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Advanced Heat Transfer

Special S. Exam – Problem 3

July 27, 2021 (18h00)

Consider combined diffusion and convection transport of thermal energy in a tube with an inner radius (R) and length (L) equal to 0.1 m and 0.5 m, respectively. A laminar flow of an incompressible and constant property fluid under fully developed hydrodynamic conditions is considered in the tube. The axial velocity profile is given by $v_z(r) = 2u_m\left[1-(r/R)^2\right]$, where r and u_m correspond to the local radial position and mean fluid velocity, respectively. (Note that for the current conditions, the radial velocity component (v_r) and $\partial v_z/\partial z$ are negligible.) At the tube inlet section ($z = 0$), the total mass flow rate ($\dot{m}$) and fluid temperature (T_{in}) are equal to $0.4\,\text{kg}\,\text{s}^{-1}$ and 100°C, respectively. The tube lateral wall (surface $r = R$) is kept at a constant temperature of 20°C. The fluid density (ρ), specific heat (c_p), and thermal conductivity (k) are equal to $13464\,\text{kg}\,\text{m}^{-3}$, $138.443\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$, and $8.883\,\text{W}\,\text{m}^{-1}\,\text{K}^{-1}$, respectively. Temperature gradients along the circumferential direction (angular coordinate – ϕ) are negligible and, consequently, the governing equation for the temperature distribution written in 2D (two-dimensional) axisymmetric (r, z) coordinates is presented below.

$$\frac{\partial}{\partial r}\left(\rho v_r c_p T\right) + \frac{\partial}{\partial z}\left(\rho v_z c_p T\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

Consider the 2D axisymmetric calculation domain discretized by a uniform mesh with $\Delta r = \Delta z = 5\,\text{mm}$. Apply the finite volume method for the following questions.

(a) (1.0 v.) Is the stated cell size ($5\,\text{mm} \times 5\,\text{mm}$) suitable for the application of the central differencing scheme to the governing equation? Justify.

Solution:

To obtain a bounded (physically realistic) solution the absolute value of the grid Peclet number (Pe) in both coordinate directions must be lower than 2.0. In the radial direction there is no bulk fluid motion ($v_r = 0$) – thermal energy transport occurs exclusively by diffusion – and, consequently, any cell spacing Δr considered does not lead to unbounded solutions. In the axial direction the relative importance of convection over diffusion differs according to the radial location in consideration. The worst case – for which the relative strength of convection over diffusion is the highest (highest Pe_z value) – is observed at the nearest node to the tube center-line ($r = 0$) because it is at the tube center-line that the highest axial velocity is observed – see Equation (1).

$$u_{\max} \equiv u_z(r=0) = 2u_m\left[1-(0/R)^2\right] \Leftrightarrow u_{\max} = \frac{2\dot{m}}{\rho\pi R^2} \tag{1}$$

Since the grid Peclet number along the (axial) flow direction must be lower than 2, the maximum possible cell spacing along the axial direction is calculated with Equation (2).

$$Pe_z \equiv \frac{F_z}{D_z} < 2 \Leftrightarrow \frac{\rho u_{\max} c_p \Delta z}{k} < 2 \Leftrightarrow \Delta z < \frac{2k}{\rho u_{\max} c_p} \Leftrightarrow \Delta z < \frac{\pi R^2 k}{\dot{m} c_p} \Leftrightarrow$$
$$\Leftrightarrow \Delta z < \frac{\pi \times 0.1^2 \times 8.883}{0.4 \times 138.443} \Leftrightarrow \Delta z < 5.039\,\text{mm} \tag{2}$$

The stated cell size is adequate for the application of the central differencing because the maximum cell spacing along the axial direction required by this scheme (5.039 mm) is higher than the stated cell spacing along the axial direction (5 mm). (The cell spacing along the radial direction does not lead to boundedness issues while applying the central differencing scheme because the velocity along the radial direction is negligible.)

For the following questions consider the upwind differencing scheme.

(b) (1.5 v.) Determine the discretized equation for a generic interior grid node. Present all intermediate calculations including the final expressions required to compute the center-point and neighboring node coefficients – a_P and a_nb, respectively – and the constant term b. (In cylindrical coordinates the infinitesimal volume is given as $dV = r\,dr\,d\phi\,dz$.)

Solution:

The discretized equation for a generic bulk node is obtained by volume integration of the governing equation followed by the application of suitable interpolation functions (profile assumptions). Volume integration is performed in Equations (3) – (7). (Note that the convective term along the radial direction is negligible ($v_r = 0$) and, consequently, it is not considered in the equations that follow. Additionally, the diffusion coefficient Γ is equal to k/c_p.)

$$\underbrace{\int_{\Delta V} \frac{\partial}{\partial z}(\rho v_z T)\,dV}_{\mathrm{A}} = \underbrace{\int_{\Delta V} \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r \frac{\partial T}{\partial r}\right) dV}_{\mathrm{B}} + \underbrace{\int_{\Delta V} \frac{\partial}{\partial z}\left(\Gamma \frac{\partial T}{\partial z}\right) dV}_{\mathrm{C}} \tag{3}$$

$$\begin{aligned}\mathrm{A} \equiv \int_{\Delta V} \frac{\partial}{\partial z}(\rho v_z T)\,dV &= \int_{r_\mathrm{w}}^{r_\mathrm{e}} \int_{z_\mathrm{s}}^{z_\mathrm{n}} \frac{\partial}{\partial z}(\rho v_z T)\,dz r dr = \left[(\rho v_z T)_\mathrm{n} - (\rho v_z T)_\mathrm{s}\right] \frac{r_\mathrm{e}^2 - r_\mathrm{w}^2}{2} = \\ &= \left[(\rho v_z T)_\mathrm{n} - (\rho v_z T)_\mathrm{s}\right] \underbrace{\frac{r_\mathrm{e} + r_\mathrm{w}}{2}}_{r_\mathrm{P}} \underbrace{r_\mathrm{e} - r_\mathrm{w}}_{\Delta r} = \left[(\rho v_z T)_\mathrm{n} - (\rho v_z T)_\mathrm{s}\right] r_\mathrm{P} \Delta r\end{aligned} \tag{4}$$

$$\begin{aligned}\mathrm{B} \equiv \int_{\Delta V} \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r \frac{\partial T}{\partial r}\right) dV &= \int_{z_\mathrm{s}}^{z_\mathrm{n}} \int_{r_\mathrm{w}}^{r_\mathrm{e}} \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r \frac{\partial T}{\partial r}\right) r dr dz = \\ = \left[\left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{e} - \left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{w}\right](z_\mathrm{n} - z_\mathrm{s}) &= \left[\left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{e} - \left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{w}\right] \Delta z\end{aligned} \tag{5}$$

$$\mathrm{C} \equiv \int_{\Delta V} \frac{\partial}{\partial z}\left(\Gamma \frac{\partial T}{\partial z}\right) dV = \int_{r_\mathrm{w}}^{r_\mathrm{e}} \int_{z_\mathrm{s}}^{z_\mathrm{n}} \frac{\partial}{\partial z}\left(\Gamma \frac{\partial T}{\partial z}\right) dz r dr = \left[\left(\Gamma \frac{\partial T}{\partial z}\right)_\mathrm{n} - \left(\Gamma \frac{\partial T}{\partial z}\right)_\mathrm{s}\right] r_\mathrm{P} \Delta r \tag{6}$$

Substituting Equations (4)–(6) into Equation (3) the following equation is obtained – see Equation (7).

$$\begin{aligned}\int_{\Delta V} \frac{\partial}{\partial z}(\rho v_z T)\,dV = \int_{\Delta V} \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r \frac{\partial T}{\partial r}\right) dV + \int_{\Delta V} \frac{\partial}{\partial z}\left(\Gamma \frac{\partial T}{\partial z}\right) dV \Leftrightarrow \\ \Leftrightarrow \left[(\rho v_z T)_\mathrm{n} - (\rho v_z T)_\mathrm{s}\right] r_\mathrm{P} \Delta r = \left[\left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{e} - \left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{w}\right] \Delta z + \\ \left[\left(\Gamma \frac{\partial T}{\partial z}\right)_\mathrm{n} - \left(\Gamma \frac{\partial T}{\partial z}\right)_\mathrm{s}\right] r_\mathrm{P} \Delta r\end{aligned} \tag{7}$$

Considering the upwind differencing scheme for the convective terms and piecewise-linear profiles (central differencing scheme) to evaluate the derivatives at the cell faces, the final discretized equation is given as follows – see Equation (8).

$$
\begin{aligned}
&[(\rho v_z T)_{\mathrm{n}} - (\rho v_z T)_{\mathrm{s}}]\, r_{\mathrm{P}}\Delta r = \left[\left(\Gamma r\frac{\partial T}{\partial r}\right)_{\mathrm{e}} - \left(\Gamma r\frac{\partial T}{\partial r}\right)_{\mathrm{w}}\right]\Delta z + \\
&\left[\left(\Gamma\frac{\partial T}{\partial z}\right)_{\mathrm{n}} - \left(\Gamma\frac{\partial T}{\partial z}\right)_{\mathrm{s}}\right] r_{\mathrm{P}}\Delta r \Leftrightarrow \rho v_z r_{\mathrm{P}}\Delta r\,(T_{\mathrm{P}} - T_{\mathrm{S}}) = \\
&\left[\frac{\Gamma_{\mathrm{e}} r_{\mathrm{e}}\Delta z}{\Delta r}(T_{\mathrm{E}} - T_{\mathrm{P}}) - \frac{\Gamma_{\mathrm{w}} r_{\mathrm{w}}\Delta z}{\Delta r}(T_{\mathrm{P}} - T_{\mathrm{W}})\right] + \left[\frac{\Gamma_{\mathrm{n}} r_{\mathrm{P}}\Delta r}{\Delta z}(T_{\mathrm{N}} - T_{\mathrm{P}}) - \frac{\Gamma_{\mathrm{s}} r_{\mathrm{P}}\Delta r}{\Delta z}(T_{\mathrm{P}} - T_{\mathrm{S}})\right] \Leftrightarrow \\
&\Leftrightarrow F_z(T_{\mathrm{P}} - T_{\mathrm{S}}) = D_{\mathrm{e}}(T_{\mathrm{E}} - T_{\mathrm{P}}) - D_{\mathrm{w}}(T_{\mathrm{P}} - T_{\mathrm{W}}) + D_{\mathrm{n}}(T_{\mathrm{N}} - T_{\mathrm{P}}) - D_{\mathrm{s}}(T_{\mathrm{P}} - T_{\mathrm{S}}) \Leftrightarrow \\
&\Leftrightarrow \underbrace{(D_{\mathrm{w}} + D_{\mathrm{e}} + D_{\mathrm{n}} + D_{\mathrm{s}} + F_z)}_{a_{\mathrm{P}}} T_{\mathrm{P}} = D_{\mathrm{w}} T_{\mathrm{W}} + D_{\mathrm{e}} T_{\mathrm{E}} + (D_{\mathrm{s}} + F_z) T_{\mathrm{S}} + D_{\mathrm{n}} T_{\mathrm{N}} \Leftrightarrow \\
&\Leftrightarrow \boxed{a_{\mathrm{P}} T_{\mathrm{P}} = a_{\mathrm{W}} T_{\mathrm{W}} + a_{\mathrm{E}} T_{\mathrm{E}} + a_{\mathrm{S}} T_{\mathrm{S}} + a_{\mathrm{N}} T_{\mathrm{N}} + b}
\end{aligned} \tag{8}
$$

The following table summarizes the expressions required to compute the coefficients and constant term for the discretized equation of any interior (bulk) node.

a_{W}	a_{E}	a_{S}	a_{N}	a_{P}	b
$D_{\mathrm{w}} \equiv \frac{\Gamma_{\mathrm{w}} r_{\mathrm{w}}\Delta z}{\Delta r}$	$D_{\mathrm{e}} \equiv \frac{\Gamma_{\mathrm{e}} r_{\mathrm{e}}\Delta z}{\Delta r}$	$D_{\mathrm{s}} + F_z \equiv \frac{\Gamma_{\mathrm{s}} r_{\mathrm{P}}\Delta r}{\Delta z} + \rho v_z r_{\mathrm{P}}\Delta r$	$D_n \equiv \frac{\Gamma_{\mathrm{n}} r_{\mathrm{P}}\Delta r}{\Delta z}$	$\sum a_{\mathrm{nb}}$	0

(c) (1.5 v.) Determine the discretized equation for the boundary node embraced by the control volume with faces coincident to $z = 0$ and $r = R$. Present all intermediate calculations as well as the final (computed) values for the center-point and neighboring node coefficients and constant term b.

Solution:

The discretized equation for such boundary grid node is obtained considering Equation (7) and taking into account the corresponding boundary conditions – see Equation (9).

$$
\begin{aligned}
&[(\rho v_z T)_{\mathrm{n}} - (\rho v_z T)_{\mathrm{s}}]\, r_{\mathrm{P}}\Delta r = \left[\left(\Gamma r\frac{\partial T}{\partial r}\right)_{\mathrm{e}} - \left(\Gamma r\frac{\partial T}{\partial r}\right)_{\mathrm{w}}\right]\Delta z + \\
&\left[\left(\Gamma\frac{\partial T}{\partial z}\right)_{\mathrm{n}} - \left(\Gamma\frac{\partial T}{\partial z}\right)_{\mathrm{s}}\right] r_{\mathrm{P}}\Delta r \Leftrightarrow \rho v_z(r_{\mathrm{P}})\, r_{\mathrm{P}}\Delta r\,(T_{\mathrm{P}} - T_{\mathrm{in}}) = \\
&\left[\frac{\Gamma_{\mathrm{wall}}\Delta z}{\ln(R/r_{\mathrm{P}})}(T_{\mathrm{wall}} - T_{\mathrm{P}}) - \frac{\Gamma_{\mathrm{w}} r_{\mathrm{w}}\Delta z}{\Delta r}(T_{\mathrm{P}} - T_{\mathrm{W}})\right] + \\
&\left[\frac{\Gamma_{\mathrm{n}} r_{\mathrm{P}}\Delta r}{\Delta z}(T_{\mathrm{N}} - T_{\mathrm{P}}) - \frac{2\Gamma_{\mathrm{in}} r_{\mathrm{P}}\Delta r}{\Delta z}(T_{\mathrm{P}} - T_{\mathrm{in}})\right] \Leftrightarrow F_z(T_{\mathrm{P}} - T_{\mathrm{in}}) = \frac{\Gamma_{\mathrm{wall}}\Delta z}{\ln(R/r_{\mathrm{P}})}(T_{\mathrm{wall}} - T_{\mathrm{P}}) - \\
&\underbrace{D_{\mathrm{w}}}_{a_{\mathrm{W}}}(T_{\mathrm{P}} - T_{\mathrm{W}}) + \underbrace{D_{\mathrm{n}}}_{a_{\mathrm{N}}}(T_{\mathrm{N}} - T_{\mathrm{P}}) - 2D_{\mathrm{in}}(T_{\mathrm{P}} - T_{\mathrm{in}}) \Leftrightarrow \\
&\Leftrightarrow \underbrace{\left[a_{\mathrm{W}} + a_{\mathrm{N}} + 2D_{\mathrm{in}} + F_z + \frac{\Gamma_{\mathrm{wall}}\Delta z}{\ln(R/r_{\mathrm{P}})}\right]}_{a_{\mathrm{P}}} T_{\mathrm{P}} = \\
&= a_{\mathrm{W}} T_{\mathrm{W}} + \underbrace{0}_{a_{\mathrm{E}}} T_{\mathrm{E}} + \underbrace{0}_{a_{\mathrm{S}}} T_{\mathrm{S}} + a_{\mathrm{N}} T_{\mathrm{N}} + \underbrace{\frac{\Gamma_{\mathrm{wall}}\Delta z}{\ln(R/r_{\mathrm{P}})} T_{\mathrm{wall}} + 2D_{\mathrm{in}} T_{\mathrm{in}} + F_z T_{\mathrm{in}}}_{b}
\end{aligned} \tag{9}
$$

To evaluate the coefficients a_{W}, a_{N}, and a_{P} and the constant term b the following is required:

$r_{\mathrm{w}} = R - \Delta r = 0.1 - 0.005 = 0.095\,\mathrm{m}$, $r_{\mathrm{P}} = R - \Delta r/2 = 0.1 - 0.005/2 = 0.0975\,\mathrm{m}$, $r_{\mathrm{e}} = R = 0.1\,\mathrm{m}$, and $\Gamma_{\mathrm{w}} = \Gamma_{\mathrm{wall}} = \Gamma_{\mathrm{in}} = \Gamma_{\mathrm{n}} = k/c_p$.

$$a_{\mathrm{W}} \equiv D_{\mathrm{w}} = \frac{\Gamma_{\mathrm{w}} r_{\mathrm{w}} \Delta z}{\Delta r} = \frac{k r_{\mathrm{w}} \Delta z}{c_p \Delta r} = \frac{8.883 \times 0.095 \times 0.005}{138.443 \times 0.005} \Leftrightarrow a_{\mathrm{W}} \approx 6.096 \times 10^{-3}\,\mathrm{kg\,s^{-1}} \tag{10}$$

$$a_{\mathrm{N}} \equiv D_{\mathrm{n}} = \frac{\Gamma_{\mathrm{n}} r_{\mathrm{P}} \Delta z}{\Delta r} = \frac{k r_{\mathrm{P}} \Delta z}{c_p \Delta r} = \frac{8.883 \times 0.0975 \times 0.005}{138.443 \times 0.005} \Leftrightarrow a_{\mathrm{N}} \approx 6.256 \times 10^{-3}\,\mathrm{kg\,s^{-1}} \tag{11}$$

$$\begin{aligned} a_{\mathrm{P}} = a_{\mathrm{W}} + a_{\mathrm{N}} + 2D_{\mathrm{in}} + F_z + \frac{\Gamma_{\mathrm{wall}} \Delta z}{\ln\left(R/r_{\mathrm{P}}\right)} = \\ = \frac{k r_{\mathrm{w}} \Delta z}{c_p \Delta r} + \frac{k r_{\mathrm{P}} \Delta r}{c_p \Delta z} + \frac{2\Gamma_{\mathrm{in}} r_{\mathrm{P}} \Delta r}{\Delta z} + \rho v_z\left(r_{\mathrm{P}}\right) r_{\mathrm{P}} \Delta r + \frac{k \Delta z}{c_p \ln\left(R/r_{\mathrm{P}}\right)} = \\ = \frac{k r_{\mathrm{w}} \Delta z}{c_p \Delta r} + \frac{k r_{\mathrm{P}} \Delta r}{c_p \Delta z} + \frac{2k r_{\mathrm{P}} \Delta r}{c_p \Delta z} + \frac{2\dot{m}}{\pi R^2}\left[1 - \left(r_{\mathrm{P}}/R\right)^2\right] r_{\mathrm{P}} \Delta r + \frac{k \Delta z}{c_p \ln\left(R/r_{\mathrm{P}}\right)} \end{aligned} \tag{12}$$

$$\begin{aligned} a_{\mathrm{P}} = \frac{k r_{\mathrm{w}}}{c_p} + \frac{k r_{\mathrm{P}}}{c_p} + \frac{2k r_{\mathrm{P}}}{c_p} + \frac{2\dot{m}}{\pi R^2}\left[1 - \left(r_{\mathrm{P}}/R\right)^2\right] r_{\mathrm{P}} \Delta r + \frac{k \Delta z}{c_p \ln\left(R/r_{\mathrm{P}}\right)} = \\ = \frac{k}{c_p}\left[r_{\mathrm{w}} + 3r_{\mathrm{P}} + \frac{\Delta z}{\ln\left(R/r_{\mathrm{P}}\right)}\right] + \frac{2\dot{m}}{\pi R^2}\left[1 - \left(r_{\mathrm{P}}/R\right)^2\right] r_{\mathrm{P}} \Delta r = \\ = \frac{8.883}{138.443} \times \left[0.095 + 3 \times 0.0975 + \frac{0.005}{\ln\left(0.1/0.0975\right)}\right] + \\ \frac{2 \times 0.4}{\pi \times 0.1^2} \times \left[1 - \left(0.0975/0.1\right)^2\right] \times 0.0975 \times 0.005 \Leftrightarrow \\ \Leftrightarrow a_{\mathrm{P}} = 3.815 \times 10^{-2}\,\mathrm{kg\,s^{-1}} \end{aligned} \tag{13}$$

$$\begin{aligned} b = \frac{\Gamma_{\mathrm{wall}} \Delta z}{\ln\left(R/r_{\mathrm{P}}\right)} T_{\mathrm{wall}} + 2D_{\mathrm{in}} T_{\mathrm{in}} + F_z T_{\mathrm{in}} = \frac{k \Delta z}{c_p \ln\left(R/r_{\mathrm{P}}\right)} T_{\mathrm{wall}} + \frac{2k r_{\mathrm{P}} \Delta r}{c_p \Delta z} T_{\mathrm{in}} + \\ \frac{2\dot{m}}{\pi R^2}\left[1 - \left(r_{\mathrm{P}}/R\right)^2\right] r_{\mathrm{P}} \Delta r T_{\mathrm{in}} = \frac{k}{c_p}\left[\frac{\Delta z}{\ln\left(R/r_{\mathrm{P}}\right)} T_{\mathrm{wall}} + 2r_{\mathrm{P}} T_{\mathrm{in}}\right] + \\ \frac{2\dot{m}}{\pi R^2}\left[1 - \left(r_{\mathrm{P}}/R\right)^2\right] r_{\mathrm{P}} \Delta r T_{\mathrm{in}} = \frac{8.883}{138.443} \times \left[\frac{0.005}{\ln\left(0.1/0.0975\right)} \times 20 + 2 \times 0.0975 \times 100\right] + \\ \frac{2 \times 0.4}{\pi \times 0.1^2} \times \left[1 - \left(0.0975/0.1\right)^2\right] \times 0.0975 \times 0.005 \times 100 \Leftrightarrow \\ \Leftrightarrow b = 1.566\,\mathrm{kg\,^{\circ}C\,s^{-1}} \end{aligned} \tag{14}$$

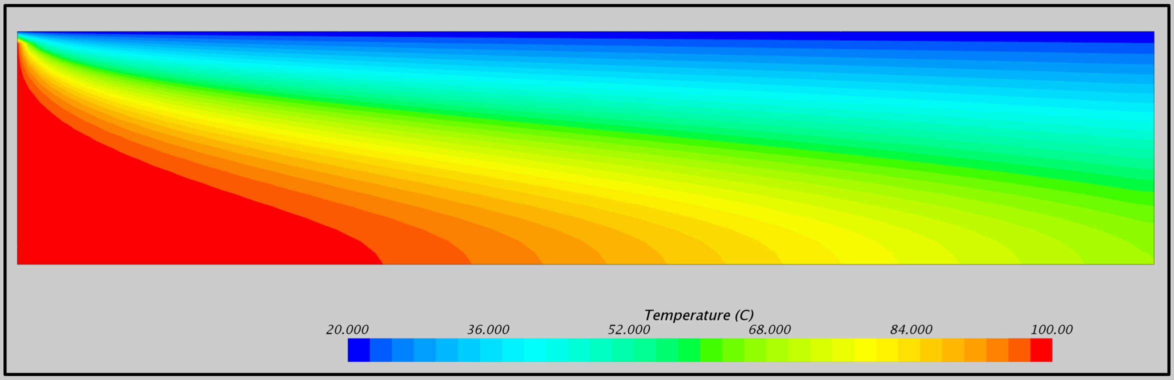
Finally, the discretized equation is given by Equation (15).

$$a_{\mathrm{P}} T_{\mathrm{P}} = a_{\mathrm{W}} T_{\mathrm{W}} + a_{\mathrm{N}} T_{\mathrm{N}} + b \Leftrightarrow \boxed{38.150 T_{\mathrm{P}} = 6.096 T_{\mathrm{W}} + 6.256 T_{\mathrm{N}} + 1566} \tag{15}$$

Solution:

Supplementary material – not necessary for previous questions (bonus).

The numerical solution for the problem under consideration is provided in the following figure.



Challenge: Formulate the finite volume discretization equation for every node and implement/solve the corresponding set of linear algebraic equations with a programming language of your choice – even MS Excel can be used to solve the problem. Compare your numerical solution with the solution presented above. With your solution for the temperature field calculate the mean fluid temperature along the axial direction ($T_m(z)$), the convection heat transfer coefficient ($h(z)$), and the Nusselt number ($Nu_D(z)$). Determine the thermal entrance length and compare the thermally fully developed Nusselt number solution with the analytical solution $Nu_D = 3.66$.