Exam 1 – 2019/2020: Problem 1 (Solution)

1. (4.0 points) Transient heat conduction experiments were performed with a plane wall with 0.2 m of thickness at the initial temperature of 20°C (T_i), immersed at different occasions in three different fluid media. The three media have the same temperature, equal to 180°C (T_∞), but different convection heat transfer coefficients (h). Heat conduction within the wall was observed only along the wall thickness (one-dimensional) and both sides of the wall were subjected to the same (symmetrical) convection boundary condition. The wall material thermal conductivity, k, is equal to $200\,\mathrm{W\cdot m^{-1}\cdot K^{-1}}$.

   Figure 1 presents the plane wall temperature profiles registered for the three media (Media 1 – 3) when the plane wall surface temperature value, $T\,(x = \pm 0.1\,\mathrm{m})$, reached 100°C. For the three profiles presented in Figure 1, the Fourier number is greater than 0.5. **Do not use the Heisler plots in the resolution.**

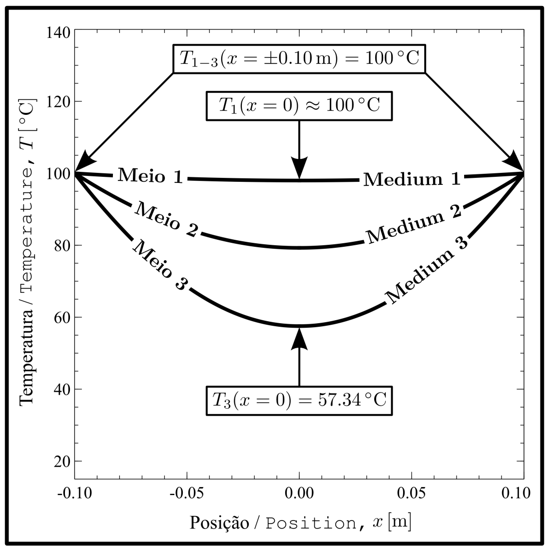

Figure 1

   (a) (2.0 points) Determine the wall volumetric heat capacity, ρc, knowing that the profile for Medium 1 is observed after 46 min of the beginning of the transient process ($t = 0$) and the convection heat transfer coefficient of Medium 1 is equal to $100\,\mathrm{W\cdot m^{-2}\cdot K^{-1}}$. (If you did not solve this question, assume $\rho c = 4\,\mathrm{MJ\cdot m^{-3}\cdot K^{-1}}$ for the following question.)

Solution:

The relevance of the spatial temperature gradients along the plane wall can be estimated with the Biot number (Bi) criterion, according to the following:

$$Bi = \frac{hL_c}{k} = \begin{cases} < 0.1, & \text{Negligible spatial temperature gradients.} \\ \geq 0.1, & \text{Relevant spatial temperature gradients.} \end{cases} \tag{1}$$

For the current case, $h = 100\,\mathrm{W\cdot m^{-2}\cdot K^{-1}}$, $L_c = L = 0.1\,\mathrm{m}$, and $k = 200\,\mathrm{W\cdot m^{-1}\cdot K^{-1}}$. Consequently, the Biot number is calculated in Eq. (2).

$$Bi = \frac{hL_c}{k} = \frac{100 \times 0.1}{200} \Leftrightarrow Bi = 0.05 \tag{2}$$

Since $Bi < 0.1$, according to the Bi criterion (Eq. (1)) the plane wall temperature distribution can be considered spatially uniform (depends only on the elapsed time): $T(x,t) \approx T(t)$. In these conditions, the lumped capacitance method can be readily applied. Therefore,

$$\begin{aligned} \frac{\rho V c}{h A_s} \ln \frac{\theta_i}{\theta} = t \Leftrightarrow \rho c = h \frac{A_s}{V} \frac{1}{\ln(\theta_i/\theta)} t \Leftrightarrow \\ \Leftrightarrow \rho c = 100 \times \frac{1}{0.1} \times \frac{1}{\ln[(20-180)/(100-180)]} \times 46 \times 60 \Leftrightarrow \\ \Leftrightarrow \boxed{\rho c \approx 3.982\,\mathrm{MJ} \cdot \mathrm{m}^{-3} \cdot \mathrm{K}^{-1}} \end{aligned} \tag{3}$$

(b) (2.0 points) Determine the required time from the beginning of the transient process to observe an average wall temperature, $\overline{T}$, equal to 100°C with Medium 3. Note that $Q/Q_o = (1/V) \int (1-\theta^*)\, dV = 1 - \overline{\theta^*}$ and in Figure 1 for the profile of Medium 3, $T(x=0) = 57.34$°C.

Solution:

Following the tip given in the problem statement:

$$\frac{Q}{Q_o} = \frac{1}{V} \int (1-\theta^*)\, dV = 1 - \frac{1}{V} \int \theta^* dV \Leftrightarrow \frac{Q}{Q_o} = 1 - \overline{\theta^*} \tag{4}$$

where, $\overline{\theta^*}$ is equal to $\left(\overline{T} - T_\infty\right) / (T_i - T_\infty)$.
On the other hand, the one-term approximation solution to Q/Q_o for a plane wall is given by Eq. (5).

$$\frac{Q}{Q_o} = 1 - \frac{\sin(\zeta_1)}{\zeta_1} \theta_o^* \tag{5}$$

Equating the R.H.S. of Eqs. (4) and (5) and after manipulation Eq. (6) is obtained for the calculation of the Fourier number.

$$\overline{\theta^*} = \frac{\sin(\zeta_1)}{\zeta_1} \theta_o^* = \frac{\sin(\zeta_1)}{\zeta_1} C_1 \exp\left(-\zeta_1^2 Fo\right) \Leftrightarrow Fo = -\ln\left(\frac{\overline{\theta^*}\zeta_1}{\sin(\zeta_1)\, C_1}\right) \frac{1}{\zeta_1^2} \tag{6}$$

Following the definition of the Fourier number, the elapsed time, t, is calculated with Eq. (7).

$$Fo = \frac{\alpha t}{L^2} = \frac{kt}{\rho c L^2} \Leftrightarrow t = \frac{Fo \rho c L^2}{k} \tag{7}$$

In Eq. (6), ζ_1 and C_1 are evaluated according to the Biot number for Medium 3. The Biot number for Medium 3 can be calculated applying Eq. (8) to a particular dimensionless position, x^* – note that the problem statement refers that the one-term approximation solution (Eq. (8)) can be applied for the three temperature profiles because, $Fo > 0.2$.

$$\theta^*(x^*) = \theta_o^* \cos(\zeta_1 x^*) \Leftrightarrow \zeta_1 = \cos^{-1}\left(\frac{\theta^*(x^*)}{\theta_o^*}\right) \frac{1}{x^*} \tag{8}$$

At the plane wall surface, x^* is equal to 1 and, consequently, Eq. (8) is written as follows.

$$\zeta_1 = \cos^{-1}\left(\frac{\theta^*(1)}{\theta_o^*}\right) = \cos^{-1}\left(\frac{100-180}{57.34-180}\right) \Leftrightarrow \zeta_1 \approx 0.8603\,\mathrm{rad} \Rightarrow \mathrm{Bi} = 1.0 \tag{9}$$

For $Bi = 1.0$, $\zeta_1 = 0.8603\,\mathrm{rad}$ and $C_1 = 1.1191$. Replacing the values for ζ_1, C_1, and $\overline{\theta}$ in Eq. (6), the actual Fourier number is determined.

$$Fo = -\ln\left(\frac{((100-180)\,/\,(20-180))\times 0.8603}{\sin(0.8603)\times 1.1191}\right)\times\frac{1}{0.8603^2} \Leftrightarrow Fo \approx 0.918 \tag{10}$$

Finally, the required time to observe $\overline{T} = 100°\mathrm{C}$ with Medium 3 is computed with Equation (7), considering $Fo = 0.918$, $\rho c = 3.982\times 10^6\,\mathrm{J\cdot m^{-3}\cdot K^{-1}}$, $L = 0.1\,\mathrm{m}$, and $k = 200\,\mathrm{W\cdot m^{-1}\cdot K^{-1}}$:

$$t = \frac{0.918\times 3.982\times 10^6\times 0.1^2}{200} \Leftrightarrow \boxed{t \approx 182.774\,\mathrm{s}\ (3.046\,\mathrm{min})} \tag{11}$$