Advanced Heat Transfer (Part IV)
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List of Problems

1. Consider the one-dimensional planar, steady-state heat diffusion equation (Equation (1)) with a constant volumetric rate of thermal energy generation, $\dot{q}$.

$$k\frac{d^2T}{dx^2} + \frac{dk}{dx}\frac{dT}{dx} + \dot{q} = 0 \tag{1}$$

The first and second terms on the LHS of Equation (1) can be approximated by the expressions provided in Equations (2) and (3), respectively. In such case, determine the conditions for which the neighboring node coefficients of the discretized equation (a_W and a_E) would become negative, violating, consequently, the boundedness property of discretization schemes. Assume dk/dx is a known quantity, which may be positive or negative.

$$k\frac{d^2T}{dx^2} = \frac{k_\text{P}\left(T_\text{E} + T_\text{W} - 2T_\text{P}\right)}{\left(\Delta x\right)^2} \tag{2}$$

$$\frac{dT}{dx} = \frac{T_\text{E} - T_\text{W}}{2\Delta x} \tag{3}$$

2. Considering the finite volume method, discretize the one-dimensional, steady-state heat diffusion equation in cylindrical coordinates given by the following equation:

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) + \dot{q} = 0 \tag{4}$$

3. The source term for a dependent variable ϕ is given by $s = A - B\left|\phi\right|\phi$, where A and B are positive constants. If this term is to be linearized as $s = s_\text{C} + s_\text{P}\phi_\text{P}$, comment on the following practices (ϕ_P^* denotes the previous iteration value):
   (a) $s_\text{C} = A - B\left|\phi_\text{P}^*\right|\phi_\text{P}^*$ and $s_\text{P} = 0$;
   (b) $s_\text{C} = A$ and $s_\text{P} = -B\left|\phi_\text{P}^*\right|$;
   (c) $s_\text{C} = A + B\left|\phi_\text{P}^*\right|\phi_\text{P}^*$ and $s_\text{P} = -2B\left|\phi_\text{P}^*\right|$; and
   (d) $s_\text{C} = A + 9B\left|\phi_\text{P}^*\right|\phi_\text{P}^*$ and $s_\text{P} = -10B\left|\phi_\text{P}^*\right|$.

4. Consider one-dimensional planar, steady-state heat conduction in a medium with constant thermal conductivity (k) and volumetric rate of thermal energy generation ($\dot{q}$). At the left and right domain boundaries ($x = 0$ and $x = L$, respectively) the boundary conditions are described by Equations (5) and (6), respectively.

$$-k\left.\frac{dT}{dx}\right|_{x=0} = h_0\left(T_\infty - T_0\right) \tag{5}$$

$$-k\frac{dT}{dx}\bigg|_{x=L} = h_L\left(T_L - T_\infty\right) \tag{6}$$

In Equations (5) and (6), h_0 and h_L are the heat transfer convection coefficients and T_0 and T_L are the corresponding boundary temperatures. Establish the discretized equations for boundary and bulk (interior) control volumes. (Boundary control volumes – control volumes adjacent to the domain boundaries.)

5. Consider one-dimensional planar, steady-state heat conduction with constant thermal conductivity (k) and volumetric rate of thermal energy generation ($\dot{q}$). Establish the discretized equations for boundary and bulk control volumes assuming a uniform mesh and taking into account the following conditions:

   (a) $\dot{q} = 2\,\mathrm{kW\,m^{-3}}$, $k = 3\,\mathrm{W\,m^{-1}\,K^{-1}}$, and the boundary conditions at the left and right boundaries of the domain given by Equations (7) and (8), respectively, where $q''_{\mathrm{A}} = 10\,\mathrm{W\,m^{-2}}$ and $T_{\mathrm{B}} = 50°\mathrm{C}$;

   $$-k\frac{dT}{dx}\bigg|_{x=0} = q''_{\mathrm{A}} \tag{7}$$ $$T(L) = T_{\mathrm{B}} \tag{8}$$

   (b) $\dot{q} = 2\,\mathrm{kW\,m^{-3}}$, $k = 3\,\mathrm{W\,m^{-1}\,K^{-1}}$, and the boundary conditions at the left and right boundaries of the domain given by Equations (9) and (10), respectively, where $T_{\mathrm{A}} = 50°\mathrm{C}$ and $q''_{\mathrm{B}} = -20\,\mathrm{W\,m^{-2}}$;

   $$T(0) = T_{\mathrm{A}} \tag{9}$$ $$k\frac{dT}{dx}\bigg|_{x=L} = q''_{\mathrm{B}} \tag{10}$$

   (c) $\dot{q} = 2\,\mathrm{kW\,m^{-3}}$, $k = 3\,\mathrm{W\,m^{-1}\,K^{-1}}$, and the boundary conditions at the left and right boundaries of the domain given by Equations (11) and (12), respectively, where $h = 10\,\mathrm{W\,m^{-2}\,K^{-1}}$, $T_\infty = 100°\mathrm{C}$, and $T_{\mathrm{B}} = 50°\mathrm{C}$;

   $$-k\frac{dT}{dx}\bigg|_{x=0} = h\left[T_\infty - T(0)\right] \tag{11}$$ $$T(L) = T_{\mathrm{B}} \tag{12}$$

   (d) $\dot{q} = 2T\,[\mathrm{K}]\ \mathrm{W\,m^{-3}}$, $k = 1\,\mathrm{W\,m^{-1}\,K^{-1}}$, and the boundary conditions at the left and right boundaries of the domain given by Equations (13) and (14), respectively, where $\epsilon = 0.1$, $T_{\mathrm{sur}} = 100°\mathrm{C}$, $h = 10\,\mathrm{W\,m^{-2}\,K^{-1}}$, and $T_\infty = 150°\mathrm{C}$; and

   $$-k\frac{dT}{dx}\bigg|_{x=0} = \epsilon\sigma\{T_{\mathrm{sur}}^4 - [T(0)]^4\} \tag{13}$$ $$-k\frac{dT}{dx}\bigg|_{x=L} = h\left[T(L) - T_\infty\right] \tag{14}$$

6. Consider one-dimensional planar, steady-state heat conduction in a medium with a thermal conductivity (k) equal to $1\,\mathrm{W\,m^{-1}\,K^{-1}}$ and a volumetric rate of thermal energy generation ($\dot{q}$) equal to $2\,\mathrm{W\,m^{-3}}$. The medium thickness (L) is equal to $4\,\mathrm{m}$. At $x = 0$ (left boundary), a heat flux into the domain equal to $5\,\mathrm{W\,m^{-2}}$ is applied and at $x = L$ the heat flux leaving the domain is equal to $13\,\mathrm{W\,m^{-2}}$. Using the finite volume method and a uniform mesh with four control volumes, discretize the equations and solve the resulting system of algebraic equations by:

   (a) Thomas algorithm (TDMA);

(b) Gauss-Seidel iteration method;

(c) TDMA considering $T = 100°\text{C}$ for the left control volume; and

(d) Gauss-Seidel iteration method considering $T = 100°\text{C}$ for the left control volume.

7. Consider one-dimensional, steady-state heat conduction in a fin of constant cross-sectional area governed by Equation (15).

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) - \frac{hP}{A}\left(T - T_\infty\right) = 0 \tag{15}$$

Assume that a temperature equal to T_b is prescribed at the fin base and that the fin tip is perfectly insulated.

(a) Discretize the equation using the finite volume method (bulk control volumes).

(b) Establish the discretized equations for the boundary control volumes.

(c) Solve the system of algebraic equations derived from the discretization process and compare the corresponding numerical solution against the analytical solution.

8. The fully developed laminar flow between two parallel plates is described by Equation (16), where u, μ, and dp/dx correspond to the flow velocity, fluid dynamic viscosity, and pressure gradient, respectively.

$$\frac{d}{dy}\left(\mu\frac{du}{dy}\right) - \frac{dp}{dx} = 0 \tag{16}$$

(a) Discretize the equation using the finite volume method (bulk control volumes).

(b) Considering stationary plates, establish the discretized equations for the boundary control volumes.

(c) Considering one of the plates stationary and the other moving with velocity U, establish the discretized equations for the boundary control volumes.

(d) Solve the system of algebraic equations derived from the discretization process and compare the corresponding numerical solution against the analytical solution.

9. Consider two-dimensional, transient heat conduction in a rectangular plate with $1\,\text{m} \times 1\,\text{m}$ cross section. In such conditions, heat conduction is described by Equation (17). The plate is subjected to the boundary conditions given by Equations (18) – (21). Consider the initial condition of the plate given by Equation (22).

$$\rho c_p \frac{\partial T}{\partial t} = \text{div}\left(k\,\text{grad}T\right) + \dot{q} \tag{17}$$

$$k\left.\frac{\partial T}{\partial x}\right|_{x=0} = 10\,\text{W}\,\text{m}^{-2} \tag{18}$$

$$T\left(x = 1\,\text{m}\right) = 350°\text{C} \tag{19}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0 \tag{20}$$

$$-k\frac{\partial T}{\partial y}\bigg|_{y=1\,\mathrm{m}} = 30\left[T\left(y=1\,\mathrm{m}\right) - 300^{\circ}\mathrm{C}\right] \tag{21}$$

$$T\left(x, y, t=0\right) = 500^{\circ}\mathrm{C} \tag{22}$$

Consider the thermophysical properties given by Equations (23) – (25) and assume a volumetric rate of thermal energy generation dependent on temperature given by Equation (26). (Equations (23) and (26) require local temperature values in °C.)

$$k = 100 + 0.1T - 0.0001T^2\,\mathrm{W\,m^{-1}\,K^{-1}} \tag{23}$$

$$\rho = 1000\,\mathrm{kg\,m^{-3}} \tag{24}$$

$$c_p = 200\,\mathrm{J\,kg^{-1}\,K^{-1}} \tag{25}$$

$$\dot{q} = 20\left(1 - 0.05T + 0.001T^2\right)\,\mathrm{W\,m^{-3}} \tag{26}$$

Using the finite volume method, a uniform mesh 10 × 10 control volumes, and the fully implicit discretization scheme with a time step size (Δt) equal to 1 s, establish the discretized equations for the control volumes containing the following grid nodes:

(a) $i = 5$ and $j = 5$;

(b) $i = 1$ and $j = 5$;

(c) $i = 5$ and $j = 1$;

(d) $i = 10$ and $j = 5$;

(e) $i = 5$ and $j = 10$;

(f) $i = 1$ and $j = 1$;

(g) $i = 10$ and $j = 1$;

(h) $i = 10$ and $j = 10$;

(i) $i = 1$ and $j = 10$; and

10. Consider the one-dimensional distribution of a scalar variable ϕ governed by convection and diffusion transport processes along the axial direction of a porous-walled duct – the corresponding governing equation is given by Equation (27).

$$\frac{d}{dx}\left(\dot{m}_x\phi\right) - \Gamma A\frac{d^2\phi}{dx^2} + \dot{m}_L\phi_L = 0 \tag{27}$$

In Equation (27), A is the duct cross-sectional area, $\dot{m}_x$ is the mass flow rate at any location x, and $\dot{m}_L$ corresponds to the rate of mass leakage through the porous wall per unit length of the duct ($\dot{m}_L \equiv -d\dot{m}_x/dx$). When $\dot{m}_L$ is positive (fluid leaking out), ϕ_L is equal to the value of ϕ within the duct and when $\dot{m}_L$ is negative (fluid leaking into the duct), ϕ_L is equal to ϕ_0 (value of ϕ in the duct external environment). For a duct length l, the boundary conditions are $\phi = \phi_1$ at

$x = 0$ and $\phi = \phi_0$ at $x = l$. Assume that $\dot{m}_L$ and ΓA are constants. Apply the central-differecing scheme for the following.

(a) Establish the discretized equations for bulk and boundary control volumes.

(b) Determine ϕ for the following two cases:

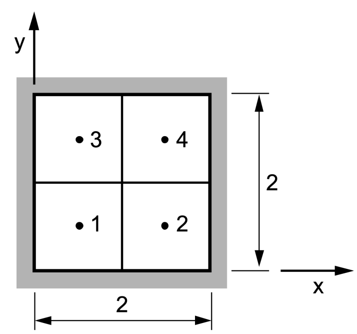
(i) $\dot{m}_x/\Gamma A = 40$ and $\dot{m}_x = 0$ at $x = 0$ and $x = l$, respectively; and

(ii) $\dot{m}_x = 0$ and $\dot{m}_x/\Gamma A = 40$ at $x = 0$ and $x = l$, respectively.

11. In a steady-state two-dimensional flow, the value of a scalar variable ϕ is governed by Equation (28), where $\rho = 1$, $\Gamma = 1$, $a = 10$, and $b = 2$. The velocity field is uniform and equal to $\mathbf{u} \equiv (u, v) = (1, 4)$. Consider $\phi = 0$ at the top and right boundaries and $\phi = 100$ at the remaining boundaries.

$$\text{div}\,(\rho\phi\mathbf{u}) = \text{div}\,(\Gamma\,\text{grad}\phi) + a - b\phi \tag{28}$$

For the uniform mesh presented in the figure, establish the finite volume discretization equations for all control volumes and calculate the values of ϕ considering for the convective term the



Problem 11

following methods:

(a) central differencing scheme;

(b) upwind differencing scheme; and

(c) hybrid differencing scheme.

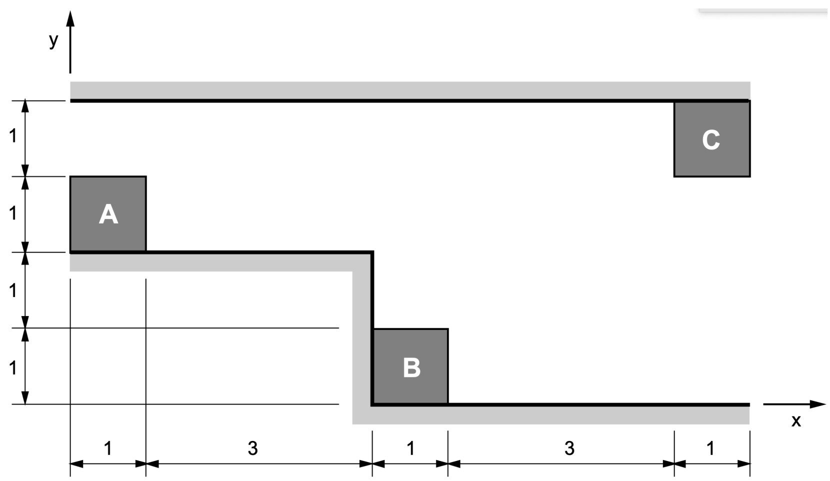
12. Consider an incompressible, two-dimensional, steady-state laminar flow in the channel shown in the figure. Assume that the velocity profile in the inlet section is uniform, where $u = 4$ and $v = 0$, and that the Reynolds number based on the velocity in the inlet section and its height is equal to 100. Assume that the velocity gradient is null at the outlet section. Apply the hybrid differecing scheme.

(a) Establish the discretized equation for the momentum in the $x-$direction for the Control Volumes A, B, and C.

(b) Establish the discretized equation for the momentum in the $y-$direction for the Control Volumes A, B, and C.

(c) Establish the discretized pressure correction equation for the Control Volumes A, B, and C.

(d) Is the exit section boundary condition indicated in the problem statement reasonable? Justify.



Problem 12