Goal programming

Introduction

Multicriteria decision making refers to situations where we have more than one objective (or goal) and these objectives conflict and we must somehow reach a decision taking them all into account. This contrasts with, for example, decision trees or linear programming, where we have a single objective – either optimise expected monetary value for decision trees or optimise a single linear objective in the case of linear programming. Here we consider goal programming, one technique used for multicriteria decision making.

Goal programming (GP)

To illustrate goal programming (GP) we consider the Two Mines problem.

The Two Mines Company owns two different mines that produce an ore which, after being crushed[1], is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting[2] plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below.

		Production (tons/day)		
Mine	Cost per day	High	Medium	Low
X	180	6	3	4
Y	160	1	1	6

How many days per week should each mine be operated to fulfil the smelting plant contract ?

■

To solve this problem introduce variables

x = number of days per week mine X is operated
y = number of days per week mine Y is operated

with $x \geq 0$ and $y \geq 0$, and formulate the problem as a linear program:

$$
\begin{array}{llll}
\text{Minimize} & 180\,x & +\,160\,y & \\
\text{subject to} & 6\,x & +\,1\,y & \geq 12 \\
 & 3\,x & +\,1\,y & \geq 8 \\
 & 4\,x & +\,6\,y & \geq 24 \\
 & x & & \leq 5 \\
 & & y & \leq 5 \\
 & & x, y \geq 0 &
\end{array}
$$

where we assume we can work no more than **five** days per week on each mine.

The solution to this linear program is:

$$x = 12/7 = 1.71$$
$$y = 20/7 = 2.86$$

from

[1] *To crush*, triturar.
[2] *To smelt*, fundir para extrair metal.

MINIMIZE									
1,714286	2,857143								
180	160						765,7143		
x	y	C+	C-	H+	M+	L+			
6	1						13,14286	>=	12
3	1						8	>=	8
4	6						24	>=	24
1							1,714286	<=	5
	1						2,857143	<=	5

Figure 1

with the value of the objective function being given by

$$z = 180\,x + 160\,y = 180(12/7) + 160(20/7) = 765.71$$

Extension to more objectives

Now, although the solution shown above is the minimum cost solution it does raise a difficulty. If we adopt this solution we will be producing precisely as much medium and low-grade ore per week as we need. This is good. However, we will be producing

$$6\,(12/7) + 1\,(20/7) = 13.14 \text{ tonnes}$$

of high-grade per week. As the contract is only for 12 tonnes we will somehow have to deal with this excess.

Hence we can see that the Two Mines company might well feel that whereas before we had a single objective problem

- minimise cost

now we have a problem that has two (conflicting) objectives:

- minimise cost
- do not produce excess ore

where, although excess ore might be taken as referring just to high-grade ore, a moment's thought will reveal that if we move from our current solution to reduce the excess of high-grade ore, we might find an excess of another grade of ore. Hence we need to consider all grades and their excess rather than just high-grade.

Goal programming is one approach to dealing with problems of this kind.

Goal programming formulation

To deal with these two objectives in our example problem via GP, we need to introduce extra variables — these variables deal with the deviation from the goal for each objective. To proceed we need to decide a numeric goal for each objective.

We now have the two objectives:

- minimise cost — our previously calculated minimum cost was 765.71 (per week). The company may consider that, in the interests of eliminating excess ore, they would be prepared to increase the cost that they incur, but they would not like this cost to increase beyond, say, 800. This figure of 800 becomes our numeric cost goal.

- do not produce excess ore — which actually relates to the three grades of ore for which the respective goals are the amount that we need: 12, 8 and 24 tonnes, respectively.

Let C^+ (≥ 0) represent the amount by which we deviate upward from our cost goal and C^- (≥ 0) represent the amount by which we deviate downward from our *cost* goal. Then we have an equation linking these new variables to our old variables:

$$180\,x + 160\,y = 800 + C^+ - C^-$$

This constraint, which must be an equality, says that whatever we decide in terms of production on mines X and Y (x and y, respectively) the cost of that ($180\,x + 160\,y$) must equal the goal (800) adjusted by the deviation variables C^+ and C^-, the plus in front of C in this equation indicating an upward movement (deviation) from the goal and the minus in front of C in this equation indicating a downward movement (deviation) from the goal.

There are two comments to make here:

- it is standard notational practice in GP to use a plus superscript to indicate upward deviation from the goal and a negative superscript to indicate downward deviation from the goal.
- one subtle point here is that the way we have written our equation including $C^+ - C^-$ opens the possibility that when we come to numerically solve the problem we get an answer like $C^+ = 100$ and $C^- = 120$, so both an upward and a downward deviation with the overall deviation being $C^+ - C^- = 100 - 120 = -20$; a downward deviation of 20.

We can now deal with our objective relating to production of excess ore. Letting H^+, M^+ and L^+ (all ≥ 0) be the upward deviation for high, medium and low-grade, respectively, and H^-, M^- and L^- (all ≥ 0) be the downward deviation for high, medium and low-grade, respectively, we have the three equality equations:

$$\begin{aligned}6\,x + 1\,y &= 12 + H^+ - H^- \\ 3\,x + 1\,y &= 8 + M^+ - M^- \\ 4\,x + 6\,y &= 24 + L^+ - L^-\end{aligned}$$

One point to note here is that the above equations leave open the possibility that the company might decide not to supply all the ore for some particular grade (i.e., we have a downward deviation from the goal level where here the goal level relates to the amount that we are contracted to supply). If the company wishes to exclude this possibility, then we simply set $H^- = M^- = L^- = 0$ (eliminate these variables from the problem).

Hence for our Two Mines problem where we wish to reconcile our conflicting objectives we have the equations (p 5):

$$\begin{aligned}&180\,x + 160\,y = 800 + C^+ - C^- \\ &6\,x + 1\,y = 12 + H^+ - H^- \\ &3\,x + 1\,y = 8 + M^+ - M^- \\ &4\,x + 6\,y = 24 + L^+ - L^- \\ &x \leq 5 \\ &y \leq 5 \\ &\qquad\text{(nonnegative variables)}\end{aligned}$$

These equations are, given our variables, equations that must be satisfied.

To proceed, there are two usual different approaches:

- to attach weights to each of the variables associated with upward/downward deviation and then to solve a single problem whose objective is to minimise this weighted deviation sum. This approach is sometimes known as **weighted goal programming.**
- to decide priority levels for the goals (priority level 1 for the most important goal, then priority level 2 for the second most important goal, etc) and first satisfy priority one goals, then priority two goals, then…, so a sequence of related problems are solved. This approach is sometimes known as **sequential goal programming**[3] or **preemptive goal programming**[4] as priorities cannot be traded off against each other (unlike the weighted goal programming approach).

We shall adopt the second approach.

Weighted approach

(pp 5–10)

Priority approach

(p 11) For the priority approach, we will again assume that we always wish to supply the amount of ore we are contracted to supply, so that $H^- = M^- = L^- = 0$ (i.e., these variables can be eliminated from the problem). This leaves variables C^+, C^-, H^+, M^+ and L^+.

As mentioned above, in the priority approach we need to decide priority levels for the goals (priority level 1 for the most important goal, then priority level 2 for the second most important goal, etc.) and first satisfy priority one goals, then priority two goals, etc., so that a sequence of related problems are solved.

Here, for the purposes of illustrating the approach, we shall assume that management consider that their priority levels are:

- priority level 1 – as little excess high-grade ore as possible
- priority level 2 – cost
- priority level 3 – as little excess combined medium-grade and low-grade ore as possible

Hence the first problem that we solve (a linear program) is (p 11):

$$
\begin{array}{llll}
\text{Minimize} & H^+ & & \\
\text{subject to} & 180\,x & +\,160\,y & = 800 + C^+ - C \\
 & 6\,x & +\,1\,y & = 12 + H^+ \\
 & 3\,x & +\,1\,y & = 8 + M^+ \\
 & 4\,x & +\,6\,y & = 24 + L^+ \\
 & x & & \le 5 \\
 & & y & \le 5 \\
 & \multicolumn{3}{c}{\text{(nonnegative variables)}}
\end{array}
$$

[3] *Sequential goal programming*, programação sequencial por objectivos.
[4] *Preemptive goal programming*, programação preemptiva por objectivos (por objectivos prioritários).

Solving the linear program represented in the above model we get (Excel p 12a&b):

First goal									
MINIMIZE									
1,333333	**4**	**0**	**0**	**0**	**0**	**5,333333**			
				1			**0**		
x	**y**	**C+**	**C-**	**H+**	**M+**	**L+**			
6	1			-1			**12**	=	12
3	1				-1		**8**	=	8
4	6					-1	**24**	=	24
1							**1,333333**	<=	5
	1						**4**	<=	5

Figure 2

This indicates that we can achieve zero excess high-grade ore, H^+ (a zero value corresponding to a zero upward deviation variable for high-grade ore), but at the price of an upward deviation from our cost goal of **80** and an upward deviation from our low-grade ore goal, L^+, of 5.333.

We can now move to our second priority level. This was cost so we have the new linear program (p 13):

$$
\begin{array}{lllll}
\text{Minimize} & C^+ - C^- & & & \\
\text{subject to} & 180\,x & +\,160\,y & = 800 + C^+ - C^- \\
 & 6\,x & +\,1\,y & = 12 + H^+ \\
 & 3\,x & +\,1\,y & = 8 + M^+ \\
 & 4\,x & +\,6\,y & = 24 + L^+ \\
 & x & & \leq 5 \\
 & & y & \leq 5 \\
 & & H^+ & = 0 \\
\end{array}
$$

(nonnegative variables)

where we are trying to minimise deviation from our cost goal but where the upward deviation variable H^+ for high-grade ore is now constrained to be zero (the value it had in our solution calculated above at the first priority level).

Note that the objective function here is minimise $C^+ - C^-$, where the positive sign before C^+ indicates that we dislike upward deviations from our cost goal but the minus sign before C^- indicates that we like downward deviations from our cost goal.

Had we wished to just minimise deviation from our cost goal our objective would have been minimise $C^+ + C^-$.

You will see: (Excel p 14a & b)

Second goal									
MINIMIZE									
1,333333	**4**	**80**	**0**	**0**	**0**	**5,333333**			
		1	**-1**				**80**		
x	**y**	**C+**	**C-**	**H+**	**M+**	**L+**			
180	160	-1	1				**800**	=	800
6	1			-1			**12**	=	12
3	1				-1		**8**	=	8
4	6					-1	**24**	=	24
1							**1,333333**	<=	5
	1						**4**	<=	5
				1			**0**	=	0

Figure 3

which is actually the solution we had before, i.e.: our previous solution that we obtained when we minimised upward deviation for high-grade ore also happened to minimise our cost goal.

We can now consider the third priority level which was: as little excess combined medium-grade and low-grade ore as possible. This implies that we minimise the combined deviation for medium and low-grade ore. The linear program for this problem is (p 15):

$$\begin{array}{llll}
\text{Minimize} & M^+ + L^+ & & \\
\text{subject to} & 180\,x & +\,160\,y & = 800 + C^+ - C^- \\
 & 6\,x & +\,1\,y & = 12 + H^+ \\
 & 3\,x & +\,1\,y & = 8 + M^+ \\
 & 4\,x & +\,6\,y & = 24 + L^+ \\
 & x & & \leq 5 \\
 & & y & \leq 5 \\
 & & H^+ & = 0 \\
 & & C^+ - C^- & = 80 \\
\end{array}$$

(nonnegative variables)

where we have added a constraint specifying that the objective at the second priority level ($C^+ - C^-$) must retain the minimum value (80) we calculated above.

Third goal									
MINIMIZE									
1,333333	**4**	**80**	**0**	**0**	**1,11E-16**	**5,333333**			
					1	**1**	**5,333333**		
x	**y**	**C+**	**C-**	**H+**	**M+**	**L+**			
180	160	-1	1				**800**	=	800
6	1			-1			**12**	=	12
3	1				-1		**8**	=	8
4	6					-1	**24**	=	24
1							**1,333333**	<=	5
	1						**4**	<=	5
				1			**0**	=	0
		1	-1				**80**	=	80

Figure 4

The constraint relating to $C^+ - C^- = 80$ has been added. (Excel p 16b)

An effect we referred to before appears — we have both an upward and downward deviation variable, upward of 90 and downward of 10, equivalent to an upward of 80. This is not what we wanted, but the resolution of this problem is easy (Excel p 17) which we can sensibly interpret and which has precisely the same objective function value and also satisfies the constraints we imposed on the first and second level priorities. The technical explanation of why this difficulty appeared here is that the underlying linear program was degenerate (there were two or more solutions with the same objective function value).

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