Single facility location

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The facility location is studied in the case of a single facility, in the continuous case. The results are surprising.

Key words: *facility location; optimization; supply chain.*

1. Fundamentals and scope

In the supply chain environment, one of the first problems to be solved, regarding the enlargement of a company's activity, is the creation of a single, new facility (warehouse, depot), which must be located in a place that minimizes the global cost of transportation of goods to the customers, according to the distance from the facility to them and to the "size" of their demand, for given unit costs of transportation. The mathematics involved is simple, but the results may look surprising.

The omnipresence of this type of problem leads us to consider it in detail. Further generalizations are, of course, very important, but the analysis of this problem is enlightening. The generalizations can be many, such as: implementation of more than one facility; and preference for discrete variables. The continuous case will be addressed in what follows.

2. The problem

Suppose the company under consideration has n customers (or cities, etc.), with known localizations, (x_i, y_i), and demands (or weights), W_i, $i = 1..n$. The question is to determine the localization of the depot in order to minimize the total cost of transportation, z, in a given period of time,

$$z(x, y) = \sum_i r_i W_i d_i(x, y) \qquad \{1\}$$

with r_i the transportation rate (cost), in units of \$ / kg-km, with \$ an arbitrary money unit, assuming that the distances will be in km and the transportation in kg. If we substitute the constants $r_i W_i = w_i$, the problem can, more specifically, be stated as

$$z(x, y) = \sum_i w_i d_i(x, y) \qquad \{2\}$$

where d_i is the distance from the depot (to be created) to every customer, or

$$d_i = \sqrt{(x - x_i)^2 + (y - y_i)^2} \qquad \{3\}$$

The objective is, of course, to find (x, y) that minimizes z:

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$$[\min] z = \sum_i w_i d_i(x, y) \qquad \{4\}$$

3. The resolution

As usual, there are two ways to solve the minimization in Eq. {4}: differentiating with respect to x and y, and making the derivatives zero; or through numerical minimization, such as using the Nelder-Mead simplex method[1].

Using derivatives, it is

$$\frac{\partial z(x, y)}{\partial x} = \sum_i w_i \frac{2(x - x_i)}{2d_i} \qquad \{5\}$$

Simplifying and considering symmetry between x and y, we shall have

$$\operatorname{grad} z = \mathbf{f}(x, y) = \begin{bmatrix} \sum_i w_i \dfrac{x - x_i}{d_i} \\ \sum_i w_i \dfrac{y - y_i}{d_i} \end{bmatrix} \qquad \{6\}$$

In order to make the gradient zero, an iterative method appears to be necessary, the Newton-Raphson method being the usual choice. Thus, writing Eq. {6} as follows

$$\mathbf{f}(\mathbf{u}) = \mathbf{0} \qquad \{7\}$$

it will be

$$\Delta\mathbf{u} = -\mathbf{J}^{-1}\mathbf{f}(\mathbf{u}) \qquad \{8\}$$

with the Jacobian, in this particular case,

$$\mathbf{J} = \frac{\partial(f_1, f_2)}{\partial(x, y)} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix} \qquad \{9\}$$

with f_1 and f_2 the two components of the gradient (Eq. {6}). This gives

$$\mathbf{J} = \begin{bmatrix} \sum_i w_i \left[d_i^{-1} - (x - x_i) d_i^{-2}\right] \dfrac{\partial d_i}{\partial x} & \sum_i w_i \left[-(x - x_i) d_i^{-2}\right] \dfrac{\partial d_i}{\partial y} \\ \sum_i w_i \left[-(y - y_i) d_i^{-2}\right] \dfrac{\partial d_i}{\partial x} & \sum_i w_i \left[d_i^{-1} - (y - y_i) d_i^{-2}\right] \dfrac{\partial d_i}{\partial y} \end{bmatrix} \qquad \{10\}$$

with

[1] This method (also named *sequential simplex*) is vastly used, *e.g.*, in Matlab function *fminsearch* or Mathematica *NMinimize*. As Dantzig's simplex is also sequential, "Nelder-Mead" is perhaps the most common designation.

$$\begin{bmatrix} \dfrac{\partial d_i}{\partial x} \\ \dfrac{\partial d_i}{\partial y} \end{bmatrix} = \begin{bmatrix} \left[(x-x_i)^2 + (y-y_i)^2\right]^{-1/2} x \\ \left[(x-x_i)^2 + (y-y_i)^2\right]^{-1/2} y \end{bmatrix} = \frac{1}{d_i}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \{11\}$$

The Jacobian becomes

$$\mathbf{J} = \begin{bmatrix} \sum_i w_i \left[d_i^{-1} - (x-x_i)d_i^{-2}\right]\dfrac{x}{d_i} & \sum_i w_i \left[-(x-x_i)d_i^{-2}\right]\dfrac{y}{d_i} \\ \sum_i w_i \left[-(y-y_i)d_i^{-2}\right]\dfrac{x}{d_i} & \sum_i w_i \left[d_i^{-1} - (y-y_i)d_i^{-2}\right]\dfrac{y}{d_i} \end{bmatrix} \qquad \{12\}$$

or

$$\mathbf{J} = \begin{bmatrix} x\sum_i \dfrac{w_i}{d_i^3}(d_i - x + x_i) & y\sum_i \dfrac{w_i}{d_i^3}(x_i - x) \\ x\sum_i \dfrac{w_i}{d_i^3}(y_i - y) & y\sum_i \dfrac{w_i}{d_i^3}(d_i - y + y_i) \end{bmatrix} \qquad \{13\}$$

The inverse of the Jacobian is in this case easily calculated (provided the determinant is not zero):

$$\mathbf{J}^{-1} = \frac{1}{j_{11}j_{22} - j_{12}j_{21}}\begin{bmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{bmatrix} \qquad \{14\}$$

A suitable initial guess can be the centroid of the destinations, but the method may fail, as do many other numerical algorithms. The bibliographical sources of the method are so numerous that no specific recommendation is made in this opuscule.

The possibility that d_i becomes zero (is it possible ? see Buescu [2009]) suggests to prefer the numerical method (*e.g.*, Nelder-Mead). See Casquilho [2008].

4. Conclusions

The facility location is a primary topic in the supply chain, regarding the minimization of the cost of transportation of goods to the customers. The mathematics involved is simple, but the results may look surprising: for a certain preponderance of the weight of a customer's location, the facility not only comes closer to it, but it will *coincide* with that location. Further generalizations are also very important.

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15-05-2012 **Plant location**

http://pt.wikipedia.org/wiki/Anexo:Lista_de_distritos_portugueses_ordenados_por_popula%C3%A7%C3%A3c
http://www.daftlogic.com/projects-google-maps-distance-calculator.htm

Castelo Branco	195433	inhabitants
Évora	168034	
Portalegre	24930	
Badajoz	151565	
Cáceres	95026	
Mérida	57797	
Santarém	32000	
Tomar	15764	

Distances from DaftLogic

	Castelo Branc	Évora	Portalegre	Badajoz	Cáceres	Mérida
Castelo Branco	0	143,9	59,8	113,6	103,1	141,1
Évora		0	88,9	88,3	166,7	141,6
Portalegre			0	61,6	93,6	102,4
Badajoz				0	85,2	53,5
Cáceres					0	62,3
Mérida						0

Coords. from gif	X	Y	astelo Branc	Évora	Portalegre	Badajoz	Cáceres	Mérida					
Castelo Branco	**176**	**325**	0	98,234414	39,319207	78,39005	70,384657	95,462034	1,4648634	1,5208852	1,4491635	1,4648079	1,4780745
Évora	**151**	**420**		0	63,529521	60,539243	112,80071	95,880134		1,3993495	1,45855804	1,4778276	1,476844
Portalegre	**181**	**364**			0	42,011903	61,188234	68,44706			1,46625112	1,5297059	1,4960467
Badajoz	**207**	**397**				0	56,400355	36,221541				1,5106288	1,4770217
Cáceres	**241**	**352**					0	41,048752					1,5177075
Mérida	**243**	**393**						0	Average, stdev:		***1,47918236***	*0,033103*	
Lisboa	78	408	for reference								Conversion factor		

Distances, calculated

	Castelo Branc	Évora	Portalegre	Badajoz	Cáceres	Mérida					
Castelo Branco	0	145,31	58,16	115,95	104,11	141,21	1,4066111	1,6397235	2,35317935	1,0117433	0,1057568
Évora		0	93,97	89,55	166,85	141,82		5,071746	1,24858073	0,1528187	0,2242017
Portalegre			0	62,14	90,51	101,25			0,54326573	3,0914437	1,1543168
Badajoz				0	83,43	53,58				1,7735906	0,0782637
Cáceres					0	60,72					1,5814111
Mérida						0	Average, stdev:		*1,43*	*1,33*	km

Coordinates and distances, km

	X	Y	astelo Branc	Évora	Portalegre	Badajoz	Cáceres	Mérida	**Weight**
Castelo Branco	**260,3**	**480,7**	0	145,3	58,2	116,0	104,1	141,2	195433
Évora	**223,4**	**621,3**		0	94,0	89,5	166,9	141,8	168034
Portalegre	**267,7**	**538,4**			0	62,1	90,5	101,2	24930
Badajoz	**306,2**	**587,2**				0	83,4	53,6	151565
Cáceres	**356,5**	**520,7**					0	60,7	95026
Mérida	**359,4**	**581,3**						0	57797

Where is the depot ?

	X	Y	Weight	Dist_to_dep
Castelo Branco	260,3	480,7	195433	88,26
Évora	223,4	621,3	168034	89,50

Depot:	X	Y	
	291,55	**563,29**	km
$Cost = r\ \Sigma\ w_i\ d_i$		Minimize	

Solver model
48840983

gif:	X	Y
	197,10076	380,81462

Portalegre	267,7	538,4	24930	34,44
Badajoz	306,2	587,2	151565	28,06
Cáceres	356,5	520,7	95026	77,67
Mérida	359,4	581,3	57797	70,25
	km	km	kg	

4,88E+07

2
100

About halfway between Portalegre & Badajoz

286,96 562,83

Rate, *r* = 1 $ / kg-km


Aveiro
Guarda
Covilhã
Coimbra
Plasencia
Castelo Branco
Leiria
Tomar
Nazaré
Cáceres
Portalegre
Santarém
Mérida
PORTUGAL
Sintra
Lisbon
Badajoz
Évora
Setúbal
Zafra
Beja
Sines


Bay of Biscay
FRANCE
Toulouse
Montpellier
Nîmes
La Coruña
Ferrol
Avilés
Gijón
Oviedo
Mieres
Torrelavega
Santander
Baracaldo
Getxo
Bilbao
Irún
San Sebastián
Santiago de Compostela
Lugo
Vitoria
Pamplona
Miranda de Ebro
León
Monforte
Ponferrada
Pontevedra
Vigo
Orense
Burgos
Logroño
Pico de Aneto 3404 m
ANDORRA
Andorra la Vella
Perpignan
Figueres
Gerona
Palencia
Tudela
Huesca
Viana do Castelo
Braga
Guimarães
Bragança
Vila Real
Valladolid
Zamora
Soria
Aranda de Duero
Zaragoza
Lérida
Manresa
Sabadell
Terrassa
Mataró
L'Hospitalet de Llobregat
Barcelona
Reus
Villanueva y Geltrú
Tarragona
ATLANTIC OCEAN
Porto
Salamanca
Segovia
Calatayud
Tortosa
Aveiro
Viseu
Guarda
Covilhã
Ávila
Guadalajara
Madrid
Móstoles
Fuenlabrada
Getafe
Alcalá de Henares
Teruel
Figueira da Foz
Coimbra
Plasencia
Aranjuez
Cuenca
Castellón de la Plana
Minorca
Mahón
Castelo Branco
Leiria
Talavera de la Reina
Toledo
Nazaré
Tomar
Cáceres
Sagunto
Palma de Mallorca
Majorca
Valencia
SPAIN
Alcázar de San Juan
Portalegre
Santarém
PORTUGAL
Mérida
Albacete
Gandía
Ibiza
Balearic Islands
Sintra
Lisbon
Cascais
Badajoz
Ciudad Real
Manzanares
Alcoy
Évora
Puertollano
Setúbal
Zafra
Elda
Benidorm
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Mediterranean Sea
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Sines
Córdoba
Murcia
Jaén
Lorca
Seville
Puente Genil
Cartagena
Guadix
Lagos
Portimão
Huelva
Alcalá de Guadaira
Utrera
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Mulhacén 3478 m
Faro
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Vélez-Málaga
Motril
Almería
Sanlúcar de Barrameda
Ronda
Málaga
El Ejido
Jerez de la Frontera
Cádiz
Chiclana
Marbella
Algiers
Bejaïa
Blida
Ech Cheliff
Gibraltar (UK)
Algeciras
Ceuta (Spain)
Tangier
MOROCCO
Tétouan
Mostaganem
ALGERIA
Oran
Relizane
Spain, Portugal, and Andorra
0 50 100 150 200
Kilometers
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Where is the depot ? **Increase the weight of Castelo Branco (or any other)**

195000, 400000, ...

	X	Y	Weight	Dist_to_dep
Castelo Branco	260,34	480,73	195433	0,00
Évora	223,36	621,26	168034	145,30
Portalegre	267,73	538,42	24930	58,16
Badajoz	306,19	587,24	151565	115,95
Cáceres	356,48	520,67	95026	104,11
Mérida	359,44	581,32	57797	141,20
	km	km	kg	

Depot:	X	Y	
	260,34	**480,74**	km

Cost = $r \, \Sigma \, w_i \, d_i$ Minimize

6,15E+07

fraction weight
28,21%
24,25%
3,60%
21,88%
13,72%
8,34%
1

Solver model
6,15E+07
2
100

Rate, r = 1 $ / kg-km

Castelo Branco 431150 critical