Advanced Heat Transfer

Part IV: Numerical Heat Transfer Methods

1. Introduction

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Introduction — Outline

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   - Analytical and Numerical Solution Methods
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1. Methods for Transport Phenomena Prediction

Prediction Approaches

Two main methods can be considered to evaluate the performance of transport (heat transfer and fluid flow) processes: (i) experimental methods; and (ii) theoretical methods.

Experimental Investigation

- More reliable than the theoretical method.
- Expensive and, eventually, not feasible for full-scale equipment.
- The representativeness of lab-scale (prototype) measurements for full-scale equipment may be unsatisfactory.
- Some physical phenomena are difficult to reproduce at a lab-scale, and consequently, neglected.
- Measuring equipment (*in-situ* sampling techniques) may influence (disturb) the actual physical processes under investigation (invasive measurements) leading to erroneous interpretations.

1. Methods for Transport Phenomena Prediction

Theoretical Approach: Analytical and Numerical Solution Methods

- Two theoretical solution methods are available: (i) analytical (exact or approximate); and (ii) numerical (approximate) methods.
- A mathematical model for the description of the underlying physical processes is required.
    - The prediction ability of the theoretical approach (theoretical results reliability) depends strongly on the validity of mathematical model.
    - For complex phenomena, modeling approximations are required.
- Analytical methods provide the solution in a continuum spatio-temporal domain. On the contrary, numerical methods provide solutions at discrete space and time locations.
- Many problems with engineering relevance cannot be solved by analytical means – due to complicated geometries, complex boundary conditions, or variable material properties. (For this class of problems, numerical methods are the only theoretical solution approach available.)

1. Methods for Transport Phenomena Prediction

Numerical Solution Methods: Advantages

- Less expensive, less harmful, and faster than experimental methods.
- Comprehensive (detailed and complete) process characterization – all model variables are determined in the domain of interest.
- Realistic and/or ideal conditions can be investigated.
- For systems with multiple phenomena, the effect of any individual physical phenomenon can be isolated and investigated.
- Data generated by theoretical models based on reliable and fundamental physical descriptions can provide insights for critical gaps of understanding.
- Continuous trend of increasing computer power allowing faster calculations and the potential to develop models from detailed fundamentals.
- Abundance of general-purpose, reliable, robust, and user-friendly software packages.

1. Methods for Transport Phenomena Prediction

Numerical Solution Methods: Cautions on Accuracy

- Numerical solutions for physical problems are not completely free from underlying errors (solution deficiencies not derived from a lack of knowledge) and uncertainties (deficiencies derived from lack of knowledge).
- Errors can be classified into numerical errors – roundoff errors, iteration errors (truncation of iterative sequences), and discretization errors –, coding errors and user errors. (If errors were completely absent the numerical solution would be equal to the analytical (exact) solution of the mathematical problem statement.)
- The main sources of uncertainties (modeling errors) are related to:
  - rough (or inadequate) approximations for geometry representation, boundary conditions, or material properties (input uncertainty); and
  - modeling approximations (simplifying assumptions) for complex physical processes (turbulence, combustion, multiphase flows, ...) that can lead to deviations of theoretical predictions from the real performance (physical model uncertainty).

2. Numerical Solution Methods

Components of a Numerical Solution Method

1. Mathematical Model
2. Discretization Method
3. Coordinate and Basis Vector Systems
4. Numerical Grid
5. Finite Approximations
6. Solution Method
7. Convergence Criteria

2. Numerical Solution Methods

Components of a Num. Sol. Method – 1. Mathematical Model

- Mathematical model — set of differential or integro-differential equations that represent the governing equations as well as the boundary conditions and, for time-dependent problems, initial conditions.
  - Governing equations — mathematical statements of conservation principles – mass conservation principle, Newton's second law of motion (momentum conservation), first law of thermodynamics (energy conservation principle) – in an infinitesimal (differential) volume element. The governing equations take into account the relevant trans. mechanisms described by the corresponding rate equations.
- The mathematical model is selected based on the target application (problem in consideration) – heat conduction through a solid wall, convection heat transfer between a wall and a adjoining fluid, *etc.*.
- Simplifying model assumptions may be applied to decrease the mathematical and numerical model complexity, save computational time, and enhance the solution success.

2. Numerical Solution Methods

Components of a Num. Sol. Method – 2. Discretization Method

- Discretization method (or equation discretization method) — method to approximate a differential equation by a system of algebraic equations for the dependent variables at discrete locations in space- and, eventually, time-coordinates. The solution for this set of algebraic equations provides the numerical solution for the dependent variable at the discrete points (computational domain).
- Most relevant discretization methods:
  - Finite Difference Method;
  - Finite Element Method; and
  - Finite Volume Method.
- The same solution is obtained by any discretization method if the grid is very fine.
- The selection of a suitable discretization method depends on the class of problems in consideration.

2. Numerical Solution Methods

Comp. of a Num. Sol. Method – 3. Coordinate and Basis Vector Systems

- Coordinate and basis vector systems – systems that define the form in which the conservation equations are written.

Components of a Num. Sol. Method – 4. Numerical Grid (1/3)

- Numerical grid (also called mesh or nodal network) — discrete representation of the geometric domain (continuum space region) by a finite number of subdomains on which the problem is to be solved.
- Different types of grids can be considered to obtain a numerical solution depending on the problem in consideration but mainly on the domain geometric complexity. (Note that different mesh types can be applied for the same problem and geometry.)
- Depending on the geometric complexity the following two main mesh categories can be considered:
  - structured grids – suitable for very simple to complex geometries; and
  - unstructured grids – recommended for very complex geometries.

2. Numerical Solution Methods

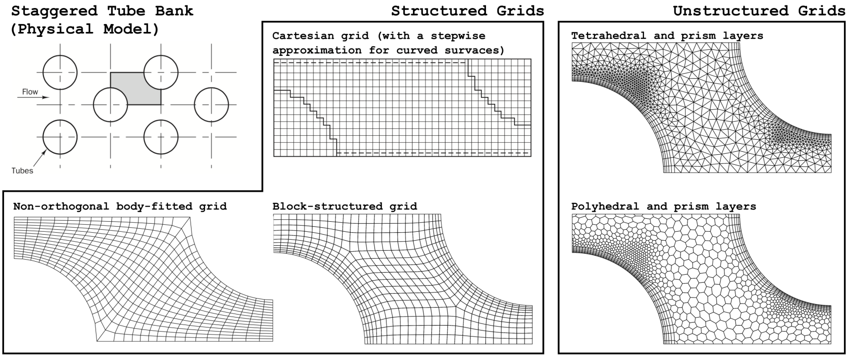
Components of a Num. Sol. Method – 4. Numerical Grid (2/3)

- For a structured grid: (1) grid points are located at the intersection of two (three) lines of different families in 2D (3D) geometries; (2) interior grid points have a fixed number of neighboring grid points; and (3) grid points location stored in a matrix can be easily identified by their indices (I, J and I, J, K for 2D and 3D, respectively).
- Structured grids can be further classified into the following categories:
  - structured (orthogonal) Cartesian, cylindrical, and spherical (uniform ou non-uniform) – very simple (orthogonal and regular) geometries;
  - structured (orthogonal and non-orthogonal) curvilinear grids (or body fitted grids) – complex geometries; and
  - block-structured grids – complex geometries.
- Unstructured grids offer great geometric flexibility and an efficient computational cost since cells can be strictly concentrated where needed.

2. Numerical Solution Methods

Components of a Num. Sol. Method – 4. Numerical Grid (3/3)

- An unstructured grid can be composed by elements (cells) of any shape and there are no constrains on the number of neighbor cells or grid points.
- An explicit specification of node locations and neighbor connectivity is required for unstruc. grids – this complicates programming and the solution method.



2. Numerical Solution Methods

Components of a Num. Sol. Method – 5. Finite Approximations

- Finite Approximations — approximations for the discretization process, namely for the derivatives at the grid points (in the framework of the finite difference method), for surface and volume integrals (finite volume method) and shape and weighting functions (for finite element method).
- Numerical (discretization) errors can be reduced by considering more accurate approximations (or by applying the approximations to finer meshes).

2. Numerical Solution Methods

Components of a Num. Sol. Method – 6. Solution Method

- Solution Method — method for solving the system of (linear or nonlinear) algebraic equations derived from the discretization process. (Depending on the nature of partial differential equations (mathematical model) from which the algebraic equations are derived (through the discretization process), the system of algebraic equations can be linear or nonlinear.)
- A suitable method of solution depends on the problem in consideration, grid type, and the number of nodes considered in each algebraic equation.
- For a system of nonlinear algebraic equations, an iterative solution technique is mandatory which involves: (1) guessing a solution; (2) linearizing the discretized equations about that solution; (3) improving the solution; and (4) repeating steps (2) and (3) until a converged solution is obtained.

2. Numerical Solution Methods

Components of a Num. Sol. Method – 7. Convergence Criteria

- Convergence Criteria – stopping criteria for iterative solution methods.
- Generally, two levels of iterative processes are considered: (1) inner iterations for solving a system of linear algebraic equations; and (2) outer iterations in which nonlinearities and coupling procedures are handled.
- From the standpoints of solution accuracy (iteration errors) and method efficiency, the criteria considered to stop the iterative method is very relevant.

3. General Transport Equation: Differential Form

Transport Equation for Property ϕ

The generic conservation equation for a scalar quantity, ϕ, in a coordinate-free form is given as follows:

$$\boxed{\underbrace{\frac{\partial(\rho\phi)}{\partial t}}_{\mathtt{A}} + \underbrace{\mathrm{div}\,(\rho\phi\mathbf{u})}_{\mathtt{B}} = \underbrace{\mathrm{div}\,(\Gamma\,\mathrm{grad}\,\phi)}_{\mathtt{C}} + \underbrace{S_\phi}_{\mathtt{D}}}$$

- Nomenclature: ρ – density; ϕ – conserved intensive property (dependent variable); t – time; $\mathbf{u}$ – velocity vector; Γ – diffusion coefficient; and S_ϕ – source term.
- Equation Terms: `A` – Rate of change (transient or unsteady) term; `B` – Convective term; `C` – Diffusive term; and `D` – Source term.
- The governing equations of fluid flow and heat transfer can be obtained from the general transport equation considering the adequate values for ϕ, $\mathbf{u}$, Γ, and S_ϕ.

3. General Transport Equation: Differential Form

Heat Diffusion Equation

The heat diffusion equation – equation that governs the temperature distribution in a medium without bulk motion (advection) – is obtained from the general transport equation considering:

$$\phi = T;\ \mathbf{u} = \mathbf{0};\ \Gamma = k/c_p;\ \text{and}\ S_\phi = \dot{q}/c_p.$$

$$\rho c_p \frac{\partial T}{\partial t} = \operatorname{div}\left(k \operatorname{grad} T\right) + \dot{q}$$

Fluid Flow Governing Equations: Continuity Equation

The overall mass conservation (continuity) equation is obtained from the general transport equation considering:

$$\phi = 1;\ \Gamma = 0;\ \text{and}\ S_\phi = 0.$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho \mathbf{u}\right) = 0$$

3. General Transport Equation: Differential Form

Fluid Flow Governing Equations: Momentum Equations (1/2)

Considering gravity as the only body force, the momentum equation for the x Cartesian coordinate is obtained from the gen. transport equation considering:

$$\phi = u \text{ ($x-$direction velocity); } \Gamma = \mu \text{ (dynamic viscosity); and}$$

$$S_\phi = \left\{ \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial w}{\partial x}\right) \right\} - \frac{2}{3}\frac{\partial}{\partial x}\left(\mu \operatorname{div} \mathbf{u}\right) - \frac{\partial p}{\partial x} + \rho g_x;$$

where g_x is the $x-$direction gravity vector component. Consequently,

$$\frac{\partial(\rho u)}{\partial t} + \operatorname{div}(\rho u \mathbf{u}) = \operatorname{div}(\mu \operatorname{grad} u) + S_\phi$$

For Cartesian coordinates, the previous equation – taking into account the expression for S_ϕ – can be written in a tensor notation, as follows:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j u_i) = \frac{\partial}{\partial x_j} \overbrace{\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\right]}^{\tau_{ij} - \text{viscous stress tensor}} - \frac{\partial p}{\partial x_i} + \rho g_i$$

3. General Transport Equation: Differential Form

Fluid Flow Governing Equations: Momentum Equations (2/2)

In a coordinate-free vector form the momentum equation is written as follows:

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \operatorname{div}(\rho\mathbf{u}\mathbf{u}) = \operatorname{div}\mathrm{T} + \rho\mathbf{g}$$

The stress tensor, T, and the rate of strain tensor, D, are provided below.

$$\mathrm{T} = -\left(p + \frac{2}{3}\mu\operatorname{div}\mathbf{u}\right)\mathrm{I} + 2\mu\mathrm{D}$$

$$\mathrm{D} = \frac{1}{2}\left[\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^{T}\right]$$

3. General Transport Equation: Differential Form

Fluid Flow Governing Equations: Energy Equation (1/2)

The energy equation is obtained from the gen. transport equation considering:

$$\phi = u_{\mathrm{t}} \text{ (specific internal (thermal) energy); } \Gamma = k \text{; and}$$
$$S_\phi = -p\,\mathrm{div}\,\mathbf{u} + \mu\Phi + \dot{q};$$

consequently,

$$\frac{\partial(\rho u_{\mathrm{t}})}{\partial t} + \mathrm{div}\,(\rho u_{\mathrm{t}}\mathbf{u}) = \mathrm{div}\,(k\,\mathrm{grad}\,T) - p\,\mathrm{div}\,\mathbf{u} + \mu\Phi + \dot{q}$$

In Cartesian coordinates, the dissipation function reads as follows:

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] +$$
$$\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right] - \frac{2}{3}(\mathrm{div}\,\mathbf{u})^2$$

3. General Transport Equation: Differential Form

Fluid Flow Governing Equations: Energy Equation (2/2)

For steady, three-dimensional flow of an incompressible fluid ($\mathrm{div}\,\mathbf{u} = 0$ and $u_\mathrm{t} = cT$) with constant properties (ρ, c, μ, and k) in a Cartesian coordinate system (x, y, z), the energy equation reads as follows:

$$\rho c \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \mu \left\{ 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right] \right\} + \dot{q}$$

3. General Transport Equation: Relevance

Relevance of the General Transport Equation

- The governing equations for fluid flow and heat transfer phenomena can be though as particular cases of the general transport equation. Therefore, the general transport equation is a convenient equation to demonstrate the application of discretization methods for the terms that are common to all conservation equations – transient, convection, diffusion, and source terms.
- For the application of the Finite Volume Method, the integral (steady or unsteady) form of the general transport equation is required – see next slides.

3. General Transport Equation: Steady Integral Form

Steady Integral Form of the Transport Equation for Property ϕ

Integrating the steady-state differential form of the general transport equation over a three-dimensional control volume (CV), one obtains

$$\int_{\mathrm{CV}} \mathrm{div}\,(\rho\phi\mathbf{u})\,dV = \int_{\mathrm{CV}} \mathrm{div}\,(\Gamma\,\mathrm{grad}\,\phi)\,dV + \int_{\mathrm{CV}} S_\phi dV$$

Applying the Gauss's divergence theorem to convert the volume integrals of the first term on the LHS (convective term) and first term on the RHS (diffusive term) into surface integrals, the integrated form of the steady-state transport equation is given by:

$$\int_A \underbrace{\mathbf{n}\cdot(\rho\phi\mathbf{u})}_{f^c\,-\,\mathrm{Convective\,Flux}} dA = \int_A \underbrace{\mathbf{n}\cdot(\Gamma\mathrm{grad}\,\phi)}_{f^d\,-\,\mathrm{Diffusive\,Flux}} dA + \int_{\mathrm{CV}} S_\phi dV$$

f^c – flux component of property ϕ due to fluid flow (convection) along the outward normal vector $\mathbf{n}$; and f^d – flux component of property ϕ due to diffusion along the inward normal vector $\mathbf{n}$.

3. General Transport Equation: Unsteady Integral Form

Unsteady Integral Form of the Transport Equation for Property ϕ

Integrating the unsteady-state differential form of the general transport equation over a three-dimensional CV – and applying the Gauss's divergence theorem to the volume integrals representing the convective and diffusive terms – and over the time interval t to $t + \Delta t$, the following equation is obtained.

$$\int_{\Delta_t} \frac{\partial}{\partial t} \left(\int_{\mathrm{CV}} \rho\phi dV \right) dt + \int_{\Delta_t} \int_A \underbrace{\mathbf{n} \cdot (\rho\phi\mathbf{u})}_{f^c} dAdt = \int_{\Delta_t} \int_A \underbrace{\mathbf{n} \cdot (\Gamma \operatorname{grad} \phi)}_{f^d} dAdt + \int_{\Delta_t} \int_{\mathrm{CV}} S_\phi dVdt$$

4. Discretization Methods: Finite Difference Method

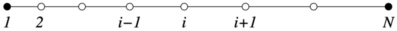
Basic Concepts (1/2)

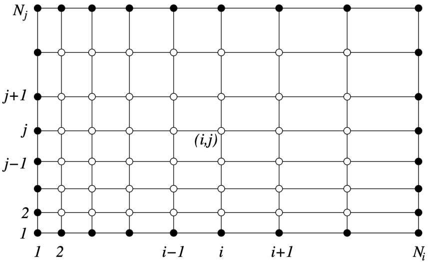
- The governing equation in the differential form is the starting point for the application of the finite difference (discretization) method.
- For each nodal point of the numerical grid, one algebraic equation is derived from the differential equation – governing equation (for interior or bulk nodes) and Neumann or convection boundary conditions (boundary nodes) – by approximating the derivatives by finite-difference approximations. Such finite-difference approximations involve the value of the dependent variable, ϕ, at each grid point as well as at its neighboring grid points. At the end, one equation for each unknown ϕ value is obtained. (For boundary nodes where Dirichlet boundary conditions apply no action is required.)
- Since the governing equation is solved in its original (differential) form – without the need to modify the governing equation into an alternative form for the application of the discretization method – the solution obtained is called the strong form solution.

4. Discretization Methods: Finite Difference Method

Basic Concepts (2/2)

- The solution domain is covered by a set of nodal points (numerical grid). See below examples of 1D (top) and 2D (bottom) grids used by finite difference methods with the corresponding (typical) node notation. Filled dots correspond to boundary nodes and blank dots correspond to interior (bulk) nodes.





4. Discretization Methods: Finite Difference Method

Application to Steady, Source-free, Const. Property, 2D Cartesian Heat Diff. Eq.

The 2D Cartesian, steady-state heat diffusion equation without sources and constant thermal conductivity (Laplace equation in 2D rectangular coordinates) is written as follows:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

For the application of the finite difference method, the second (partial) derivatives must be approximated. The usual procedure to derive such (finite-difference) approximations involves the use of the Taylor series expansion around a reference interior (bulk) point (x_i, y_j). Particularly, for the derivative $\partial^2 T/\partial x^2$, the Taylor series expansion at points (x_{i-1}, y_j) and (x_{i+1}, y_j) are considered as follows

$$T_{i-1,j} = T_{i,j} - (x_i - x_{i-1}) \left(\frac{\partial T}{\partial x}\right)_{i,j} + \frac{(x_i - x_{i-1})^2}{2} \left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} - \dots$$

$$T_{i+1,j} = T_{i,j} + (x_{i+1} - x_i) \left(\frac{\partial T}{\partial x}\right)_{i,j} + \frac{(x_{i+1} - x_i)^2}{2} \left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} + \dots$$

4. Discretization Methods: Finite Difference Method

Application to Steady, Source-free, Const. Property, 2D Cartesian Heat Diff. Eq.

Truncating the series after the third term and combining both equations yields,

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} \approx \frac{T_{i+1,j}\left(x_i - x_{i-1}\right) + T_{i-1,j}\left(x_{i+1} - x_i\right) - T_{i,j}\left(x_{i+1} - x_{i-1}\right)}{\frac{1}{2}\left(x_{i+1} - x_{i-1}\right)\left(x_{i+1} - x_i\right)\left(x_i - x_{i-1}\right)}$$

This equation corresponds to the central-difference approximation for the second derivative. An alternative procedure to arrive at the previous equation consists in the application of approximations for the first derivative as follows

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} \approx \frac{\left(\frac{\partial T}{\partial x}\right)_{i+1/2,j} - \left(\frac{\partial T}{\partial x}\right)_{i-1/2,j}}{\frac{1}{2}\left(x_{i+1} - x_{i-1}\right)}$$

where the first derivatives are approx. by the central-difference form as shown below.

$$\left(\frac{\partial T}{\partial x}\right)_{i-1/2,j} \approx \frac{T_{i,j} - T_{i-1,j}}{x_i - x_{i-1}} \qquad \left(\frac{\partial T}{\partial x}\right)_{i+1/2,j} \approx \frac{T_{i+1,j} - T_{i,j}}{x_{i+1} - x_i}$$

(These approximations for the first derivatives are obtained using Taylor series expansion – as previously, for the second derivative approximation but at points (x_{i-1}, y_j) and (x_i, y_j) and points (x_i, y_j) and (x_{i+1}, y_j).)

4. Discretization Methods: Finite Difference Method

Application to Steady, Source-free, Const. Property, 2D Cartesian Heat Diff. Eq.

Considering a constant grid spacing (equidistant grid points – uniform grid) along the x axis, $x_i - x_{i-1} = x_{i+1} - x_i = \Delta x$, the last equation can be written as

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} \approx \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{(\Delta x)^2}$$

In a similar fashion, for the derivative $\partial^2 T/\partial y^2$ and considering a constant grid spacing along the y axis, the following equation is obtained

$$\left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j} \approx \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{(\Delta y)^2}$$

Substituting the approximations obtained for the second derivatives into the governing equation, one obtains the discretized (or discrete) equation for the grid point (x_i, y_j):

$$\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{(\Delta x)^2} + \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{(\Delta y)^2} = 0$$

4. Discretization Methods: Finite Difference Method

Application to Steady, Source-free, Const. Property, 2D Cartesian Heat Diff. Eq.

For a uniform 2D grid ($\Delta x = \Delta y$), the last equation simplifies to

$$T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1} - 4T_{i,j} = 0$$

The procedure to obtain the discretized equation applies for the interior (bulk) nodal points. Such discretized equations correspond to approximate algebraic equations for the governing equation.

For boundary (external) nodes at which non-Dirichlet boundary conditions are applied, the corresponding discrete equation is obtained considering simultaneously the boundary condition and the governing equation.
For instance, consider that at the left boundary of the domain the following (convection) boundary condition applies:

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=x_1} = h\left[T_\infty - T\left(x_1, y_j\right)\right]$$

4. Discretization Methods: Finite Difference Method

Application to Steady, Source-free, Const. Property, 2D Cartesian Heat Diff. Eq.

The discretized equation for a nodal point located at the left boundary where the previous convection boundary condition applies – generic point (x_1, y_j), with $2 \leq j \leq N_j - 1$ (see node locations in Slide 26) – is obtained as follows:

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{1,j} = \frac{\left(\frac{\partial T}{\partial x}\right)_{1+1/2,j} - \left(\frac{\partial T}{\partial x}\right)_{1,j}}{x_{1+1/2,j} - x_{1,j}} =$$

$$\frac{\frac{T_{2,j} - T_{1,j}}{\Delta x} - \frac{h\left(T_{1,j} - T_\infty\right)}{k}}{\Delta x/2} = \frac{2T_{2,j} - 2\left(1 + h\Delta x/k\right)T_{1,j} + 2\left(h\Delta x/k\right)T_\infty}{(\Delta x)^2}$$

The approximation for $\partial^2 T/\partial y^2$ presented in Slide 29 still holds for the current boundary nodes. Substituting the approximations for the second derivatives into the governing equation (first equation presented in Slide 27) and assuming a uniform 2D grid ($\Delta x = \Delta y$), the discretized equation for these boundary nodes corresponds to

$$2T_{2,j} + T_{1,j+1} + T_{1,j-1} + \frac{2h\Delta x}{k} T_\infty - 2\left(\frac{h\Delta x}{h} + 2\right) T_{1,j}$$

4. Discretization Methods: Weighted Residual Method

Basic Concepts

- The differential equation is expressed in the form $L(\phi) = 0$, where $\phi(\mathbf{x})$ and L correspond to the dependent variable and differential operator, respectively. Considering $\overline{\phi}(\mathbf{x})$ an approximate solution of $\phi(\mathbf{x})$, the residual $R \equiv L(\overline{\phi}) \neq 0$ is multiplied by a weighting function w – forming the weighted residual (wR) – and the corresponding integration over the domain of interest Ω is set to zero

$$\int_{\Omega} w(\mathbf{x}) R(\mathbf{x}) \, d\Omega = 0$$

$$\overline{\phi}(\mathbf{x}) = \sum_{i=1}^{N} a_i \Psi_i(\mathbf{x}) \qquad\qquad w(\mathbf{x}) = \sum_{i=1}^{N} b_i w_i(\mathbf{x})$$

- Finite element method and finite volume method are particular versions of the weighted residual method.
- Finite element method: the same set of basis functions is considered for the definition of the approximate solution and weighting function, *i.e.*, $w_i = \Psi_i$.
- Finite volume method: no assumption on the functional dependence of $\overline{\phi}$ is considered and $w(\mathbf{x})$ is set equal to one.

4. Discretization Methods: Finite Volume Method

Basic Concepts (1/2)

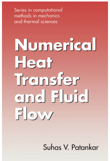
- The entire solution domain (computational domain) is divided into a finite (discrete) number of contiguous (adjacent and non-overlapping) control volumes (cells) forming a mesh (grid) – step known as space discretization (or mesh generation).
- In general, control volumes are triangles, quadrilaterals, or arbitrary polygons in 2D and tetrahedral and hexahedral elements or arbitrary polyhedra in 3D geometries.
- At the centroid (geometrical center) of each control volume, a nodal point is placed at which the dependent variable is computed – other choices for nodal point locations are possible.
- The governing equation in the integral form is the starting point for the application of the finite volume method – since the governing differential equation is not solved directly (it is integrated, approximated, and then solved) the solution obtained is called the weak form solution.

4. Discretization Methods: Finite Volume Method

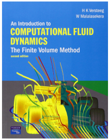
Basic Concepts (2/2)

- The integral form of the transport (conservation) equation is applied to each control volume yielding one discretized algebraic equation.
- Piecewise profiles for the variation of the dependent variable between grid points are considered to evaluate integrals.
- Each discretized equation represents a mathematical statement of the conservation principle over the corresponding control volume.
- The numerical solution obtained using the finite volume method satisfies local and global conservation independently of grid characteristics – even in a coarse mesh conservation is assured.
- The numerical solution of the system of algebraic equations derived by applying the finite volume discretization method yields an approximate solution to the original governing (differential) equation at only the grid nodes. There is no underlying assumption concerning the numerical solution at other points embraced by the computational domain.

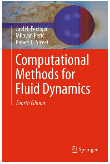
Further Reading



- Chapter 1: Introduction
- Chapter 2: Mathematical Description Physical Phenomena
- Chapter 3: Discretization Methods



- Chapter 1: Introduction
- Chapter 2: Conservation Laws of Fluid Motion and Boundary Conditions



- Chapter 1: Basic Concepts of Fluid Flow
- Chapter 2: Introduction to Numerical Methods
- Chapter 3: Finite Difference Methods