#1 $ http://www.physics.fsu.edu/users/ng/courses/phy2048c/lab/appendixI/App1.htm

Appendix I: Estimates for the Reliability of Measurements

In any measurement there is always some error or uncertainty in the result. This uncertainty may be large or small, sometimes small enough to completely neglect, but it is always there. Uncertainties and errors can come from any number of sources such as human errors in reading and recording data, uncertainties of instrument resolution, changes in the environment of the experiment, and many others. In any experiment one of the main jobs of the experimenter is to determine the size of the uncertainty in the measurement and when possible to identify the causes of any possible errors.

In general when one makes a number of measurements of the same quantity in an experiment, one usually obtains different results. What we need then is a method of determining from the different results the best value of the measured quantity and with what certainty we are able to call this value the best. These various measurements may be the results of completely different experiments or they may be the results of the same experiment repeated several times. Differences between the measurements can be due systematic (such as, those errors resulting from the method of the measurement) or random (such as, those errors resulting from the limited accuracy of the equipment). As an experimenter you try to eliminate the former and minimize the latter.

The discussion that follows presents two methods for estimating errors. The first is used when it is not possible to make repeated measurements. In such cases one must resort to making a reasonable estimate of the uncertainties. When multiple measurements are possible a well defined mathematical formalism can be used to estimate the errors in the measured quantities. This appendix also discusses the presentation of results which should be guided by the size of the certainty in the final result. Finally, this appendix presents the mathematical formalism needed to estimate how uncertainties in measured quantities effect uncertainties in quantities deduced from measurements. This propagation

of error is necessary whenever measured results must be combined to determine the quantity of interest.

Estimated Uncertainties or Guesstimated Uncertainties

It is not always possible to calculate the uncertainty in the result of an experiment using the results of a number of separate measurements of the same quantity. One's ability to make multiple measurements is often limited by time and money. In such cases an uncertainty can be estimated or *guesstimated* by observation of how close a measurement can be made or by a crude determination from variations in similar measurements. *Guesstimated* uncertainties are usually a good estimate of the random uncertainties, e.g., the standard deviation.

Whenever possible during this course uncertainties calculated from multiple measurements should be used. However, you will find in most cases this will not be possible due to time limitations. In these cases an estimated uncertainty should always be made and the basis for this estimate should be stated.

Calculated Uncertainties

Assume that you have made n different measurements of a quantity x. Usually the results of these measurements will vary; call them $x_1, x_2, ..., x_n$. We define the mean or average of these measurements as

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

.

For each measurement you can now calculate the deviation from the average, namely $x_i - \bar{x}$. From the definition of the mean it follows that the sum of these deviations must be zero.

There are several ways of estimating uncertainty. First we will discuss a very simple method, then we will present the most commonly used method. This latter method provides the basis for determining uncertainties in most of our experiments. More details concerning this method can be found in the section on *Elements of Statistical Inference*. Another common method is to simply guess the error, this also has it place in our laboratory and it is important that you learn to estimate the error in a measurement simply by examining how the measurement was made.

1) A simple estimate of the uncertainty of your measurements can be obtained by adding the sum of the absolute values of the differences between the individual measurements and the average value of the measurements. This sum, divided by the number of measurements is called the average absolute deviation, α.

$$\alpha = \frac{1}{n}\sum_{i=1}^{n}\left|x_i - \bar{x}\right|$$

The parameter α is an approximate measure of the typical deviation of any one of the measurements from the average.

The uncertainty in the average result of a set of *n* measurements can be estimated by computing the value of α and dividing it by $\sqrt{n}$, where again *n* is the number of measurements. Thus one can write the result of *n* experimental determinations of *x* as

$$x = \bar{x} \pm \frac{\alpha}{\sqrt{n}}$$

2) The most commonly used method of estimating uncertainty, and the method you should use in this course, is based on statistical considerations. In this system uncertainty is defined in terms of *the root-mean-square* or *standard deviation*, σ,

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$$

which is related to the square of the difference of each measured value from the mean. The division by *n-1* is discussed in the *Elements of Statistical Inference* section. As stated in that section, the chances are roughly two out of three (more exactly, 68.26%) that the mean of a sample of *n* measurements of a quantity differs by less than $\sigma/\sqrt{n}$ from the true mean value for that quantity. The quantity $\sigma/\sqrt{n}$ is the standard deviation of the mean. Once you have computed the mean and standard deviation of a set of measurements, the results are quoted as follows:

$$result = \bar{x} \pm \frac{\sigma}{\sqrt{n}}$$

Example

The following example demonstrates how this is done in practice.

#3 $ http://www.physics.fsu.edu/users/ng/courses/phy2048c/lab/appendixI/App1.htm

Assume you have five measurements of distance x. The table below shows how to compute $\bar{x}$, α, and σ, and how to present the final result.

Length Measurements

n	x_i	$(\bar{x} - x_i)$	$\lvert\bar{x} - x_I\rvert$	$(\bar{x} - x_i)^2 \times 10^{-4}$
1	45.12	-0.002	0.002	0.04
2	45.09	-0.032	0.032	10.23
3	45.14	+0.018	0.018	3.24
4	45.16	+0.038	0.038	14.42
5	45.10	-0.022	0.022	4.48
sum	225.61	0.00	0.112	32.77

First calculate the average of the measurements, $\bar{x}$:

$$\bar{x} = \frac{1}{5} \cdot 225.61$$

$$\bar{x} = 45.122$$

Next calculate the average deviation, α:

$$\alpha = \frac{1}{5} \cdot 0.112 = 0.0224$$

$$\frac{\alpha}{\sqrt{5}} = 0.0101$$

Then calculate the standard deviation, σ:

$$\sigma^2 = \frac{32.77 \cdot 10^{-4}}{5 - 1} = 8.19 \cdot 10^{-4}$$

$$\sigma = 2.86 \cdot 10^{-2}$$

$$\frac{\sigma}{\sqrt{5}} = 1.28 \cdot 10^{-2}$$

From these values we can state the result of the measurement in one of two ways:

Using the average absolute deviation:

$$x = 45.122 \pm 0.010 \; cm$$

and using the standard deviation:

$$x = 45.122 \pm 0.013 \; cm$$

Methods of Stating Error

In giving the result of an experiment, it is clearly meaningless to state the result to much greater precision than is indicated by your estimate of its error. Thus, it is nonsensical to give a result like the following:

$$A = 13.25432 \pm 0.104372 \; cm \text{ (!)}$$

The meaningful result would be stated:

$$A = 13.3 \pm 0.1 \; cm$$

or perhaps

$$A = 13.2_5 \pm 0.1_0 \; cm$$

The digits 132 are said to be significant because they lie within the range of reliability as measured by the stated error.

A rule of thumb regarding the carrying of significant figures though a series of arithmetic operations is that you should carry one more than the minimum number of significant digits in any of the contributing factors.

It should be noted that zeroes give rise to some ambiguity here since they are used to indicate the position of the decimal point and may not be significant digits at all. Therefore you should learn to state results in terms of number between one and ten, times an appropriate power of ten. Thus:

$$(1.25 \pm 0.04) \cdot 10^5$$

not

$$125{,}000 \pm 4{,}000$$

One final word of caution: In taking and recording individual measurements do not round off the numbers according to your estimate or guess as to the reliability of each measurement. If you do, you will force σ to be larger than you estimate (i.e., carry at least one more figure than you think significant.) Only round off the final results in your reports.

Propagation of Uncertainty

We will first describe a *simple* method of calculating the uncertainty in the final result of an experiment which involves several measurements from the uncertainties in the measurements of the contributions. To take a very simple example, suppose we have measured two lengths, and that the final result of our experiment is to be the sum or difference of these two lengths. Let the measured lengths and their respective actual uncertainties be $A \pm D A$ and $B \pm D B$. Then the final result will have an uncertainty which may lie anywhere between $(D A + D B)$. Thus the sum is written as

$$S = (A + B) \pm (D A + D B)$$

while the difference is written as

$$D = (A - B) \pm (D A + D B).$$

Notice the fractional uncertainty in the difference D is much greater than that in either A or B alone if A and B are nearly equal.

If we want to find the uncertainty in the product of A and B we proceed as follows:

$$P = (A \pm D A)^{x} (B \pm D B)$$

that is

$$P = A^{x} B \pm (A^{x} D B + B^{x} D A \pm D A^{x} D B)$$

Since ΔA and ΔB are usually small compared with A and B, we can neglect the product $\Delta A^{x} D B$. By neglecting this term we find

$$P » A^{x} B \pm (A^{x} D B + B^{x} D A).$$

Instead of giving the absolute uncertainty as written above, one usually gives the fractional uncertainty by dividing the uncertainty terms by $A^{x} B$. We thus define the fractional uncertainty

$$\frac{\Delta P}{P} = \frac{A \cdot \Delta B + B \cdot \Delta A}{A \cdot B} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$$

and write the product as

$$P = A^{x} B \pm D P.$$

It can be shown that the same relationship will hold for the quotient $Q = A/B$, so that one can write

$$Q = \frac{A}{B} \pm \left(\frac{\Delta A}{A} + \frac{\Delta B}{B} \right).$$

These formulas overestimate the uncertainty somewhat, since the probability that, in a given experiment, the uncertainty in A and B would each have the same sign is only 50%. A better estimate (and the one which you should use in this course) can be obtained by using the differentiating procedure outlined below. A detailed derivation of these formulae is presented at the end of the section for those who are familiar with multivariable calculus. In any event you should use the formulas given at the end of this write-up whenever possible.

Mathematical Derivation of Error Propagation Formulae

Small uncertainties in a function of one variable:

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Suppose we have a function *F*, which depends on only one variable, *x*, i.e. $F = F(x)$. Now when the uncertainty in *x* (which we call Δx) is small, we can use differentials to estimate the relationship between the uncertainty in *x* and the uncertainty in *F* (designated as ΔF). From the definition of a differential

$$dF = \frac{dF}{dx}dx \quad [1]$$

where $\frac{dF}{dx}$ is the first derivative of *F* with respect to *x* evaluated at *x*. By definition *dx* is infinitely small, as is *dF*. However, if Δx is small compared to *x*, then, to a good approximation, we can replace *dx* and *dF* in the expression above by Δx and ΔF, which are finite in size. In addition, we can assume that $\frac{dF}{dx}$ does not vary over the interval Δx, even though *x* and *F* may vary.

For example, suppose $F = ax^2$ where *a* is a constant. Then

$dF = 2ax\, dx$ [2]

Converting this to finite changes gives $\Delta F = 2a\, x\, D\, x$. In terms of uncertainty this says: If the quantity *x* has an uncertainty Δx, then the uncertainty in *F* (where $F = ax^2$) will be $2ax\, D\, x$.

Quite often it is desirable to talk about fractional uncertainty, ΔF. For the example above

$$\frac{\Delta F}{F} = \frac{2ax\Delta x}{ax^2} = 2\left(\frac{\Delta x}{x}\right). \quad [3]$$

This equations states the fractional uncertainty in F is twice the fractional uncertainty in *x*.

Small uncertainties in functions of several variables.

Suppose *F* is a function of several variables, i.e., $F = F(x,y,z)$ and each of these variables has its own uncertainty (Δx, Δy, and Δz). The problem can be analyzed using the same procedure as described above, except that we must take the differential of *F* with respect to several variables. In analogy to the procedure for one variable one gets

$$\Delta F = \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y + \frac{\partial F}{\partial z}\Delta z \quad [4]$$

where $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$ are *partial derivatives*. A partial derivative means that the derivative is taken with respect to one variable, while all the other variables are considered constant.

For example, take $F = ax^2y^3/z$ where *a* is a constant.

$$\frac{\partial F}{\partial x} = \frac{2axy^3}{z}, \frac{\partial F}{\partial y} = \frac{3ax^2y^2}{z} \text{ and } \frac{\partial F}{\partial z} = -\frac{ax^2y^3}{z^2}$$

Using these partial derivatives in Equation. 4, one obtains

$$\Delta F = \frac{2axy^3}{z}\Delta x + \frac{3ax^2y^2}{z}\Delta y - \frac{ax^2y^3}{z^2}\Delta z \quad [5]$$

or expressed in terms of fractional uncertainties:

$$\frac{\Delta F}{F} = \frac{2\Delta x}{x} + \frac{3\Delta y}{y} - \frac{\Delta z}{z} \quad . [6]$$

Propagation of the Mean Square Uncertainty

The method described in Equations. 1 through 6 gives us a way of estimating the uncertainty in a function of several variables each with its own uncertainty. However, as we argued when discussing the statistical treatment of data, it is the mean-square uncertainty which in general gives the best estimate. In order to calculate this using the present method, we first square Equation 4.

$$(\Delta F)^2 = \left(\frac{\partial F}{\partial x}\Delta x\right)^2 + \left(\frac{\partial F}{\partial y}\Delta y\right)^2 + \left(\frac{\partial F}{\partial z}\Delta z\right)^2$$
$$+ 2\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\Delta x\Delta y + 2\frac{\partial F}{\partial x}\frac{\partial F}{\partial z}\Delta x\Delta z + 2\frac{\partial F}{\partial y}\frac{\partial F}{\partial z}\Delta y\Delta z \quad [7]$$

Now let Δx, Δy, and Δz take on all possible values within their allowed ranges of variation. The derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$ are essentially constants if Δx, Δy, and Δz are small.

If we further assume that no change in one variable will affect the change in any other variable, then the crossterms will be zero. This can be seen by noting that the uncertainty, when not squared, has a sign associated with it. For each $+\Delta x$ there is *a - D x* value, and for each $+\Delta y$, *a -D y*. Averaging over these four possibilities to find the average ΔxDy crossterm gives

$\Delta xDy + \Delta x(-\Delta y) + (-\Delta x)(-\Delta y) + (-\Delta x)\Delta y = \Delta xDy(1-1+1-1) = 0.$

A similar argument can be made for the other crossterms. Then the general form for the mean square uncertainty (and the one used in propagating uncertainties) is

$$(\Delta F)^2 = \left(\frac{\partial F}{\partial x}\Delta x\right)^2 + \left(\frac{\partial F}{\partial y}\Delta y\right)^2 + \left(\frac{\partial F}{\partial z}\Delta z\right)^2 \quad [8]$$

Examples:

Suppose $F = x - y$. Then differentiating we get $\Delta F = Dx - Dy$. From above we see that the uncertainty in F is given by

$$\Delta F^2 = (\Delta x)^2 + (\Delta y)^2$$

or

$$\Delta F = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

As a second example suppose $F = xy$, then

$$\Delta F = y\Delta x + x\Delta y = \frac{F\Delta x}{x} + \frac{F\Delta y}{y}$$

and

$$\Delta F^2 = \left(\frac{F\Delta x}{x}\right)^2 + \left(\frac{F\Delta y}{y}\right)^2$$

The uncertainty in F becomes

$$\Delta F = F\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2} .$$

Another way of solving this same problem is to first take the natural log of each side

$\ln F = \ln x + \ln y$

then differentiate

$$\frac{\Delta F}{F} = \frac{\Delta x}{x} + \frac{\Delta y}{y}$$

next square each side, dropping the crossterms

$$\Delta F^2 = F^2\left(\frac{\Delta x}{x}\right)^2 + F^2\left(\frac{\Delta y}{y}\right)^2$$

and finally obtaining

$$\Delta F = F\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

As a final example, suppose $F = x/y$, then taking natural log of both sides.

$\ln F = \ln x - \ln y$

Differentiating:

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$$\frac{\Delta F}{F} = \frac{\Delta x}{x} - \frac{\Delta y}{y}$$

Squaring and summing:

$$\Delta F^2 = F^2\left(\frac{\Delta x}{x}\right)^2 + F^2\left(\frac{\Delta y}{y}\right)^2$$

then

$$\Delta F = F\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

This result is the same as the result for $F = xy$.

Formulae for Standard Cases

Following is a list of some formulae for propagating uncertainties.

1. $F = x \cdot y$

$$\Delta F = F\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

2. $F = Cx^n y^m z^k$ in which C, n, m, and k are constants.

$$\Delta F = F\sqrt{\left(n\frac{\Delta x}{x}\right)^2 + \left(m\frac{\Delta y}{y}\right)^2 + \left(k\frac{\Delta z}{z}\right)^2}$$

3. $F = Cx^y$ in which C is a constant.

$$\Delta F = F\sqrt{\left(y\frac{\Delta x}{x}\right)^2 + \left(\Delta y \ln x\right)^2}$$

4. $F = Cx \log_{10} y$ in which C is a constant.

$$\Delta F = F\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y \log_{10} e}{y \log_{10} y}\right)^2}$$

5. $F = Cx \sin y$ in which C is a constant.

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$$\Delta F = F\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{\tan y}\right)^2}$$

6. $F = Cx \cos y \sin z$ in which C is a constant.

$$\Delta F = F\sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{\cos y}\right)^2 + \left(\frac{\Delta z}{\tan z}\right)^2}$$