Tank wagons: volume

$$g(v) = av + b \qquad \{1\}$$

$$[a] = v^{-2} \qquad [b] = v^{-1} \qquad \{2\}$$

(See a better formulation in the end.)

$$G(v) = \int_{v_\mathrm{L}}^{v} (at + b)\,\mathrm{d}\,t = \left[\frac{a}{2}t^2 + bt\right]_{v_\mathrm{L}}^{v} = \\ = \frac{a}{2}\left(v^2 - v_\mathrm{L}^2\right) + b\left(v - v_\mathrm{L}\right) \qquad \{3\}$$

(This might advantageously become

$$G(v) = \left[\frac{a}{2}\left(v + v_\mathrm{L}\right) + b\right]\left(v - v_\mathrm{L}\right) \qquad \{4\}$$

and so on, confirming that, for $v = v_\mathrm{L}$, it is $G = 0$.)

$$G(v) = \frac{a}{2}v^2 - \frac{a}{2}v_\mathrm{L}^2 + bv - bv_\mathrm{L} = \\ = \frac{a}{2}v^2 + bv - \left(\frac{a}{2}v_\mathrm{L}^2 + bv_\mathrm{L}\right) \qquad \{5\}$$

$$G(v) = \frac{a}{2}v^2 + bv - \left(\frac{a}{2}v_\mathrm{L}^2 + bv_\mathrm{L}\right) \qquad \{6\}$$

$$\frac{a}{2}v_\mathrm{U}^2 + bv_\mathrm{U} - \left(\frac{a}{2}v_\mathrm{L}^2 + bv_\mathrm{L}\right) = 1 \qquad \{7\}$$

$$\frac{a}{2}\left(v_\mathrm{U}^2 - v_\mathrm{L}^2\right) + b\left(v_\mathrm{U} - v_\mathrm{L}\right) = 1 \qquad \{8\}$$

$$b = \frac{1}{v_\mathrm{U} - v_\mathrm{L}} - \frac{a}{2}\left(v_\mathrm{L} + v_\mathrm{U}\right) \qquad \{9\}$$

$$b = \frac{1}{\Delta v} - \frac{a}{2}\Sigma v \qquad \{10\}$$

$$b + av_\mathrm{L} = \frac{1}{\Delta v} - \frac{a}{2}\Sigma v + av_\mathrm{L} = \frac{1}{\Delta v} - \frac{a}{2}\left(v_\mathrm{L} + v_\mathrm{U}\right) + av_\mathrm{L} = \\ = \frac{1}{\Delta v} + \frac{a}{2}v_\mathrm{L} - \frac{a}{2}v_\mathrm{U} \qquad \{11\}$$

$$b + av_\mathrm{L} = \frac{1}{\Delta v} - \frac{a}{2}\Delta v \qquad \{12\}$$

$$\frac{a}{2}v^2 + bv - \left(\frac{a}{2}v_\mathrm{L}^2 + bv_\mathrm{L} + G\right) = 0 \qquad \{13\}$$

$$2\left(\frac{a}{2}\right)v = -b \pm \sqrt{b^2 + 4\frac{a}{2}\left(\frac{a}{2}v_{\mathrm{L}}^2 + bv_{\mathrm{L}} + G\right)} \tag{14}$$

$$\begin{aligned} av &= -b \pm \sqrt{b^2 + 4\frac{a^2}{4}v_{\mathrm{L}}^2 + 4\frac{a}{2}bv_{\mathrm{L}} + 4\frac{a}{2}G} = \\ &= -b \pm \sqrt{b^2 + 2bav_{\mathrm{L}} + (av_{\mathrm{L}})^2 + 2aG} \end{aligned} \tag{15}$$

$$av = -b \pm \sqrt{b^2 + 2bav_{\mathrm{L}} + (av_{\mathrm{L}})^2 + 2aG} \tag{16}$$

$$av = -b + \sqrt{(b + av_{\mathrm{L}})^2 + 2aG} \tag{17}$$

$$\begin{aligned} av &= -b + \sqrt{\left(\frac{1}{\Delta v} - \frac{a}{2}\Delta v\right)^2 + 2aG} = \\ &= -b + \sqrt{\frac{1}{(\Delta v)^2} - a + \left(\frac{a}{2}\right)^2 (\Delta v)^2 + 2aG} \end{aligned} \tag{18}$$

$$\boxed{av = -\frac{1}{\Delta v} + \frac{a}{2}\Sigma v + \sqrt{\left(\frac{1}{\Delta v} - \frac{a}{2}\Delta v\right)^2 + 2aG}} \tag{19}$$

Verify:

$G = 0$:

$$av = -\frac{1}{\Delta v} + \frac{a}{2}\Sigma v + \left(\frac{1}{\Delta v} - \frac{a}{2}\Delta v\right) = \frac{a}{2}2v_{\mathrm{L}} = av_{\mathrm{L}} \tag{20}$$

$G = 1$:

$$av = -\frac{1}{\Delta v} + \frac{a}{2}\Sigma v + \left(\frac{1}{\Delta v} + \frac{a}{2}\Delta v\right) = \frac{a}{2}2v_{\mathrm{U}} = av_{\mathrm{U}} \tag{21}$$

A better formulation would possibly be (dimensionless style)

$$g(v) = \frac{1}{A}\frac{v}{v_0} + \frac{1}{B} \tag{22}$$

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