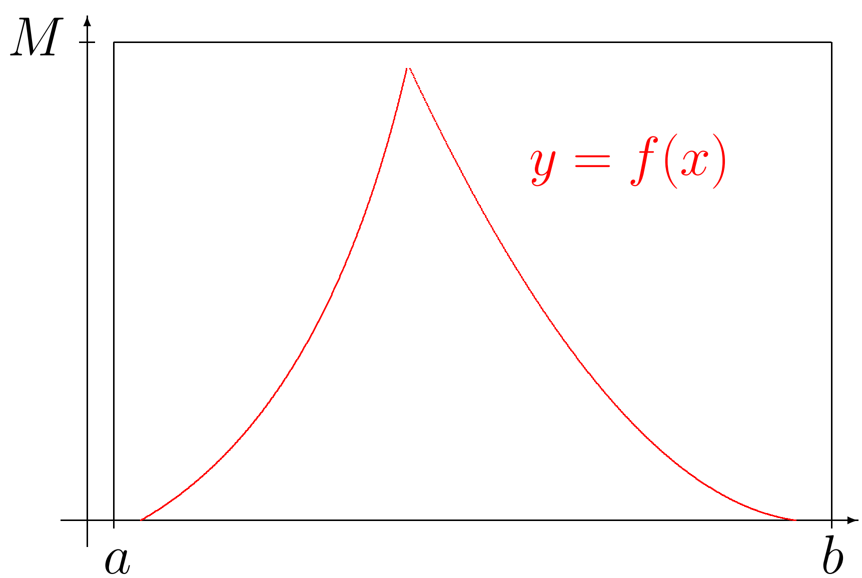
Rejection Sampling Method for Simulation

First Setting:

Suppose we wish to simulate a value W from a probability distribution having density f which satisfies the following conditions:

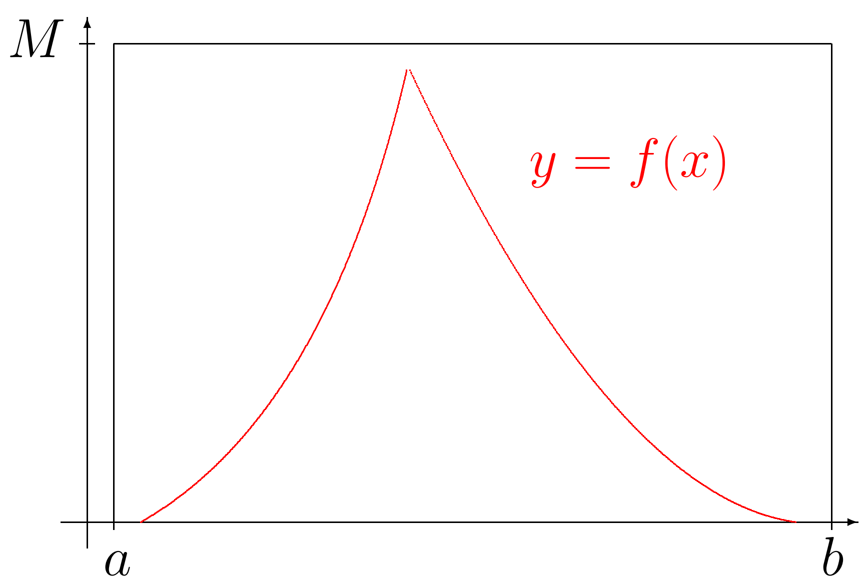
- $f(x) > 0$ only for x in an interval $[a, b]$; and
- $f(x) \leq M$ for some value of M.



Rejection Sampling Method for Simulation

Simulation Approach:

- Generate a Uniform value on $[a, b]$; call it X.
- Independently generate a Uniform value on $[0, M]$; call it Y.
- $$\begin{cases} \text{Accept the point } (X,Y) \text{ if } Y \le f(X); \\ \text{the simulated value is } W = X; \\ \\ \text{or} \\ \\ \text{Reject the point } (X,Y) \text{ if } Y > f(X); \\ \text{repeat the first two steps again.} \end{cases}$$



Rejection Sampling Method for Simulation

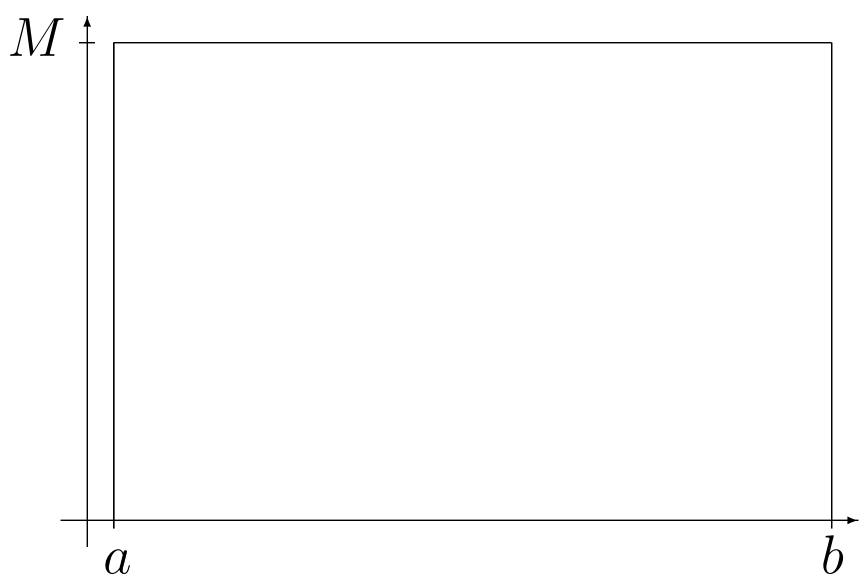
Probabilistic Foundation:

- The density of X is $\dfrac{1}{b-a}$ for $a \le x \le b$.

- The density of Y is $\dfrac{1}{M}$ for $0 \le y \le M$.

- The *joint* density of (X, Y) is

$$f(x, y) = \frac{1}{M(b-a)}\,, \quad a \le x \le b, 0 \le y \le M.$$

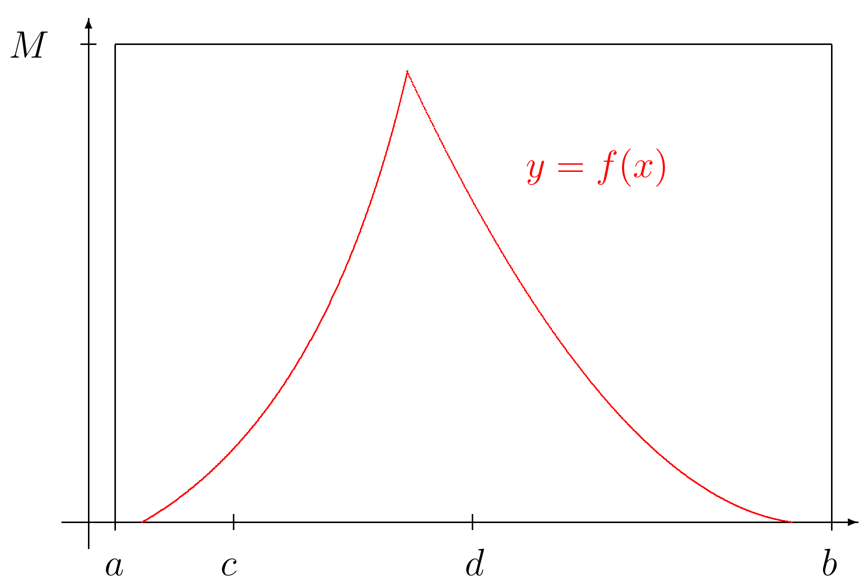


Rejection Sampling Method for Simulation

Probabilistic Foundation:

- Observe that

$$
\begin{aligned}
P(c \le W \le d) &= P\left(c \le X \le d \,\middle|\, Y \le f(X)\right) \\
&= \frac{P(c \le X \le d, Y \le f(X))}{P(Y \le f(X))} \\
&= \frac{\int_{x=c}^{x=d} \int_{y=0}^{y=f(x)} \frac{1}{M(b-a)}\, dy\, dx}{\int_{x=a}^{x=b} \int_{y=0}^{y=f(x)} \frac{1}{M(b-a)}\, dy\, dx}
\end{aligned}
$$



Rejection Sampling Method for Simulation

QUESTIONS:

How many uniform values must be simulated to produce a single value of W?

How many uniform values must be simulated to produce 1000 values of W?

Rejection Sampling Method for Simulation

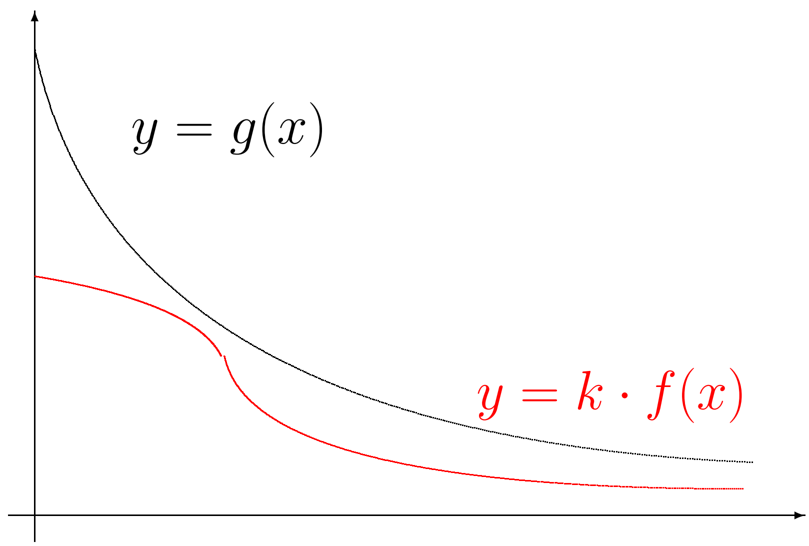
SECOND SETTING:

Suppose we wish to simulate a value W from a probability distribution having density f which satisfies the condition: there is

- some constant k and
- probability density g which is easy to simulate

such that

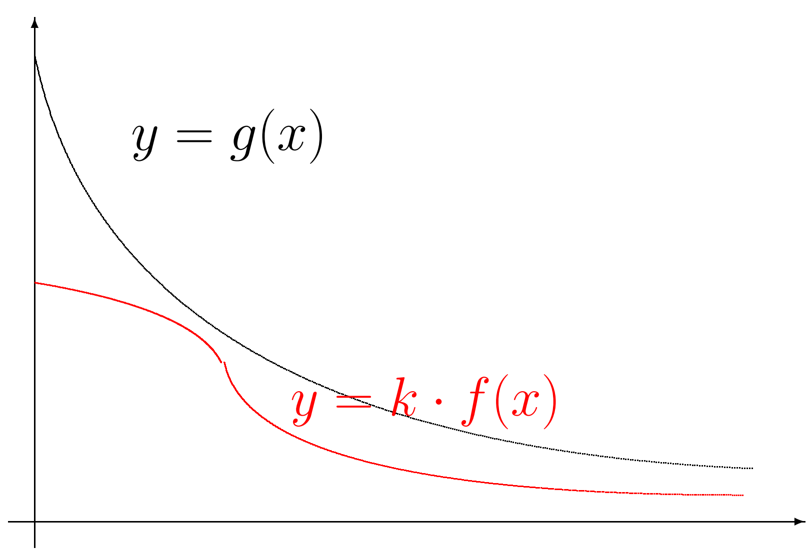
$$k \cdot f(x) \leq g(x), \qquad -\infty < x < \infty.$$



Rejection Sampling Method for Simulation

Simulation Approach:

- Generate a value according to the density g; call it X.
- Independently generate a Uniform value on $[0, g(X)]$; call it Y.
- $\begin{cases} \text{Accept the point } (X,Y) \text{ if } Y \le k \cdot f(X); \\ \text{the simulated value is } W = X; \\ \\ \text{or} \\ \\ \text{Reject the point } (X,Y) \text{ if } Y > k \cdot f(X); \\ \text{repeat the first two steps again.} \end{cases}$



Rejection Sampling Method for Simulation

PROBABILISTIC FOUNDATION:

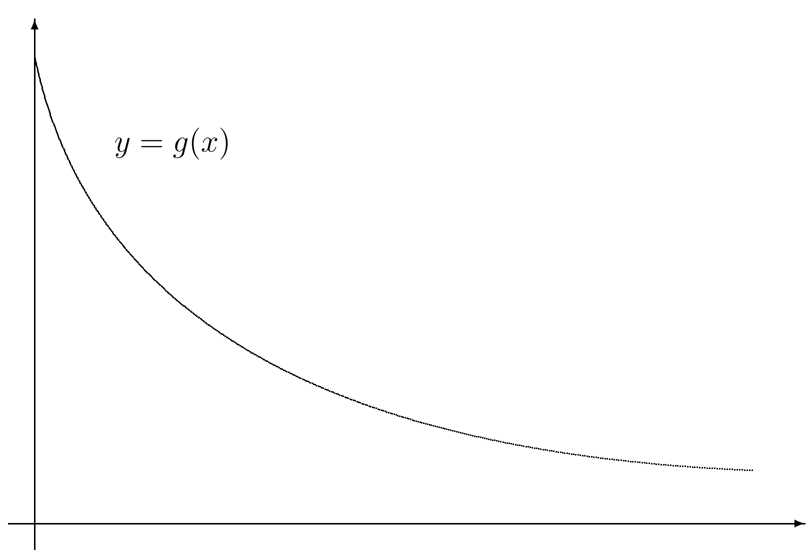
- The density of X is $g(x)$.

- The *conditional density* of Y given $X = x$ is

$$f(y|X = x) = \frac{1}{g(x)} \text{ for } 0 \le y \le g(x).$$

- The *joint* density of (X, Y) is

$$\begin{aligned} f(x, y) &= f_1(x) f(y|x) \\ &= g(x) \cdot \frac{1}{g(x)}, \qquad 0 \le y \le g(x), 0 \le x < \infty \\ &= 1, \qquad\qquad\quad 0 \le y \le g(x), 0 \le x < \infty \end{aligned}$$



Rejection Sampling Method for Simulation

Probabilistic Foundation:

- Observe that

$$\begin{aligned} P(c \le W \le d) &= P\left(c \le X \le d \,\middle|\, Y \le k \cdot f(X)\right) \\ &= \frac{P(c \le X \le d, Y \le k \cdot f(X))}{P(Y \le k \cdot f(X))} \\ &= \frac{\displaystyle\int_{x=c}^{x=d} \int_{y=0}^{y=kf(x)} 1\,dy\,dx}{\displaystyle\int_{x=0}^{x=\infty} \int_{y=0}^{y=kf(x)} 1\,dy\,dx} \end{aligned}$$