) + M\sum x_i^+ + M\sum x_i^- & = z \geqslant z' = & \min & \sum c(x_j) + \sum c_i^+ x_i^+ + \sum c_i^- x_i^- \\ s.t. & \sum a_{ij}x_j + x_i^+ - x_i^- = b_i, \qquad \text{(II)} & & s.t. & \sum a_{ij}x_j + x_i^+ - x_i^- = b_i, \qquad \text{(IIa)} \\ & x_j, x_i^+, x_i^- \geqslant 0. & & & x_j, x_i^+, x_i^- \geqslant 0. \end{array}$$

Proof. Similar to Theorem 1 (if $c(x_j) =_{\text{def}} -f(x_j)$, Theorems 1 and 2 are equivalent). □

Theorem 2 is the main result in Charnes et al. (1987) (though the proof here is more succinct). With the context of the well-known diet problem, with foods x_j, minimum dietary requirements b_i and unit costs c_j, it states the apparently paradoxical fact that in certain circumstances a diet *exceeding* minimum dietary requirements may be cheaper than one exactly meeting those requirements. (For more on this example see Charnes et al. (1987)).

Another class of special cases of Theorem 2 are those which conditionally correspond to the distribution problem (Theorem 2*).

Theorem 2*. *If a feasible solution exists for programme* $(II)^*$ *then*:

$$\begin{aligned}
\min \quad & \sum_i\sum_j c_{ij}x_{ij} + \sum_i Mx_i^+ + \sum_j Mx_i^- + \sum_j My_j^+ + \sum_j My_j^- \\
s.t. \quad & \sum_j x_{ij} + x_i^+ - x_i^- = a_i, && \text{(II)}^* \\
& \sum_i x_{ij} + y_j^+ - y_j^- = b_j, \\
& \sum_i a_i = \sum_j b_j, \quad x_{ij}, x_i^+, x_i^-, y_j^+, y_j^- \geqslant 0, \\
= z \geqslant z' = & \\
\min \quad & \sum_i\sum_j c_{ij}x_{ij} + \sum_i c_i^+x_i^+ + \sum_i c_i^-x_i^- + \sum_j d_j^+y_j^+ + \sum_j d_j^-y_j^- \\
s.t. \quad & constraints\ of\ (II)^*. && \text{(IIa)}^*
\end{aligned}$$

Proof. As for Theorem 2. (To exclude trivial cases $c_{ij} > 0$ all *i,j* will be assumed throughout the rest of the paper.) □

Clearly feasible solutions exist for (II), (IIa)* with $x_i^+, x_i^-, y_j^+, y_j^- = 0$ all *i,j*. In those cases programmes (II), (IIa)* are each equivalent to the distribution problem in its standard form (see Charnes and Cooper, 1961; Shogan, 1988). As another class of special cases, Theorem 2* admits the MFL and MFN cases as considered by Charnes and Klingman (1971), Szwarc (1971), Ryan (1980) and Charnes et al. (1980) with $x_i^+, y_j^+ = 0$ all *i,j* and $x_i^-, y_j^- > 0$ some *i,j* at an optimum. But Theorem 2* also admits cases in which $x_i^+, y_j^+ > 0$ some *i,j* at an optimum in (IIa). In that way it not only includes the distribution model and MFL and MFN cases as two classes of specializations, but also potentially generalizes both of these classes of distribution model related cases within a more comprehensive goal programming framework.

In Section 3, I show how the degeneracy–decomposability result for MFL and MFN cases as applied to the distribution model in Ryan (1980) and Charnes et al. (1980) can be correspondingly generalized using the goal programming approach implicit in Theorem 2*. (Incidentally, while more work on the MFL or MFN paradoxes has subsequently been done by others, including Arshan (1992) and Gupta and Puri (1995), that work focuses on partial post optimality analyses and provides no market related economic interpretations. Nor does it use a goal programming approach.)

3. Duality, degeneracy decomposability and MFL/MFN

Associating dual variables R_i and K_j, respectively with the origin and destination constraints of programme (IIa)* in Theorem 2* its dual is

$$\begin{aligned} \max \quad & \sum_i R_i a_i + \sum_j K_j b_j \\ & R_i + K_j \leqslant c_{ij}, \qquad\qquad (\mathrm{IIa})^{*\prime} \\ & -c_i^- \leqslant R_i \leqslant +c_i^+, \\ & -d_j^- \leqslant K_j \leqslant +d_j^+. \end{aligned}$$

If $c_i^-, c_i^+, d_j^-, d_j^+$ are sufficiently large and positive, optimal solutions to the potentially MFL or MFN formulation (IIa)*′ are equivalent to optimal solutions to the standard not more for less (nothing) formulation of the distribution problem. But in other cases $c_i^-, c_i^+, d_j^-, d_j^+$ may be such that, while (II)* is a feasible solution to (IIa)*, it is not optimal. In particular, if c_i^-, d_j^- are of appropriate magnitudes, and if $x_i^- = y_j^- = \delta > 0$ for some nonbasic route i,j in (IIa)*, an optimising MFL or MFN solution may be attained to (IIa)*′ with $R_i = -c_i^-, K_j = d_j^-$ by complementary slackness.

For a solution *maximising* a potential for MFL or MFN in (IIa)*, δ will be set at its maximal level consistent with maintenance of the initial set of basic routes. Degeneracy* follows immediately. A basic solution to (IIa)* would then have at most $m+n-2$ positive shipments, there being $m+n$ constraints with $x_i^- = y_j^- > 0$. Decomposability of such a solution follows from the fact that a basis nondegenerate* in shipments x_{ij} for (IIa)* (minimally) spans that system so that, conversely, a degenerate* basis does not.

In the degenerate* case, it is possible to set (at least) *two* distinct dual variables R_i, K_j arbitrarily e.g., such that a pair of dual variables are equated to relatively external values via $R_i = -c_i^-, K_j = d_i^-$. While such cases are possible and useful in some applications, it is not necessary to set a pair of values R_i, K_j equal to relatively external magnitudes to obtain a MFL (MFN) result. All that is necessary are conditions consistent with $R_i + K_j \leqslant 0$ for some nonbasic route at an optimum. In Section 4, I consider various classes of special cases, including spatially competitive cases, together with a numerical example.

4. A more for less (nothing) example

Consider an example in which supplies a_i at two factories, demands at two markets b_j, planned shipments x_{ij} and unit shipping costs c_{ij} are as indicated in Scheme 1.

Using the North West Corner Rule the initial basis is as in Scheme 1. Due to the degeneracy of (II)*, at an optimum one dual variable can be selected arbitrarily. Setting $R_1 = 0$ the values of the other dual variables follow directly since, by complementary slackness, $R_i + K_j = c_{ij}$ for all basic routes i,j. In this case this initial dual pair of solutions is feasible and thence optimal with a total shipping cost of 230.

Parenthetically, as I noted in Ryan (1980), for data organized routinely from top to bottom (North to South) and from left to right (West to East), a North West Corner Rule may be more efficient than other starting rules since it naturally corresponds to the adjacencies inherent in a pre-existing pattern of ship-

	$K_1 = 4$	$K_2 = 10$	
$R_1 = 0$	$c_{11}=4$ $x_{11}=$ 10	$c_{12}=10$ $x_{12}=$ 10	$a_1=20$
$R_2 = -7$	$c_{21}=6$ $x_{21}=$ 0	$c_{22}=3$ $x_{22}=$ 30	$a_2=30$
	$b_1=$ 10	$b_2=$ 40	

Scheme 1.

$$a_1 \longrightarrow b_1$$

$$a_2 \longrightarrow b_2$$

Fig. 1.

ments. (Try it with origins and destinations being Seattle and New York. A North West Rule starts with a shipment Seattle–Seattle, whereas a South West Rule would start with a cross country shipment New York–Seattle.)

The solution in Scheme 1 being nondegenerate* with $R_2 + K_1 = -3 < 0$ for the nonbasic route {2,1} exhibits the preconditions for MFL. (With $c_{22} = 6$ and thence $R_2 = -4, R_2 + K_1 = 0$, they would be preconditions for MFN.)

Fig. 1 will help to clarify how and why the more for less case arises in Scheme 1 by showing how a nondegenerate* basis with three positive shipments requires a "cross country" shipment, in this case $x_{12} = 10$. Further, shipment costs in Scheme 1 are such that the sum of the unit costs of the two "local" shipments x_{11},x_{22} are less than the unit cost of the cross country shipment x_{12}. That is

$$c_{11} - c_{12} - c_{22} = R_2 + K_1 = -3 < 0. \tag{1}$$

It follows that

$$\delta(c_{11} + c_{22}) < \delta c_{12}. \tag{2}$$

That is, if supply at origin 1 and demand at market 2 are both increased by δ, overall shipping cost can be reduced by correspondingly decreasing "cross country" shipments x_{12} by δ. Evidently, the maximal feasible value of δ consistent with nondegeneracy is $10 - \epsilon_{12}$ in this case (with ϵ_{12} arbitrarily small). Overall transport costs are then reduced by $3(10 - \epsilon_{12})$. If $\epsilon_{12} = 0$ then cross country shipments are reduced to zero and the initially connected pairs of factories and markets become disconnected. That is, an initially connected basis in x_{ij} becomes degenerate* and decomposable.

Clearly, more general MFL and MFN examples are also available via Theorem 2. However, from the perspective of this paper, the significant points are: first, that costs may be reduced *both* by increasing the connectedness of markets i.e., by increasing opportunities to generate economies of scope *and* by increasing the scale of operations of particular factories within a given structure of already connected factories and markets; second, the optimizing approach in this paper yields optimizing trade-offs between these two means of reducing overall costs.

The numerical example in Scheme 1 was chosen in part because the initial connectedness and subsequent disconnectedness of the pairs $\{O_1, D_1\}$ and $\{O_2, D_2\}$, suggests further interpretations in relation to potential competition and monopoly since in this case an *actual* entry condition $x_{12} > 0$ into market 2 from factory 1 in the initial solution becomes a *potential* entry condition $\epsilon_{12} > 0$ in the MFL solution. These ideas are pursued in the next Section.

5. The MFL (MFN) paradox and spatial competition

One condition of spatial competition is that, with unit transport costs c_{ij} for any connected pair of markets *i*,*j*, origin prices p_i, and destination prices p^j are such that

$$x_{ij} > 0 \Rightarrow p^j - p_i = c_{ij}. \tag{3}$$

(Another and stronger condition would be that all markets are connected).

Condition (5.1) in turn suggest that, rather than starting with an arbitrary valuation, say $R_1 = 0$, one dual variable would be chosen as a *base price*, say $R_1 = -c_1^- =_{\mathrm{def}} p_1$. In this way prices for all markets would be generated consistent with conditions of spatial competition and such that, for all basic routes

$$x_{ij} > 0 \Rightarrow p^j - p_i = R_i + K_j = c_{ij} \tag{4}$$

and for all nonbasic routes

$$x_{ij} > 0 \Rightarrow p^j - p_i \leqslant R_i + K_j \leqslant c_{ij}. \tag{5}$$

From Eqs. (3)–(5) it follows that competitive price regimes are potentially consistent *both* with conditions exhibiting the MFL or MFN paradox in the distribution model *and* with exploitations of those conditions. Nevertheless, as I showed using an example in Ryan (1980), conditions of the MFL or MFN paradox are not *inevitably* consistent with conditions of spatial competition. This can be demonstrated formally using a variant of (IIa)* with explicit incremental supply and demand goals x_i^{-*}, y_j^{-*}as follows.

$$\begin{aligned}
\min \quad & \sum_j - p^j y_j^- + \sum_i \sum_j c_{ij} x_{ij} + \sum_i p_i x_i^- + M \sum_i \sum_j (x_i^+ + y_j^+) \\
& \quad + \sum_i \sum_j (c_i^{-+} x_i^{-+} + c_i^- x_i^- + d_j^{-+} y_j^{-+} + d_j^- y_j^-), \\
& \sum_j x_{ij} + x_i^+ - x_i^- = a_i, \\
& \sum_i x_{ij} + y_j^+ - y_j^- = b_j, \qquad\qquad\qquad\qquad\qquad\qquad \text{(III)} \\
& x_i^- + x_i^{-+} - x_i^- = x_i^{-*}, \\
& y_j^- + y_j^{-+} - y_j^- = y_j^{-*}, \\
& \sum_i a_i = \sum_j b_j \qquad x_{ij}, x_i^+, x_i^-, y_j^+, y_j^-, x_i^{\mp}, x_i^-, y_j^{-+}, y_j^- \geqslant 0.
\end{aligned}$$

Associating variables $R_i, K_j, \theta_i, \varphi_j$ with its constraints (III) generates the dual:

$$\begin{aligned}
\max \quad & \sum_i R_i a_i + \sum_j K_{j_i} b_j + \sum_i \theta_i x_i^* + \sum_j \varphi_j y^{j*}, \\
& R_i + K_j \leqslant c_{ij}, \\
& - c_i^- \leqslant \theta_i \leqslant + c_i^{-+}, \qquad\qquad\qquad\qquad \text{(III)}' \\
& - d_j^- \leqslant \varphi_j \leqslant + d_j^{-+}, \\
& - R_i + \theta_i \leqslant p_i, \\
& - K_j + \varphi_j \leqslant - p_j, \\
& R_i, K_j \leqslant M.
\end{aligned}$$

If in effect $R_i = -p_i, K_j = p_j$ with $\theta_i = 0$ and $\varphi_j = 0$ at an optimum, then (III),(III)$'$ are potentially consistent with conditions of spatial competition as defined in Eqs. (3)–(5). But, by considering cases for which optimally $x_i^+, x_i^-, y_j^+, y_j^- > 0$, so that by complementary slackness $\theta_i \neq 0$ and/or $\varphi_j \neq 0$ some *i,j* in (III), (III)$'$, those systems may also yield *noncompetitive* interpretations of θ_i and/or φ_j as relative taxes and/or subsidies since

$$x_i > 0 \Rightarrow -R_i + \theta_i = p_i, \tag{6}$$

$$y_i > 0 \Rightarrow -K_j + \varphi_j = -p^j, \tag{7}$$

so

$$x_i \text{ and } y^j > 0 \Rightarrow R_i + K_j = p^j - p_i + \varphi_j + \theta_i. \tag{8}$$

Specifically if $\theta_i = \varphi_j = 0$, all *i,j* conditions (8) are consistent with spatial competition. But with $\theta_i \neq 0$ and/or $\varphi_j \neq 0$ they are also consistent with relative demand and supply *taxes,* if y^j (resp. x_i) is above target, and relative *subsidies* if y^j (resp. x_i) is below target. (These interpretations are particular applications of general goal related tax/subsidy interpretations in Ryan (1992). Note that a relative tax and a relative subsidy may optimally apply to the same shipment.)

In either case if MFL or MFN conditions are fully exploited the resulting more for less optimum is degenerate* and decomposable since it is then consistent with *two* base prices. Such an optimum will be correspondingly consistent with *potential* spatial competition (if optimally $\varphi_j = \theta_i = 0$) or with essentially noncompetitive conditions involving tariffs and/or subsidies (e.g., import tariffs/ subsidies) otherwise. But with $\theta_i \neq 0$ and/or $\varphi_j \neq 0$, Eqs. (6)–(8) could also be consistent with tax and subsidy related non-competitive systems, including second best related regulatory systems. Second best interpretations are especially germane here since they suggest the potential, which is in fact inherent in this approach, for interpretations in relation to economies of scale and scope.

6. MFL and MFN cases and economies of scale and scope

So far emphasis has been on *how* conditions may arise under which it might become optimal to connect markets and/or to decompose a set of connected markets into sub-markets. Now focus on two reasons *why* it might be optimal to seek to optimise within a multiple market structure as distinct from a single market structure. One reason is that, to the extent that opportunities to gain are increased by increases in the numbers of potential suppliers (factories) and demanders involved, as distinct from increases in the quantities of product which might be offered by suppliers, or required by existing demanders, there may be opportunities for gains due to increases in *scope* (numbers of suppliers and/or demanders). Secondly, opportunities to gain may be increased by increases in *scale*. That is, by increases in quantities supplied and demanded in one or more markets within a given collection of markets. Now consider these two ideas more formally.

Theorem 3 (*Economies of Scope*). *Assume two alternative cost regimes* $\{c_{ij}, M\}$ *and* $\{c_{ij}, c'_{ij}\}$ *for potential shipments between sub-markets* $i, j \in (I_1, J_1), i, j \in (I_2, J_2)$, *total availabilities* a_i *and requirements* b_j *being the same in each case. Then if a feasible solution exists for* (IV):

$$\begin{aligned} \min \quad & -\sum_j f(y_j^-) + \sum_{I_1}\sum_{J_1} c_{ij}x_{ij} + M\sum_{I_2 J_2} x_{ij} + \sum_i p_i x_i + M\sum_i\sum_j (x_i^+ + y_j^+) \\ & + \sum_i\sum_j (c_i^{-+}x_i^{-+} + c_i^- x_i^- + d_j^{-+}y_j^{-+} + d_j^- y_j^-) \\ & \textit{subject to the constraints of } (III) \end{aligned} \tag{IV}$$

$$= z \geqslant z' =$$

$$\begin{aligned}\min \quad & -\sum_j f(y_j^-) + \sum_{I_1}\sum_{J_1} c_{ij}x_{ij} + \sum_{I_2 J_2} c'_{ij}x_{ij} + \sum_i p_i x_i + M\sum_i\sum_j (x_i^+ + y_j^+) \\ & + \sum_i\sum_j (c_i^{-+}x_i^{-+} + c_i^- x_i^- + d_j^{-+}y_j^{-+} + d_j^- y_j^-) \\ & \textit{subject to the constraints of (III).} \qquad \text{(IVa)}\end{aligned}$$

Proof. Any feasible solution to (IV) is a feasible solution to (IVa) and conversely. But an optimal solution to (IV) is a feasible but not necessarily an optimal solution to (IVa). It follows that there may exist optimal solutions to (IVa) such that $z' < z$ or $z' = z$ with $x_{ij} > 0$ some $i, j \in (I_2, J_2)$. □

Theorem 4 (*Economies of Scale*). *Consider two distinct regulatory regimes, one associating prohibitive penalties M and the other non prohibitive penalties* $(-p^j, p_i)$ *with potentially marginal increases in sub-market demand and supply levels* y_j^-, x_i^- *in (III). Then, if a feasible solution exists for (V):*

$$\begin{aligned}\min \quad & \sum_j M y_j^- + \sum_{I_1}\sum_{J_1} c_{ij}x_{ij} + M\sum_{I_2 J_2} x_{ij} + \sum_i M x_i^- + M\sum_i\sum_j (x_i^+ + y_j^+) \\ & + \sum_i\sum_j (c_i^{-+}x_i^{-+} + c_i^- x_i^- + d_j^{-+}y_j^{-+} + d_j^- y_j^-) \\ & \textit{subject to the constraints of (III)} \qquad \text{(V)} \\ = z \geqslant z' = & \\ \min \quad & \sum_j - f^j y_j^- + \sum_{I_1}\sum_{J_1} c_{ij}x_{ij} + M\sum_{I_2 J_2} x_{ij} + \sum_i p_i x_i^- + M\sum_i\sum_j (x_i^+ + y_j^+) \\ & + \sum_i\sum_j (c_i^{-+}x_i^{-+} + c_i^- x_i^- + d_j^{-+}y_j^{-+} + d_j^- y_j^-) \\ & \textit{subject to the constraints of (III).} \qquad \text{(Va)}\end{aligned}$$

Proof. Any feasible solution to (V) is a feasible solution to (Va), and conversely. But an optimal solution to (V) is a feasible but not necessarily an optimal solution to (Va) with $f^j =_{\text{def}} p^j$. It follows that there may exist optimal solutions to (Va) such that $z' < z$ or $z' = z$ with $x_i^-, y_j^- > 0$ some x_i^-, y_j^-. □

Clearly, in general, a potentially connected set of markets may exhibit economies of scale and then of scope or, conversely, of scope and then of scale. In each case the resulting configuration will conform to an optimal solution to an overall model of the form of (III) with the appropriate parameters. In that sense (III) potentially includes all of (IV), (IVa), (V) and (Va) as special cases.

7. A homogeneous product example

Consider an initial allocation from origins a_i to destinations d_j as in Scheme 2.

Since the solution in Scheme 2 is both primal and dual feasible for (II)*, this market structure is consistent with optimal solutions to two *disjoint* distribution models of the standard type and thence potentially consistent with optimal solutions to (III) and (IV) with preemptive weights M associated with shipments between factories 1 and 2 and market 1, and between factory 3 and markets 2, 3 and 4. The overall shipping cost associated with this specification is 99 + 121 = 220 units.

	$K_1=9$	$K_2=6$	$K_3=3$	$K_4=5$	
$R_1=0$	$\boxed{M}$	$\boxed{6}$ 13	$\boxed{3}$ 3	$\boxed{5}$ 4	$a_1=20$
$R_2=-2$	$\boxed{M}$	$\boxed{4}$ A	$\boxed{1}$ 14	$\boxed{6}$	$a_2=14$
$R_3=0$	$\boxed{9}$ 11	$\boxed{M}$	$\boxed{M}$	$\boxed{M}$	$a_3=11$
	$b_1=11$	$b_2=13$	$b_3=17$	$b_4=4$	45

Scheme 2.

If preemptive weights in Scheme 2 now become nonpreemptive the solution in Scheme 2 is no longer potentially optimal in (III) and (IV): there are opportunities for economies of scope, as in the movement from conditions of programme (IV) to those of programme (IVa) in Theorem 3. A potentially optimal solution to the revised problem, with revised shipping costs asterisked is as shown in Scheme 3.

For this example a shipping cost reduction of 78 (from 220 to 142 units) due to *economies of scope* is gained by linking sub-markets (O_3, D_1) and $(O_1, O_2, D_2, D_3, D_4)$. One interpretation of this reduction is that, while initially market 1 is supplied from factory 3, given relatively freer opportunities for exchange as in Scheme 2, market 1 becomes wholly supplied from factory 1.

Now notice that cell {2,1} is a MFL cell with $R_2 + K_1 = -1 < 0$ and consider the same numerical example with reference to opportunities for gains to *economies of scale*. If the conditions of programme (V) in Theorem 4 correspond to a potentially optimal solution to (III) (as in Scheme 3 above) and if supplies at O_2 are increased and demands at D_1 are also increased by a positive amount $\delta \leqslant 9$ in such a way that the initial basis remains unchanged, then shipment costs are actually reduced. If this more for less opportunity is fully exploited then $\delta = 9$ and the set of markets decomposes into disjoint sub-markets as in Scheme 4.

The reader can verify that the solution in Scheme 4 is potentially optimal for (Va) and so for (III) and that in this case the economies of scale resulting from the increased operations at factory 2 amount to -9 units.

Now reconsider the various stages in this numerical example with specific reference to potential interpretations in relation to conditions of spatial monopoly and spatial competition.

In the initial Scheme 2, market 1 is wholly supplied by plant 3 and prohibitively large weights M attach to potential entry from plants 1 and 2. That is, there is initially *spatial monopoly* in market 3 in the senses both of a sole provider and of no potential entry. By contrast the allocation in Scheme 3 is potentially consistent with *spatial competition* in both senses. This is because in Scheme 3 there is more than one

	$K_1=1$	$K_2=6$	$K_3=3$	$K_4=2$	
$R_1=0$	$\boxed{\mathbf{1*}}$ 11	$\boxed{6}$ 9	$\boxed{3}$	$\boxed{5}$	$a_1=20$
$R_2=-2$	$\boxed{\mathbf{7*}}$	$\boxed{4}$ 4	$\boxed{1}$ 10	$\boxed{6}$	$a_2=14$
$R_3=2$	$\boxed{9}$	$\boxed{\mathbf{4*}}$	$\boxed{\mathbf{5*}}$ 7	$\boxed{\mathbf{4*}}$ 4	$a_3=11$
	$b_1=11$	$b_2=13$	$b_3=17$	$b_4=4$	45

Scheme 3.

	$K_1=1$	$K_2=6$	$K_3=3$	$K_4=2$	
$R_1=0$	**1*** 20	6 ε	3	5	$a_1=20$
$R_2=-2$	**7***	4 13	1 10	6	$a_2=23$
$R_3=2$	9	**4***	**5*** 7	**4*** 4	$a_3=11$
	b_1 =20	b_2 =13	b_3 =17	b_4 =4	54

Scheme 4.

provider in market 3 and with conditions $p^j - p_i = c_{ij}$ for all basic routes *i,j*, the allocation in Scheme 3 is potentially consistent via (Va) with the easy entry condition of spatial competition. With reference to the scale related MFL refinement of Scheme 3 via Scheme 4, there is also an interpretation in relation to *potential* spatial competition in the sense of potential for easy entry into the relatively isolated market (O_1, D_1) via (increases in) the shipment ϵ_{12} on route {1,2}. (Alternatively, noticing that the number of suppliers in each region is small and the actions of each varying in a perceptibly interdependent fashion with the change in market conditions from Scheme 3 to Scheme 4, these various conditions in Scheme 3 are consistent with standard conditions of spatial *oligopoly*.

As another homogeneous product interpretation of the example in Schemes 2–4, consider two unconnected markets for a given type of labour which are initially separated (as in Scheme 2) by prohibitive transportation costs. If they then become optimally connected (as in Schemes 3 and 4) in consequence of reductions in transportation costs, the optimizing solutions would be consistent with circumstances in which workers in region 3 are initially dedicated wholly to factory 3, as in Scheme 1, but are induced to choose to be redeployed to factory 1 and replaced at factory 3 by workers from regions 1 and 2, as in Scheme 3.

In both of these products and labour market cases, transport costs have been reduced *twice.* First, there is a reduction of 142 units of cost due to market connection related *economies of scope* and then a further reduction of 9 units of cost stemming from more for less related *economies of scale* associated with simultaneous increases of 9 units in the supply at origin 2 and demand at destination 1.

8. A heterogeneous product example

By interpreting the costs in Schemes 2–4 as unit costs (including unit transport costs) of supplying workers for work and the objective of the associated optimization problems as that of maximizing the net overall gain to supplying factories with workers as in (III), those schemes were given interpretations in relation to *homogeneous* labour markets. (Note that even in that case the labour market might be considered as spatially *heterogeneous* insofar as net returns to labour will differ between individuals due to differences in transport costs for journeys to and from work.)

Now consider an explicitly (skill-based) *heterogeneous* labour market interpretation with the context of an proposed bank reorganization as follows: assume that, prior to the proposed reorganization, available skilled workers $a_i, i = 1,2,3$, correspond to 20 managers, 14 clerks and 11 tellers, as in Scheme 1, and that after the reorganization workers will be redeployed, re-skilled and retrained as necessary to fill positions $b_j, j = 1,2,3,4$, corresponding to 11 telebusiness workers, 13 managers, 17 clerical workers and 4 tellers.

With these interpretations costs in Scheme 2 refer to potential transition specific adjustment costs. In that context initially prohibitively large weights M might relate *inter alia* to existing union agreements which if not renegiotiated would prohibit telebusiness related work practices for managers and tellers and would implicitly require retraining of tellers alone for the specialization of telebusiness worker (as in Scheme 2).

Next assume that, as part of the proposed bank reorganization, and due to new technology and renegotiation of job descriptions with workers, initially large transition costs M become reduced to the corresponding figures in Schemes 3 and 4. Then the prospective gain from *economies of scope* stemming from retraining and redeploying workers as in the transition from the optimal solution in Scheme 3 to the potentially optimal solution to (III) in Scheme 4 is 78 units.

Finally, there is potentially a third (telebusiness and MFL related) *economies of scale* stage in this heterogeneous labour market related interpretation. Having already attained a prospective gain of 78 units via economies of scope reorganization may be pursued further to consider a further expansion of telebusiness, from the output of 11 retrained managers as telebusiness workers, to the output of all the 20 managers as telebusiness workers. In more detail: the transition from Scheme 3 to Scheme 4 indicates that the cheapest way of getting an additional 9 telebusiness related workers would be to acquire an additional 9 tellers and reorganise the retraining plan, with the remaining 9 managers being retrained for telebusiness, as in Scheme 4, rather than being retained as managers as in Scheme 3.

Parenthetically, even though the numbers in this example are hypothetical, this example has features which might apply to a real bank reorganization. These include the fact that with increasingly sophisticated databases and communication software remote supervision of clerks and tellers has become possible and exclusively old style managerial/white collar occupations have become some of the most easily replaced. Secondly, the example is consistent with the fact that in retail banking telebusiness activities have become an increasingly significant means of dealing with valued customers. Thirdly, this example demonstrates that, even if minimization of net retraining cost is the objective, it will not always be optimal to minimise the *number* of individuals being retrained. In Schemes 2 and 3 and 4, respectively 18 of 45, 22 of 45 and 40 of 54 workers are retrained.

The latter point is a particular application of two more general properties of this optimizing approach. First, it illustrates *economies of scope* related interpretations of Theorem 1 to the effect that other things equal, labour market costs will not be increased and may be reduced as the costs of potential transitions of workers between occupations are reduced. Second, together with the transition between Scheme 3 and Scheme 4 this example also illustrates a labour market related *economies of scale* interpretation of Theorem 2 to the effect that, other things equal, labour market costs may be reduced if number of vacancies for workers of a particular skill are increased in such a way that overall retraining/reorganization costs can be reduced.

More technically, economists commonly attribute the term economies of scale either to homogeneous production cases or to heterogeneous product cases with a fixed product mix. With such contexts they then restrict the application of the term economies of scale to cases for which, when all inputs are increased by a factor λ, total costs increase by a factor *less* than λ. While this definition is not inconsistent with the theorems and examples which have been considered here the present analysis potentially includes other and stronger kinds of economies of scale and scope in which *total* costs may actually *decrease* in absolute value, even when quantities of inputs and outputs of just one type of product are increased by a factor λ. In cases where part (or all) of this overall cost reduction is attributable only to the product bringing it about, the two types of definition can be reconciled since in that case an increase δ in inputs and an equal increase δ in outputs of at least one type of product (e.g., telebusiness workers and telebusiness output) may not simply lead to a proportionately lower increase in cost attributable to that product. Thus, for the heterogeneous examples such as the spatially or professionally differentiated MFL labour market cases which have just been considered such increases led to an *absolute reduction* in total cost so that the increment in overall cost attributable to increased production of telebusiness the marginal output is *negative* and average costs will be falling *a fortiori* due to the increased production of telesales.

Still in the context of the heterogeneous labour market example, the dual variables in Schemes 2–4 are consistent with potentially competitive interpretations analogous to the spatially competitive interpretations of Section 5 via particular interpretations of (III)′ and (IV)′. In such spatially competitive labour market cases all post reorganization wages would be related in such a way that an initial wage (e.g., the initial wage for managers w_1) is taken as a *base wage* $R_1 =_{\text{def}} w_1$ and all other wages set via net retraining cost differentials $R_i + K_j = c_{ij}$ for $x_{ij} > 0$ at an optimum. In that way these labour market examples are also potentially consistent with spatially and intertemporally competitive labour market interpretations.

Recalling that specializations of (IV)′ are open to interpretations as relative taxes and subsidies, it follows that variants of solutions in Schemes 2–4 may correspond to elements of payroll taxes and initial hiring related subsidies at least in part, reflecting scope and scale related labour market advantages of including an increased *variety* of potential transitions between types of workers and/or increased *numbers* of a particular type of workers, respectively. (Corresponding interpretations in relation to regionally monopolistic or oligopolistic labour markets would then follow in a manner analogous to the single commodity case considered in Section 6.)

9. Conclusion

In this paper, I have introduced a new goal programming approach to the representation and resolution of the MFL and MFN paradoxes in the distribution model and to definitions and associated theorems relating to economies of scale and economies of scope in that context. Clearly, by relating scale and scope phenomena to the conditions of programme (I) rather than the conditions of program (III) the definitions of economies of scale and of scope in Theorems 3 and 4 could be correspondingly generalized to comprehend nonlinear and explicitly multiple input production processes too.

I close with two remarks. First, for simplicity the definitions of economies of scale and scope in Section 6 have been given as if these concepts would apply on an all or nothing basis. But clearly they could apply on a partial basis. If any one (or more) of the quantities M in (IV) were reduced to a nonpreemptive magnitude c'_{ij} in (IVa) and/or if any one (or more) of the quantities M in (V) were reduced to a non preemptive magnitudes $-p^j, p_i$ in (Va) then relatively enhanced economies, respectively of scope and of scale may become attainable.

Secondly, apart from potentially yielding interpretations in relation to economies of scale and scope, distribution problems exhibiting the MFL and/or MFN paradox also have other properties, some of which are yet to be fully explored. Among these is the fact that there may be a variety of potential MFL and MFN solutions each with distinct potentials for economies of scope and scale and consequently distinct decomposition patterns. To illustrate this reconsider Scheme 2. The potential for MFL evident in that Scheme, when fully exploited, generated Scheme 4. But Scheme 4 itself exhibits an as yet unexploited potential (via cell {4,2}) for attaining a MFN solution. If that potential is fully exploited an additional 7 units could be shipped from origin 2 to destination 4 at no additional overall cost. In that case, among other things, 11 units would be shipped wholly from origin 3 to market 4 and that origin–destination pair would become optimally isolated, giving a three way partition of origins and destinations and a correspondingly still more concentrated spatial market arrangement. (This example illustrates the more general point that, while nondegeneracy* of a distribution problem together with $R_i + K_j \leq 0$ for some nonbasic cell at an optimum may be a sufficient condition for MFL/MFN, as in the Charnes–Klingman theorem cited in Section 2, it is not always necessary.)

Finally, a different more for less solution and subsequent more for nothing solution with different associated patterns of decomposition into disjoint subsets of origins and destinations would follow if the economies of scale and scope related applications in Sections 7 and 8 had started with the alternative optimum signalled by the A in cell {2,2} of Scheme 2.

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