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ACCEPTANCE–REJECTION SAMPLING MADE EASY*

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Abstract. A simple proof, consisting of three lemmas, is given for the acceptance–rejection method in Monte Carlo sampling from a density f. The proof is valid for random variables in general dimension, for bounded as well as for unbounded support, and it is not affected by discontinuities or infinite peaks of f.

Key words. random variable generation, stochastic simulation, Monte Carlo techniques

AMS(MOS) subject classifications. 62E25, 65C05

1. Introduction. Suppose we wish to generate a random variable $\mathbf{X}$ with density $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^p$. Suppose furthermore that there exists a density function $g(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^p$, and a constant $c \geqq 1$ such that $cg(\mathbf{x}) \geqq f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^p$. (Typically, g is a density such that Monte Carlo sampling from g is easy, for instance, a uniform distribution on a rectangular area.) Then the well-known acceptance–rejection algorithm proceeds as follows:

Step 1. Generate $\mathbf{Y}$ from density g.
Step 2. Generate U from the uniform distribution on the interval $(0, cg(\mathbf{Y}))$.
Step 3. If $U \leqq f(\mathbf{Y})$, put $\mathbf{X} \leftarrow \mathbf{Y}$, deliver $\mathbf{X}$
else, return to Step 1.

Proving that the algorithm produces $\mathbf{X}$ with the desired density amounts to showing that, in any given execution of Steps 1 to 3, the algorithm either delivers $\mathbf{X}$ with the correct density, or else goes on to the next iteration. In other words, it is to be shown that, conditional on acceptance in Step 3, $\mathbf{X}$ follows density f.

The reason for this note is the author's dissatisfaction with the proofs commonly found in textbooks on simulation. They tend to be purely technical, without giving much insight into why the method works (e.g., Rubinstein [1981, p. 46]), or lengthy and verbal (e.g., Morgan [1984, p. 100ff]), or incomplete and unnecessarily intertwined with examples (e.g., Kalos and Whitlock [1986, p. 61ff]). Moreover, I found the frequent distinction between the univariate case and the multivariate case (Rubinstein [1981, Thms. 3.4.1, 3.4.2]) superfluous, as the proof for general dimension p is just as simple as the proof for the univariate case. A recent paper by Grzesik (1989) gives an apparently new and unusual proof, but it lacks simplicity and is unnecessarily constrained to the univariate case and to finite support of the density f.

2. The proof. We shall state three lemmas, each of them rather trivial, but together they provide an elegant proof that the acceptance–rejection method works. The only prerequisite for understanding the proof is the notion of conditional distribution.

LEMMA 1. *Let* $\mathbf{X}$ *denote a p-variate random variable with density* $g(\mathbf{x})$, *let* $c > 0$ *denote a real constant, and let* Y *denote a random variable such that the conditional*

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distribution of Y, *given* $\mathbf{X}=\mathbf{x}$, *is uniform in the interval* $(0, cg(\mathbf{x}))$. *Then the joint distribution of the* $(p+1)$*-dimensional random variate* $\binom{\mathbf{X}}{Y}$ *is uniform in the set*

$$\mathscr{A}=\left\{\begin{pmatrix}\mathbf{x}\\ y\end{pmatrix};\mathbf{x}\in\mathbb{R}^p, 0<y<cg(\mathbf{x})\right\}.$$

Proof. Let $h(\mathbf{x}, y)$ denote the joint density of $\mathbf{X}$ and Y. Then

$$h(\mathbf{x},y)=g(\mathbf{x})\cdot\frac{1}{cg(\mathbf{x})}=\frac{1}{c}\quad\text{if}\begin{pmatrix}\mathbf{x}\\ y\end{pmatrix}\in\mathscr{A}$$

$$=0,\qquad\qquad\text{else.}$$

In Lemma 2, $V(\cdot)$ will denote the m-dimensional volume of a set.

LEMMA 2. *Suppose the* m*-dimensional random variable* $\mathbf{Z}$ *has a uniform distribution in* $\mathscr{A}\subset\mathbb{R}^m$, *where* $0<V(\mathscr{A})<\infty$. *Let* $\mathscr{B}\subset\mathscr{A}$, $V(\mathscr{B})>0$. *Then the conditional distribution of* $\mathbf{Z}$, *given* $\mathbf{Z}\in\mathscr{B}$, *is uniform in* $\mathscr{B}$.

Proof. The proof is obvious.

LEMMA 3. *Suppose the* $(p+1)$*-dimensional random variable* $\binom{\mathbf{X}}{Y}$, *where* $\mathbf{X}$ *has dimension* p, *follows a uniform distribution in the set* $\mathscr{B}=\{\binom{\mathbf{x}}{y}:\mathbf{x}\in\mathbb{R}^p,\ 0<y<f(\mathbf{x})\}$, *where* f *is the density function of a* p*-variate random variable. Then the marginal distribution of* $\mathbf{X}$ *has density* f.

Proof. The density of $\mathbf{X}$ at the point $\mathbf{x}\in\mathbb{R}^p$ is $\int_0^{f(\mathbf{x})}dy=f(\mathbf{x})$.

Putting the lemmas together, with $m=p+1$ in Lemma 2, proves the acceptance–rejection method.

3. An example. In an advanced course the three lemmas can be stated as evident, without proof. In a course on a moderate level it is useful to illustrate the procedure graphically with an example of dimension $p=1$. A good and simple example is simulation from the half-normal distribution, using exponentially distributed random numbers, as shown in Fig. 1. Here,

$$f(x)=\sqrt{\frac{2}{\pi}}\exp\left[-\frac{x^2}{2}\right],\qquad x\geqq 0,$$

$$g(x)=\exp[-x],\qquad x\geqq 0,$$

$$c=\sqrt{\frac{2}{\pi}}\exp\left[\frac{1}{2}\right]\approx 1.3155,$$

$$\mathscr{A}=\left\{\begin{pmatrix}x\\ y\end{pmatrix}:x\geqq 0, 0<y<\sqrt{\frac{2}{\pi}}\exp\left[\frac{1}{2}-x\right]\right\},$$

$$\mathscr{B}=\left\{\begin{pmatrix}x\\ y\end{pmatrix}:x\geqq 0, 0<y<\sqrt{\frac{2}{\pi}}\exp\left[-\frac{x^2}{2}\right]\right\}.$$

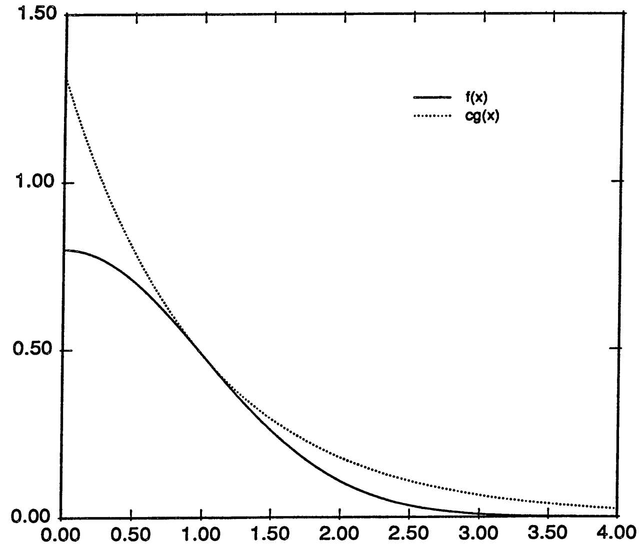

FIG. 1. *Acceptance–rejection sampling from the half-normal distribution, using the exponential distribution.*

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