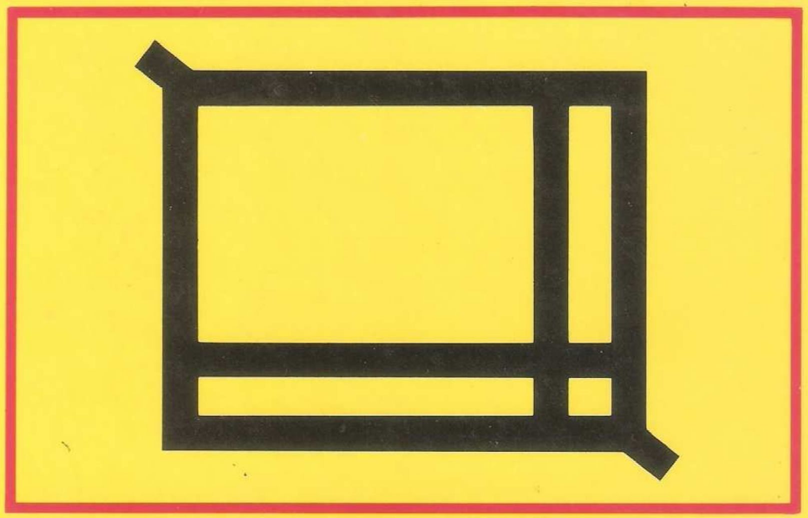
Undergraduate Texts in Mathematics

James K. Strayer

Linear Programming and Its Applications



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Note that the basic solutions of the second and third tableaus of step (3) are feasible while the basic solution of the first tableau is not feasible. There is a good reason for this—only the second and third tableaus of step (3) are maximum basic feasible tableaus! Recall that, in general, the simplex algorithm transition between maximum basic feasible tableaus is designed so that the objective function ($-g$ in this particular case) is not decreased. Hence each such transition maintains or increases the value of the objective function, usually until either a maximum value is reached or the algorithm detects unboundedness. In rare cases, a phenomenon known as cycling occurs; we discuss this phenomenon now.

§8. Cycling

We begin with an example due to E.M.L. Beale ([B2]).

EXAMPLE 18. Consider the linear programming problem

$$\begin{aligned} \text{Maximize} \quad & f(x_1,x_2,x_3,x_4) = 3/4x_1 - 20x_2 + 1/2x_3 - 6x_4 \\ \text{subject to} \quad & 1/4x_1 - 8x_2 - x_3 + 9x_4 \leqq 0 \\ & 1/2x_1 - 12x_2 - 1/2x_3 + 3x_4 \leqq 0 \\ & x_3 \leqq 1 \\ & x_1, x_2, x_3, x_4 \geqq 0. \end{aligned}$$

Six simplex algorithm pivots are performed below. While it is not intended that you verify these computations, make sure that you see that all pivots have been made in accordance with the simplex algorithm.

x_1	x_2	x_3	x_4	-1	
1/4*	−8	−1	9	0	$= -t_1$
1/2	−12	−1/2	3	0	$= -t_2$
0	0	1	0	1	$= -t_3$ →
3/4	−20	1/2	−6	0	$= f$

t_1	x_2	x_3	x_4	-1	
4	−32	−4	36	0	$= -x_1$
−2	4	3/2	−15	0	$= -t_2$
0	0	1	0	1	$= -t_3$ →
−3	4	7/2	−33	0	$= f$

t_1	t_2	x_3	x_4	-1	
−12	8	8*	−84	0	$= -x_1$
−1/2	1/4	3/8	−15/4	0	$= -x_2$
0	0	1	0	1	$= -t_3$ →
−1	−1	2	−18	0	$= f$

t_1	t_2	x_1	x_4	-1	
−3/2	1	1/8	−21/2	0	$= -x_3$
1/16	−1/8	−3/64	3/16*	0	$= -x_2$
3/2	−1	−1/8	21/2	1	$= -t_3$ →
2	−3	−1/4	3	0	$= f$

t_1	t_2	x_1	x_2	-1	
2*	−6	−5/2	56	0	$= -x_3$
1/3	−2/3	−1/4	16/3	0	$= -x_4$
−2	6	5/2	−56	1	$= -t_3$ →
1	−1	1/2	−16	0	$= f$

x_3	t_2	x_1	x_2	-1	
1/2	−3	−5/4	28	0	$= -t_1$
−1/6	1/3*	1/6	−4	0	$= -x_4$
1	0	0	0	1	$= -t_3$ →
−1/2	2	7/4	−44	0	$= f$

x_3	x_4	x_1	x_2	-1	
−1	9	1/4	−8	0	$= -t_1$
−1/2	3	1/2	−12	0	$= -t_2$
1	0	0	0	1	$= -t_3$
1/2	−6	3/4	−20	0	$= f$

Note that the seventh tableau above is the same as the initial tableau up to a rearrangement of the first four columns. Hence the seventh tableau is no closer to an optimal solution than the initial tableau!! (Needless to say, this is quite frustrating!!) Since transitions between maximum basic feasible tableaus maintain or increase the value of the objective function, it is not surprising that the basic solutions of successive tableaus above have not increased the value of the objective function at all—it remains at 0 in all seven tableaus.

The phenomenon in the example above is known as *cycling*. Cycling is rare; in fact, until quite recently, it was thought that cycling never occurred in practical problems, all of the pertinent examples having been artificially constructed. Then, in 1977, Kotiah and Steinberg ([K1]) discovered a nonartificial class of linear programming problems involving queueing theory which cycled. Hence, in this section, we give rules which prevent cycling. Inasmuch as cycling is a rare phenomenon, we make no guarantee of constant adherence to these rules in this book. While anticycling rules should certainly be a part of any computer implementation of the simplex algorithm, we treat cycling as an unfortunate infrequent occurrence rather than something that warrants constant special attention.

In each tableau of Example 18, the pivot choice is not uniquely determined by the simplex algorithm. (For example, 1/2 and 1 are acceptable alternate pivot entries in the initial tableau, 3/2 is an acceptable alternate pivot entry in the second tableau, 3/8 is an acceptable alternate pivot entry in the third tableau, etc.) In a sense, our particular pivot choices contributed to the cycling! We can remedy the phenomenon of cycling by placing additional requirements on the choice of pivot entries in those instances when more than one entry meets the pivoting requirements of the simplex algorithm. These pivoting rules are due to R.G. Bland ([B3]).

Simplex Algorithm Anticycling Rules

List all variables, both independent and dependent, appearing in the initial tableau. (The ordering of the variables in the list is not important as long as the rules below are implemented in a manner consistent with this list.) Any pivot entry is determined uniquely by a pivot row and a pivot column. The rules below determine this row and column.

Rule #1 (Determination of pivot row). Whenever there is more than one possible choice of pivot row in accordance with the simplex algorithm, choose the row corresponding to the variable that appears nearest the top (or front) of the list.

Rule #2 (Determination of pivot column). Whenever there is more than one possible choice of pivot column in accordance with the simplex algorithm, choose the column corresponding to the variable that appears nearest the top (or front) of the list.

A reminder. Don't get too engrossed in the application of these rules—before proceeding with any choice of pivot row or pivot column, examine the tableau for infeasibility or unboundedness.

We now illustrate how the anticycling rules eliminate the problem of cycling in Example 18.

EXAMPLE 19. Apply the simplex algorithm with anticycling rules to the initial maximum tableau of Example 18 below:

x_1	x_2	x_3	x_4	-1	
1/4	−8	−1	9	0	$= -t_1$
1/2	−12	−1/2	3	0	$= -t_2$
0	0	1	0	1	$= -t_3$
3/4	−20	1/2	−6	0	$= f$

We choose $x_1, x_2, x_3, x_4, t_1, t_2, t_3$ as our list of variables. In the initial tableau, we have two choices for a pivot column, namely the first column (corresponding to $c_1 = 3/4$) and the third column (corresponding to $c_3 = 1/2$). Since the first column corresponds to the variable x_1 and the third column corresponds to the variable x_3, we choose the first column as our pivot column in accordance with Rule #2 of the anticycling rules. We now have two choices for the pivot row, namely the first row ($b_1/a_{11} = 0/(1/4) = 0$) and the second row ($b_2/a_{21} = 0/(1/2) = 0$). Since the first row corresponds to the variable t_1 and the second row corresponds to the variable t_2, we choose the first row as our pivot row in accordance with Rule #1 of the anticycling rules. Hence we pivot on 1/4 as in Example 18 to obtain the second tableau

t_1	x_2	x_3	x_4	-1	
4	−32	−4	36	0	$= -x_1$
−2	4*	3/2	−15	0	$= -t_2$
0	0	1	0	1	$= -t_3$
−3	4	7/2	−33	0	$= f$

In this new tableau, we have two choices for a pivot column, namely the second column (corresponding to $c_2 = 4$) and the third column (corresponding to $c_3 = 7/2$). Since the second column corresponds to the variable x_2 and the third column corresponds to the variable x_3, we choose the second column as our pivot column in accordance with Rule #2 of the anticycling rules. The choice for the pivot row is then determined by the simplex algorithm and we pivot on 4 as in Example 18 to obtain the third tableau

t_1	t_2	x_3	x_4	-1	
-12	8	8	-84	0	$= -x_1$
$-1/2$	$1/4$	$3/8$	$-15/4$	0	$= -x_2$
0	0	1	0	1	$= -t_3$
-1	-1	2	-18	0	$= f$

In this new tableau, the choice for the pivot column is determined by the simplex algorithm but there are two choices for the pivot row, namely the first row ($b_1/a_{13} = 0/8 = 0$) and the second row ($b_2/a_{23} = 0/(3/8) = 0$). Since the first row corresponds to the variable x_1 and the second row corresponds to the variable x_2, we choose the first row as our pivot row in accordance with Rule #1 of the anticycling rules. Hence we pivot on 8 as in Example 18 to obtain the fourth tableau

t_1	t_2	x_1	x_4	-1	
$-3/2$	1	$1/8$	$-21/2$	0	$= -x_3$
$1/16$	$-1/8$	$-3/64$	$3/16$	0	$= -x_2$
$3/2$	-1	$-1/8$	$21/2$	1	$= -t_3$
2	-3	$-1/4$	3	0	$= f$

In this new tableau, we have two choices for a pivot column, namely the first column (corresponding to $c_1 = 2$) and the fourth column (corresponding to $c_4 = 3$). Since the first column corresponds to the variable t_1 and the fourth column corresponds to the variable x_4, we choose the fourth column as our pivot column in accordance with Rule #2 of the anticycling rules. The choice for the pivot row is then determined by the simplex algorithm and we pivot on 3/16 as in Example 18 to obtain the fifth tableau

t_1	t_2	x_1	x_2	-1	
2	-6	$-5/2$	56	0	$= -x_3$
$1/3$	$-2/3$	$-1/4$	$16/3$	0	$= -x_4$
-2	6	$5/2$	-56	1	$= -t_3$
1	-1	$1/2$	-16	0	$= f$

In this new tableau, we have two choices for a pivot column, namely the first column (corresponding to $c_1 = 1$) and the third column (corresponding to $c_3 = 1/2$). Since the first column corresponds to the variable t_1 and the third column corresponds to the variable x_1, we choose the third column as our pivot column in accordance with Rule #2 of the anticycling rules. (Note that the first column was chosen as the pivot column at this point in Example 18.)

The choice for the pivot row is then determined by the simplex algorithm and we pivot on 5/2 (instead of 2 as in Example 18) to obtain the sixth tableau

t_1	t_2	t_3	x_2	-1	
0	0	1	0	1	$= -x_3$
$2/15$	$-1/15$	$1/10$	$-4/15$	$1/10$	$= -x_4$
$-4/5$	$12/5$	$2/5$	$-112/5$	$2/5$	$= -x_1$
$7/5$	$-11/5$	$-1/5$	$-24/5$	$-1/5$	$= f$

The pivot in this new tableau is uniquely determined by the simplex algorithm so that no anticycling rules are necessary. The reader should verify that a pivot on $a_{21} = 2/15$ results in a tableau whose basic solution is optimal. Hence, the cycling problem of Example 18 has been remedied.

§9. Concluding Remarks

We have now developed the simplex algorithm with anticycling rules, a complete procedure for solving canonical maximization and canonical minimization linear programming problems. Canonical maximization and canonical minimization linear programming problems fall into four classes:

(i) infeasible linear programming problems,
(ii) unbounded linear programming problems,
(iii) linear programming problems having bounded constraint sets for which the optimal values of the objective functions are attained at extreme points, and
(iv) linear programming problems having unbounded constraint sets for which the optimal values of the objective functions are attained at extreme points.

The simplex algorithm with anticycling rules effectively handles all four classes above. In classes (i) and (ii), the algorithm terminates with a tableau indicating the infeasibility or unboundedness; in classes (iii) and (iv), the algorithm terminates with a tableau whose basic solution is optimal irrespective of the boundedness or unboundedness of the constraint set. In addition, the simplex algorithm is much more efficient than the geometric approach of Chapter 1. For example, the geometric approach of Chapter 1 applied to a canonical linear programming problem with 15 main constraints and 10 variables would involve finding and testing up to

$$\binom{25}{10} > 3200000$$

candidates for extreme points. The simplex algorithm, on the other hand, would only require between about 13 and 50 pivot steps. The simplex algorithm is also easily implemented on a computer.

Exercises

1. Consider the canonical maximum tableau below:

x	y	-1	
1	2	3	$= -t_1$
4	5	6	$= -t_2$
7	8	9	$= f$

   a. In the notation of (1) of §1, state the canonical maximization linear programming problem represented by the tableau above.

   b. Explain why the initial tableau for the simplex algorithm solution of the linear programming problem

   Maximize $f(x, y) = 7x + 8y - 9$
   subject to $x + 2y \geqq 3$
   $4x + 5y \geqq 6$
   $x, y \geqq 0$

   is *not* the tableau above.

   c. Pivot on 4 in the tableau above.

   d. Describe the tableau transition of part c in terms of the "solving and replacing every occurrence of" procedure demonstrated in §2.

2. Consider the canonical minimum tableau below:

x	1	2	3
y	4	5	6
-1	7	8	9
	$= t_1$	$= t_2$	$= g$

   a. In the notation of (3) of §1, state the canonical minimization linear programming problem represented by the tableau above.

   b. Explain why the initial tableau for the simplex algorithm solution of the linear programming problem

   Minimize $g(x, y) = 3x + 6y - 9$
   subject to $x + 4y \leqq 7$
   $2x + 5y \leqq 8$
   $x, y \geqq 0$

   is *not* the tableau above.

   c. Pivot on 4 in the tableau above.

   d. Describe the tableau transition of part c in terms of the "solving and replacing every occurrence of" procedure demonstrated in §2.

3. a. Describe the tableau transitions in Example 5 in terms of the "solving and replacing every occurrence of" procedure demonstrated in §2.

   b. Interpret the condition $t_P = t_F = 0$ in the optimal solutions of Example 5.

4. Solve each of the linear programming problems in Exercise 3 of Chapter 1 by using the simplex algorithm. In each problem, illustrate the movement in the constraint set diagram exhibited by the basic solutions of successive tableaus. [Note: When illustrating the movement in a minimization problem, ignore the negative transposition step to maximum tableau form.]

5. Solve each of the canonical linear programming problems below by using the simplex algorithm.

   a. Maximize $f(x, y) = x$
   subject to $x + y \leqq 1$
   $x - y \geqq 1$
   $y - 2x \geqq 1$
   $x, y \geqq 0$

   b. Minimize $g(x, y) = y - 5x$
   subject to $x - y \geqq 1$
   $y \leqq 8$
   $x, y \geqq 0$

   c. Minimize $g(x, y, z) = -x - y$
   subject to $3x + 6y + 2z \leqq 6$
   $y + z \geqq 1$
   $x, y, z \geqq 0$

   d.

x	y	-1	
1	-1	3	$= -t_1$
-2	1	2	$= -t_2$
2	-1	0	$= f$

   e.

x	-2	1	-3
y	1	-2	-2
-1	1	0	0
	$= t_1$	$= t_2$	$= g$

   f.

x	y	-1	
-1	-1	-2	$= -t_1$
1	-2	0	$= -t_2$
-2	1	1	$= -t_3$
-1	3	0	$= f$

6. a. Solve the canonical linear programming problem below by using the simplex algorithm with anticycling rules corresponding to the list x, y, t_1, t_2, t_3.

x	y	-1	
3	2	1	$= -t_1$
-9	-2	0	$= -t_2$
3	1	0	$= -t_3$
3	2	1	$= f$

b. Sketch the constraint set corresponding to the problem in part a and illustrate the movement in the constraint set diagram exhibited by the basic solutions of the successive tableaus in part a.

7. The canonical programming problem below (due to H.W. Kuhn and given in [B1]) will cycle after six particular simplex algorithm pivots. (The ambitious reader is invited to find these pivots and confirm this.) Solve the problem by using the simplex algorithm with anticycling rules.

x_1	x_2	x_3	x_4	-1	
-2	-9	1	9	0	$= -t_1$
1/3	1	-1/3	-2	0	$= -t_2$
2	3	-1	-12	2	$= -t_3$
2	3	-1	-12	0	$= f$

8. Each of the canonical linear programming problems below has infinitely many optimal solutions. Solve each of the linear programming problems by using the simplex algorithm and find all optimal solutions. [Note: In the exercises of Chapter 3 and Chapter 4, an increasing emphasis will be made on finding *all* optimal solutions of linear programming problems having infinitely many optimal solutions. For this reason, complete discussions of the problems below may be found in the answers section in the back of this book.]

a.

x	y	z	w	-1	
0	1	1	-1	3	$= -t_1$
1	1	1	-1	3	$= -t_2$
1	2	2	-4	0	$= f$

b.

x	-1	-1	-1
y	-1	1	-1
-1	-2	1	0
	$= t_1$	$= t_2$	$= g$

9. Solve each of the linear programming problems below.

a. A nut company makes three different mixtures of nuts having the following compositions and profits per pound:

	Peanuts	Cashews	Pecans	Profit
Mixture 1	100%	0%	0%	\$2
Mixture 2	80%	15%	5%	\$1.50
Mixture 3	60%	30%	10%	\$1

The management of the company decides that it wants to produce at least twice as much of mixture 3 as of mixture 2 and at least twice as much of mixture 2 as of mixture 1. The company has 500 pounds of peanuts, 250 pounds of cashews, and 100 pounds of pecans available. If all production can be sold, how many pounds of each mixture should be produced so as to maximize profits?

b. A hotel rental service needs to have clean towels for each day of a three-day period. Some of the clean towels may be purchased new and some may be dirty towels from previous days that have been washed by a laundry service. The cost of new towels is \$1 per towel, the cost of a fast one-day laundry serice is 40¢ per towel, and the cost of a slow two-day laundry service is 25¢ per towel. If the rental service needs 300, 200, and 400 clean towels for each of the next three days (respectively), how many towels should the rental service buy new and how many should the rental service have washed by the different laundry services so as to minimize total costs?

10. Consider the canonical maximum tableau below:

x_1	x_2	-1	
a_{11}	a_{12}	b_1	$= -t_1$
a_{21}	a_{22}	b_2	$= -t_2$
c_1	c_2	d	$= f$

If $a_{ij} \neq 0$, prove that pivoting on a_{ij} is equivalent to solving the i^{th} equation of the tableau for the j^{th} variable and replacing every occurrence of this variable in the other equations of the tableau by the resulting expression.

11. (This problem is an application of the pivot transformation to linear algebra.) Let $A = [a_{ij}]_{n \times n}$ be a square matrix. Form the tableau

x_1	x_2	$\cdots$	x_n	-1	
a_{11}	a_{12}	$\cdots$	a_{1n}	0	$= -t_1$
a_{21}	a_{22}	$\cdots$	a_{2n}	0	$= -t_2$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
a_{n1}	a_{n2}	$\cdots$	a_{nn}	0	$= -t_n$
0	0	$\cdots$	0	0	$= f$

Then it is a fact that it is possible to transform the tableau above into the tableau

	t_1	t_2	$\cdots$	t_n	-1	
	a_{11}'	a_{12}'	$\cdots$	a_{1n}'	0	$= -x_1$
	a_{21}'	a_{22}'	$\cdots$	a_{2n}'	0	$= -x_2$
	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
	a_{n1}'	a_{n2}'	$\cdots$	a_{nn}'	0	$= -x_n$
	0	0	$\cdots$	0	0	$= f$

via a sequence of pivot transformations and possibly a rearrangement of rows and/or columns if and only if $A' = [a_{ij}']_{nxn} = A^{-1}$. Use this fact to invert the matrices below if possible. [Note: Do not use the simplex algorithm here to determine pivot entries; choose pivot entries that will move the x's from north to east and the t's from east to north. Also, since the last row and the last column of all tableaus will always be zero, they may be deleted without any harm.]

a. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$

12. a. Find a necessary and sufficient condition for the minimum tableau

x_1	a_{11}	a_{21}	$\cdots$	a_{m1}	c_1
x_2	a_{12}	a_{22}	$\cdots$	a_{m2}	c_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_n	a_{1n}	a_{2n}	$\cdots$	a_{mn}	c_n
-1	b_1	b_2	$\cdots$	b_m	d
	$= t_1$	$= t_2$	$\cdots$	$= t_m$	$= g$

to have a feasible basic solution.

b. Does the tableau satisfying the condition in part a but viewed as a maximum tableau necessarily have a feasible basic solution?

c. Find a necessary and sufficient condition for the tableau

	x_1	x_2	$\cdots$	x_n	-1	
y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	b_1	$= -t_1$
y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	b_2	$= -t_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	b_m	$= -t_m$
-1	c_1	c_2	$\cdots$	c_n	d	$= f$
	$= s_1$	$= s_2$	$\cdots$	$= s_n$	$= g$	

viewed as a maximum tableau to have a feasible basic solution *and* viewed as a minimum tableau to have a feasible basic solution.

13. In the simplex algorithm for maximum basic feasible tableaus, the choice of positive c_j in step (3) is unrestricted. It is shown in [R2] that restrictions on the choice of positive c_j in this step effectively reduce the number of pivoting operations required in the simplex algorithm, especially for large linear programming problems. Two such restrictions are discussed below.

a. Replace step (3) of the simplex algorithm for maximum basic feasible tableaus with

(3′) Choose the most positive $c_j > 0$.

Apply this new simplex algorithm for maximum basic feasible tableaus to the linear programming problem of Example 11. Illustrate the movement in the constraint set exhibited by the basic solutions of successive tableaus above and compare this movement with the movement exhibited in Example 11.

b. Replace step (3) of the simplex algorithm for maximum basic feasible tableaus with

(3′) For each $c_j > 0$, compute

$$\mu_j = \min_{1 \leq i \leq m} \{b_i/a_{ij} : a_{ij} > 0\}.$$

Choose the c_j for which $\mu_j c_j$ is most positive.

Apply this new simplex algorithm for maximum basic feasible tableaus to the canonical linear programming problem below:

x	y	z	-1	
1	2	1	4	$= -t_1$
2	1	5	5	$= -t_2$
3	2	0	6	$= -t_3$
1	2	3	0	$= f$

f. Max $f = 100$ at $(x, y, z) = (100, 0, 0)$
g. Min $g = -425$ at $(x, y, z) = (0, 150, 125)$
h. Let x and y be the number of units of the Monitor and the number of units of the Recorder respectively and let P be the total profits. Then max $P = 10000/3$ at $(x, y) = (20/3, 20/3)$; since fractional magazines are not realistic, the "rounded-off" optimal solution is max $P = 3333$ at $(x, y) = (6.66, 6.67)$.
i. Let x and y be the number of days for the first operation and the number of days for the second operation respectively and let C be the total costs. Then min $C = 4750$ at $(x, y) = (10, 5)$.
j. Let x, y, and z be the number of units of the first formulation, the number of units of the second formulation, and the number of units of the third formulation respectively and let R be the total sales revenue. Then max $R = 475$ at $(x, y, z) = (25, 0, 150)$ or at $(x, y, z) = (0, 50, 125)$.

6. Max $f = 2$ at $(x, y, z) = (1, 0, 0)$

7. 35

8. b. $(0,0,0,1/2)$, $(0,0,1/3,0)$, $(0,1/4,0,0)$, $(1/5,0,0,0)$, $(0,0,0,0)$
   c. Min $g = -2$ at $(x, y, z, w) = (0, 0, 0, 1/2)$

9. c. $(0,0,1,1/2)$, $(0,1,1,0)$, $(3,0,0,1/2)$, $(3,1,0,0)$, $(0,0,0,5/4)$, $(0,0,5/3,0)$, $(0,5/2,0,0)$, $(5,0,0,0)$, $(0,0,0,1/2)$, $(0,1,0,0)$, $(0,0,0,0)$
   d. The actual extreme points of part c are $(0,0,1,1/2)$, $(0,1,1,0)$, $(3,0,0,1/2)$, $(3,1,0,0)$, $(0,0,5/3,0)$, and $(5,0,0,0)$; min $g = 1$ at each of the first four of these extreme points.

11. a. TRUE
    b. FALSE

Chapter 2

1. a. Maximize $f(x, y) = 7x + 8y - 9$
   subject to $x + 2y \leqq 3$
   $4x + 5y \leqq 6$
   $x, y \geqq 0$

c.

t_2	y	-1	
$-1/4$	$3/4$	$3/2$	$= -t_1$
$1/4$	$5/4$	$3/2$	$= -x$
$-7/4$	$-3/4$	$-3/2$	$= f$

2. a. Minimize $g(x, y) = 3x + 6y - 9$
   subject to $x + 4y \geqq 7$
   $2x + 5y \geqq 8$
   $x, y \geqq 0$

c.

x	$-1/4$	$3/4$	$3/2$
t_1	$1/4$	$5/4$	$3/2$
-1	$-7/4$	$-3/4$	$-3/2$
	$= y$	$= t_2$	$= g$

5. a. Infeasible
   b. Unbounded
   c. $x = 4/3$, $y = 0$, $z = 1$, $t_1 = 0$, $t_2 = 0$, min $g = -4/3$
   d. Unbounded
   e. Infeasible
   f. Unbounded

6. a. $x = 0$, $y = 0$, $t_1 = 1$, $t_2 = 0$, $t_3 = 0$, max $f = 1$

7. $x_1 = 2$, $x_2 = 0$, $x_3 = 2$, $x_4 = 0$, $t_1 = 2$, $t_2 = 0$, $t_3 = 0$, max $f = 2$

8. For definiteness of pivots, the anticycling rules were implemented in both problems below.

   a. The final tableau is

t_2	x	z	w	-1	
-1	-1	0	0	0	$= -t_1$
1	1	1	-1	3	$= -y$
-2	-1	0	-2	-6	$= f$

Now t_2, x, and w must be 0 since the coefficients of these variables in the objective function are negative, i.e., any positive choice for any of these variables decreases f. Note, however, that the coefficient of z in the objective function is 0 and hence z is not forced to be 0 in order for f to be optimal. To see what possible values $z \geqq 0$ can assume, examine the main constraints of the final tableau (remembering that $t_2 = x = w = 0$):

$$0 = -t_1$$
$$z - 3 = -y.$$

The first constraint gives $t_1 = 0$. The second equation gives $y = 3 - z$; since $y \geqq 0$, we have $3 - z \geqq 0$, i.e., $z \leqq 3$. Hence, all optimal solutions for this problem may be expressed as follows:

$$t_2 = x = w = 0, \quad t_1 = 0, \quad 0 \leqq z \leqq 3, \quad y = 3 - z, \quad \max f = 6.$$

   b. The final tableau is

t_1	t_2	-1	
$1/2$	$1/2$	$1/2$	$= -x$
$1/2$	$-1/2$	$3/2$	$= -y$
-1	0	-2	$= -g$

Now t_1 must be 0 since the coefficient of t_1 in the objective function is negative, i.e., any positive choice for t_1 decreases $-g$. Note, however, that the coefficient of t_2 in the objective function is 0 and hence t_2 is not forced to be 0 in order for $-g$ to be optimal. To see what possible values $t_2 \geqq 0$ can assume, examine the main constraints of the final tableau (remembering that $t_1 = 0$):

$$1/2t_2 - 1/2 = -x$$
$$-1/2t_2 - 3/2 = -y.$$

The first equation gives $x = 1/2 - 1/2t_2$; since $x \geqq 0$, we have $1/2 - 1/2t_2 \geqq 0$. i.e., $t_2 \leqq 1$. The second equation gives $y = 1/2t_2 + 3/2$; since $y \geqq 0$, we have $1/2t_2 + 3/2 \geqq 0$, i.e., $t_2 \geqq -3$ which we already know. Hence, all optimal solutions for this problem may be expressed as follows:

$$t_1 = 0, \quad 0 \leqq t_2 \leqq 1, \quad x = 1/2 - 1/2t_2, \quad y = 1/2t_2 + 3/2, \quad \min g = -2.$$

9. a. Let x, y, and z be the number of pounds of the first mixture, the number of pounds of the second mixture, and the number of pounds of the third mixture respectively and let P be the total profits. Then $\max P = 900$ at $(x, y, z) = (100, 200, 400)$.
   b. Let x, y, z, and w be the number of towels purchased new, the number of towels washed by the one-day service after the first day, the number of towels washed by the one-day service after the second day, and the number of towels washed by the two-day service after the first day respectively and let C be the total costs. Then $\min C = 570$ at $(x, y, z, w) = (400, 100, 200, 200)$.

11. a. $\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$
   b. The given matrix is noninvertible.
   c. $\begin{bmatrix} 1/6 & 0 & -1/2 & 2/3 \\ 0 & -1/2 & 1 & -1/2 \\ -1/2 & 1 & -1/2 & 0 \\ 2/3 & -1/2 & 0 & 1/6 \end{bmatrix}$

12. a. $b_1, b_2, \ldots, b_m \leqq 0$
   b. No
   c. $b_1, b_2, \ldots, b_m \geqq 0$ and $c_1, c_2, \ldots, c_n \leqq 0$

13. b. $x = 0, \quad y = 5/3, \quad z = 2/3, \quad t_1 = 0, \quad t_2 = 0, \quad t_3 = 8/3, \quad \max f = 16/3$

Chapter 3

1. a. $t_1 = t_3 = 0, x \leqq 3, y = 2 - x, z = 8 - 2x, t_2 = 3 - x, \max f = 6$
   b. Unbounded
   c. $x = 0, y = 6, z = 4, t_1 = 0, \min g = 14$
   d. Unbounded
   e. $t_1 = 0, 0 \leqq x \leqq 1/2, y = 1 - 2x, z = x + 1/2, t_2 = x, \max f = 3/2$
   f. $x = 8, y = 2, z = 0, t_1 = 0, \max f = 20$
   g. $t_1 = 2, t_2 = 0, x = x, y = x - 3, z = -2x, \min g = -3$
   h. Infeasible

2. a. FALSE
   b. FALSE

3. $t_1 = 0, 0 \leqq x \leqq 1/2, y = 2/3 - 4/3x, z = 1/3x + 1/3, \max f = 0$

5. a. The solutions are the same ($x = 4, y = 2, t_1 = 0, t_2 = 0, \max f = 6$).
   b. The solutions are not the same. Canonical solution: $x = 1, y = 0, t_1 = 0, t_2 = 3, \min g = 1$. Noncanonical solution: Unbounded.
   c. The solutions are not the same. Canonical solution: Infeasible. Noncanonical solution: $x = -3, y = -4, t_1 = 0, t_2 = 0, \max f = -5$.
   d. The solutions are the same (unbounded).

Chapter 4

1. a. Minimize $g(y_1, y_2) = 4y_1 + 6y_2$
   subject to $y_1 + 3y_2 \geqq 1$
   $2y_1 + y_2 \geqq 1$
   $y_1, y_2 \geqq 0$
   c. Max: $x_1 = 8/5, x_2 = 6/5, t_1 = 0, t_2 = 0, \max f = 14/5$
   Min: $y_1 = 2/5, y_2 = 1/5, s_1 = 0, s_2 = 0, \min g = 14/5$
   d. Yes

2 a. Maximize $f(x_1, x_2) = x_1 + 2x_2$
   subject to $x_1 - x_2 \leqq 0$
   $-x_1 + x_2 \leqq -1$
   $x_1, x_2 \geqq 0$
   c. Max: Infeasible
   Min: Infeasible

3. a. Minimize $g(y_1, y_2, y_3) = y_1 - y_2 - y_3$
   subject to $y_1 - y_2 + 2y_3 \geqq 1$
   $y_1 + y_2 - y_3 \geqq 0$
   $y_1, y_2, y_3 \geqq 0$
   c. Max: Infeasible
   Min: Unbounded

5. a. Max: $x_1 = 0 x_2 = 1, t_1 = 0, t_2 = 0, \max f = -2$
   Min: $s_2 = 0, 0 \leqq s_1 \leqq 1, y_1 = 1/2s_1 + 3/2, y_2 = 1/2 - 1/2s_1, \min g = -2$
   b. Max: Unbounded
   Min: Infeasible
   c. Max: Infeasible
   Min: Infeasible
   d. Max: $x_1 = 0, x_2 = 0, t_1 = 0, t_2 = 1, t_3 = 0, \max f = 0$
   Min: $y_2 = 0, y_1 \geqq 0, s_2 \geqq 0, y_3 = 2y_1 + s_2 + 2, s_1 = 15y_1 + 3s_2 + 3, \min g = 0$
   e. Max: Infeasible
   Min: Unbounded
   f. Max: Unbounded
   Min: Infeasible