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# Advanced Heat Transfer

## Extraordinary Exam – Problem 3

## September 4, 2021 (17h00)

Consider combined diffusion and convection transport as well as generation of thermal energy in a tube with an inner radius ($R$) and length ($L$) equal to 0.2 m and 0.5 m, respectively. A laminar flow of an incompressible and constant property fluid under fully developed hydrodynamic conditions is considered in the tube. The axial velocity profile is given by $v_z\left(r\right) = 2u_m\left[1-\left(r/R\right)^2\right]$, where $r$ and $u_m$ correspond to the local radial position and mean fluid velocity, respectively. (Note that for the current conditions, the radial velocity component ($v_r$) and $\partial v_z/\partial z$ are negligible.) The volumetric rate of thermal energy generation ($\dot{q}$) is given by $500\left(T\,[\mathrm{K}]\right)^2\ \mathrm{W\,m^{-3}}$, where $T\,[\mathrm{K}]$ corresponds to the local temperature provided in the Kelvin scale. At the tube inlet section ($z=0$), the total mass flow rate ($\dot{m}$) and fluid temperature ($T_{\mathrm{in}}$) are equal to $1.0\,\mathrm{kg\,s^{-1}}$ and 350 K, respectively. The fluid density ($\rho$), specific heat ($c_p$), and thermal conductivity ($k$) are equal to $13464\,\mathrm{kg\,m^{-3}}$, $138.443\,\mathrm{J\,kg^{-1}\,K^{-1}}$, and $8.883\,\mathrm{W\,m^{-1}\,K^{-1}}$, respectively. Temperature gradients along the circumferential direction (angular coordinate – $\phi$) are negligible and, consequently, the governing equation for the temperature distribution written in 2D (two-dimensional) axisymmetric ($r,z$) coordinates is presented below.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho v_r c_p T\right) + \frac{\partial}{\partial z}\left(\rho v_z c_p T\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}$$

Consider the 2D axisymmetric calculation domain discretized with a uniform mesh ($\Delta r = \Delta z$). Apply the finite volume method for the following questions.

(a) (1.0 v.) Determine the maximum cell size along both coordinate directions to ensure a reliable application of the central differencing scheme to the governing equation.

**Solution:**

To obtain a bounded (physically realistic) solution the absolute value of the grid Peclet number ($Pe$) in both coordinate directions must be lower than 2.0. In the radial direction there is no bulk fluid motion ($v_r = 0$) – thermal energy transport occurs exclusively by diffusion – and, consequently, any cell spacing $\Delta r$ considered does not lead to unbounded solutions. In the axial direction the relative importance of convection over diffusion differs according to the radial location in consideration. The worst case – for which the relative strength of convection over diffusion is the highest (highest $Pe_z$ value) – is observed at the nearest node to the tube center-line ($r=0$) because it is at the tube center-line that the highest axial velocity is observed – see Equation (1).

$$u_{\max} \equiv v_z\left(r=0\right) = 2u_m\left[1-\left(0/R\right)^2\right] \Leftrightarrow u_{\max} = \frac{2\dot{m}}{\rho\pi R^2} \tag{1}$$

Since the grid Peclet number along the (axial) flow direction must be lower than 2, the maximum possible cell spacing along the axial direction is calculated with Equation (2).

$$\begin{aligned} Pe_z \equiv \frac{F_z}{D_z} < 2 \Leftrightarrow \frac{\rho u_{\max} c_p \Delta z}{k} < 2 \Leftrightarrow \Delta z < \frac{2k}{\rho u_{\max} c_p} \Leftrightarrow \Delta z < \frac{\pi R^2 k}{\dot{m} c_p} \Leftrightarrow \\ \Leftrightarrow \Delta z < \frac{\pi \times 0.2^2 \times 8.883}{1.0 \times 138.443} \Leftrightarrow \Delta z < 8.063\,\mathrm{mm} \end{aligned} \tag{2}$$

The cell size along both coordinate directions ($\Delta r = \Delta z$) should be less than 8.063 mm. Otherwise, non-positive (discretized equation) coefficients can arise leading to unbounded (physically unrealistic) solutions.

(b) (2.0 v.) Considering the central differencing scheme and a mesh featuring $5\,\mathrm{mm} \times 5\,\mathrm{mm}$ uniform control volumes, determine the discretized equation for the node whose coordinates are given by $(r_\mathrm{P}, z_\mathrm{P}) = (0.1025\,\mathrm{m}, 0.2525\,\mathrm{m})$. Present all intermediate calculations as well as the final (computed) values for center-point and neighboring node coefficients – $a_\mathrm{P}$ and $a_\mathrm{nb}$, respectively – and the constant term $b$. (In cylindrical coordinates the infinitesimal volume is given as $dV = r\,dr\,d\phi\,dz$.)

**Solution:**

The discretized equation for a generic bulk node is obtained by volume integration of the governing equation followed by the application of suitable interpolation functions (profile assumptions). Volume integration is performed in Equations (3) – (8). (Note that the convective term along the radial direction is negligible ($v_r = 0$) and, consequently, it is not considered in the equations that follow. Additionally, the diffusion coefficient $\Gamma$ is equal to $k/c_p$.)

$$\underbrace{\int_{\Delta V} \frac{\partial}{\partial z}(\rho v_z T)\,dV}_{\mathrm{A}} = \underbrace{\int_{\Delta V} \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r \frac{\partial T}{\partial r}\right) dV}_{\mathrm{B}} + \underbrace{\int_{\Delta V} \frac{\partial}{\partial z}\left(\Gamma \frac{\partial T}{\partial z}\right) dV}_{\mathrm{C}} + \underbrace{\int_{\Delta V} \frac{\dot{q}}{c_p}\,dV}_{\mathrm{D}} \tag{3}$$

$$\begin{aligned} \mathrm{A} \equiv \int_{\Delta V} \frac{\partial}{\partial z}(\rho v_z T)\,dV &= \int_{r_\mathrm{w}}^{r_\mathrm{e}} \int_{z_\mathrm{s}}^{z_\mathrm{n}} \frac{\partial}{\partial z}(\rho v_z T)\,dz r dr = \left[(\rho v_z T)_\mathrm{n} - (\rho v_z T)_\mathrm{s}\right] \frac{r_\mathrm{e}^2 - r_\mathrm{w}^2}{2} = \\ &= \left[(\rho v_z T)_\mathrm{n} - (\rho v_z T)_\mathrm{s}\right] \underbrace{\frac{r_\mathrm{e} + r_\mathrm{w}}{2}}_{r_\mathrm{P}} \underbrace{r_\mathrm{e} - r_\mathrm{w}}_{\Delta r} = \left[(\rho v_z T)_\mathrm{n} - (\rho v_z T)_\mathrm{s}\right] r_\mathrm{P} \Delta r \end{aligned} \tag{4}$$

$$\begin{aligned} \mathrm{B} \equiv \int_{\Delta V} \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r \frac{\partial T}{\partial r}\right) dV &= \int_{z_\mathrm{s}}^{z_\mathrm{n}} \int_{r_\mathrm{w}}^{r_\mathrm{e}} \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r \frac{\partial T}{\partial r}\right) r dr dz = \\ = \left[\left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{e} - \left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{w}\right] (z_\mathrm{n} - z_\mathrm{s}) &= \left[\left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{e} - \left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{w}\right] \Delta z \end{aligned} \tag{5}$$

$$\mathrm{C} \equiv \int_{\Delta V} \frac{\partial}{\partial z}\left(\Gamma \frac{\partial T}{\partial z}\right) dV = \int_{r_\mathrm{w}}^{r_\mathrm{e}} \int_{z_\mathrm{s}}^{z_\mathrm{n}} \frac{\partial}{\partial z}\left(\Gamma \frac{\partial T}{\partial z}\right) dz r dr = \left[\left(\Gamma \frac{\partial T}{\partial z}\right)_\mathrm{n} - \left(\Gamma \frac{\partial T}{\partial z}\right)_\mathrm{s}\right] r_\mathrm{P} \Delta r \tag{6}$$

$$\mathrm{D} \equiv \int_{\Delta V} \frac{\dot{q}}{c_p}\,dV = \frac{\dot{q}}{c_p} \int_{z_\mathrm{s}}^{z_\mathrm{n}} \int_{r_\mathrm{w}}^{r_\mathrm{e}} r dr dz = \frac{\dot{q}}{c_p} r_\mathrm{P} \Delta r \Delta z \tag{7}$$

Substituting Equations (4)–(7) into Equation (3) the following equation (Equation (8)) is obtained.

$$\begin{aligned} \int_{\Delta V} \frac{\partial}{\partial z}(\rho v_z T)\,dV = \int_{\Delta V} \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r \frac{\partial T}{\partial r}\right) dV + \int_{\Delta V} \frac{\partial}{\partial z}\left(\Gamma \frac{\partial T}{\partial z}\right) dV + \int_{\Delta V} \frac{\dot{q}}{c_p}\,dV \Leftrightarrow \\ \Leftrightarrow \left[(\rho v_z T)_\mathrm{n} - (\rho v_z T)_\mathrm{s}\right] r_\mathrm{P} \Delta r = \left[\left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{e} - \left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{w}\right] \Delta z + \\ \left[\left(\Gamma \frac{\partial T}{\partial z}\right)_\mathrm{n} - \left(\Gamma \frac{\partial T}{\partial z}\right)_\mathrm{s}\right] r_\mathrm{P} \Delta r + \frac{\dot{q}}{c_p} r_\mathrm{P} \Delta r \Delta z \end{aligned} \tag{8}$$

Considering the central differencing scheme for the convective terms and piecewise-linear profiles (central differencing scheme) to evaluate the derivatives at the cell faces, the final discretized equation is given as follows – see Equation (9).

$$\begin{aligned}
&\left[(\rho v_z T)_\mathrm{n}-(\rho v_z T)_\mathrm{s}\right] r_\mathrm{P}\Delta r=\left[\left(\Gamma r\frac{\partial T}{\partial r}\right)_\mathrm{e}-\left(\Gamma r\frac{\partial T}{\partial r}\right)_\mathrm{w}\right]\Delta z+\\
&\left[\left(\Gamma\frac{\partial T}{\partial z}\right)_\mathrm{n}-\left(\Gamma\frac{\partial T}{\partial z}\right)_\mathrm{s}\right] r_\mathrm{P}\Delta r+\frac{\dot{q}}{c_p}r_\mathrm{P}\Delta r\Delta z\Leftrightarrow\frac{\rho v_z r_\mathrm{P}\Delta r}{2}\left[(T_\mathrm{P}+T_\mathrm{N})-(T_\mathrm{S}+T_\mathrm{P})\right]=\\
&\left[\frac{\Gamma_\mathrm{e}r_\mathrm{e}\Delta z}{\Delta r}(T_\mathrm{E}-T_\mathrm{P})-\frac{\Gamma_\mathrm{w}r_\mathrm{w}\Delta z}{\Delta r}(T_\mathrm{P}-T_\mathrm{W})\right]+\left[\frac{\Gamma_\mathrm{n}r_\mathrm{P}\Delta r}{\Delta z}(T_\mathrm{N}-T_\mathrm{P})-\frac{\Gamma_\mathrm{s}r_\mathrm{P}\Delta r}{\Delta z}(T_\mathrm{P}-T_\mathrm{S})\right]+\\
&\frac{(s_\mathrm{C}+s_\mathrm{P}T_\mathrm{P})}{c_p}r_\mathrm{P}\Delta r\Delta z\Leftrightarrow\frac{F_z}{2}(T_\mathrm{N}-T_\mathrm{S})=D_\mathrm{e}(T_\mathrm{E}-T_\mathrm{P})-D_\mathrm{w}(T_\mathrm{P}-T_\mathrm{W})+D_\mathrm{n}(T_\mathrm{N}-T_\mathrm{P})-\\
&D_\mathrm{s}(T_\mathrm{P}-T_\mathrm{S})+(S_\mathrm{C}+S_\mathrm{P}T_\mathrm{P})\Leftrightarrow\underbrace{\left[D_\mathrm{w}+D_\mathrm{e}+\left(D_\mathrm{s}+\frac{F_z}{2}\right)+\left(D_\mathrm{n}-\frac{F_z}{2}\right)-S_\mathrm{P}\right]}_{a_\mathrm{P}}T_\mathrm{P}=\\
&D_\mathrm{w}T_\mathrm{W}+D_\mathrm{e}T_\mathrm{E}+\left(D_\mathrm{s}+\frac{F_z}{2}\right)T_\mathrm{S}+\left(D_\mathrm{n}-\frac{F_z}{2}\right)T_\mathrm{N}+S_\mathrm{C}\Leftrightarrow\\
&\Leftrightarrow\boxed{a_\mathrm{P}T_\mathrm{P}=a_\mathrm{W}T_\mathrm{W}+a_\mathrm{E}T_\mathrm{E}+a_\mathrm{S}T_\mathrm{S}+a_\mathrm{N}T_\mathrm{N}+b}
\end{aligned}\tag{9}$$

For the grid node in consideration, the discretized equation coefficients and constant term are calculated below.

$$\begin{aligned}
a_\mathrm{W}\equiv D_\mathrm{w}=\frac{\Gamma_\mathrm{w}r_\mathrm{w}\Delta z}{\Delta r}=\frac{k\left(r_\mathrm{P}-\Delta r/2\right)\Delta z}{c_p\Delta r}=\frac{8.883\times(0.1025-0.005/2)\times0.005}{138.443\times0.005}\Leftrightarrow\\
\Leftrightarrow a_\mathrm{W}\approx 6.416\times10^{-3}\,\mathrm{kg\,s^{-1}}
\end{aligned}\tag{10}$$

$$\begin{aligned}
a_\mathrm{E}\equiv D_\mathrm{e}=\frac{\Gamma_\mathrm{e}r_\mathrm{e}\Delta z}{\Delta r}=\frac{k\left(r_\mathrm{P}+\Delta r/2\right)\Delta z}{c_p\Delta r}=\frac{8.883\times(0.1025+0.005/2)\times0.005}{138.443\times0.005}\Leftrightarrow\\
\Leftrightarrow a_\mathrm{E}\approx 6.737\times10^{-3}\,\mathrm{kg\,s^{-1}}
\end{aligned}\tag{11}$$

$$\begin{aligned}
a_\mathrm{S}\equiv D_\mathrm{s}+\frac{F_z}{2}=\frac{\Gamma_\mathrm{s}r_\mathrm{P}\Delta r}{\Delta z}+\frac{\rho v_z r_\mathrm{P}\Delta r}{2}=\frac{\Gamma_\mathrm{s}r_\mathrm{P}\Delta r}{\Delta z}+\frac{\rho r_\mathrm{P}\Delta r}{2}\frac{2\dot{m}}{\rho\pi R^2}\left[1-\left(\frac{r_\mathrm{P}}{R}\right)^2\right]=\\
\frac{8.883\times0.1025\times0.005}{138.443\times0.005}+\frac{13464\times0.1025\times0.005}{2}\times\frac{2\times1.0}{13464\times\pi\times0.2^2}\times\\
\left[1-\left(\frac{0.1025}{0.2}\right)^2\right]\Leftrightarrow a_\mathrm{S}\approx 9.584\times10^{-3}\,\mathrm{kg\,s^{-1}}
\end{aligned}\tag{12}$$

$$\begin{aligned}
a_\mathrm{N}\equiv D_\mathrm{n}-\frac{F_z}{2}=\frac{\Gamma_\mathrm{n}r_\mathrm{P}\Delta r}{\Delta z}-\frac{\rho v_z r_\mathrm{P}\Delta r}{2}=\frac{\Gamma_\mathrm{n}r_\mathrm{P}\Delta r}{\Delta z}-\frac{\rho r_\mathrm{P}\Delta r}{2}\frac{2\dot{m}}{\rho\pi R^2}\left[1-\left(\frac{r_\mathrm{P}}{R}\right)^2\right]=\\
\frac{8.883\times0.1025\times0.005}{138.443\times0.005}-\frac{13464\times0.1025\times0.005}{2}\times\frac{2\times1.0}{13464\times\pi\times0.2^2}\times\\
\left[1-\left(\frac{0.1025}{0.2}\right)^2\right]\Leftrightarrow a_\mathrm{N}\approx 3.570\times10^{-3}\,\mathrm{kg\,s^{-1}}
\end{aligned}\tag{13}$$

Source-term linearization (required to compute the center-point coefficient and constant term): There are different procedures to linearize the source term, *i.e.*, the constant part ($s_\mathrm{C}$) and the coefficient of $T_\mathrm{P}$ ($s_\mathrm{P}$) may assume different values leading to the same final (converged) solution. However, the considered $s_\mathrm{P}$ value cannot be positive to avoid divergence issues during the iterative solution procedure. The following source-term linearization is herein considered – see Equation (14). (In Equation (14), $T_\mathrm{P}^*$ corresponds to the dependent variable value obtained at the previous iteration (or initially guessed).)

$$s = s_\mathrm{C} + s_\mathrm{P} T_\mathrm{P} \Leftrightarrow s = \underbrace{500\left(T_\mathrm{P}^*\right)^2}_{s_\mathrm{C}} + \overbrace{0}^{s_\mathrm{P}} T_\mathrm{P} \tag{14}$$

In such conditions, the center-point coefficient ($a_\mathrm{P}$) and the constant term ($b$) are calculated as shown in Equations (15) and (16), respectively.

$$\begin{aligned} a_\mathrm{P} = a_\mathrm{W} + a_\mathrm{E} + a_\mathrm{S} + a_\mathrm{N} - S_\mathrm{P} \Leftrightarrow a_\mathrm{P} = 6.416 \times 10^{-3} + 6.737 \times 10^{-3} + 9.584 \times 10^{-3} + \\ 3.570 \times 10^{-3} - 0 \Leftrightarrow a_\mathrm{P} = 2.631 \times 10^{-2}\,\mathrm{kg\,s^{-1}} \end{aligned} \tag{15}$$

$$\begin{aligned} b = S_\mathrm{C} \equiv \frac{s_\mathrm{C}}{c_p} r_\mathrm{P} \Delta r \Delta z \Leftrightarrow b = \frac{500\left(T_\mathrm{P}^*\right)^2}{138.443} \times 0.1025 \times 0.005 \times 0.005 \Leftrightarrow \\ \Leftrightarrow b \approx 9.255 \times 10^{-6}\left(T_\mathrm{P}^*\,[\mathrm{K}]\right)^2\,\mathrm{K\,kg\,s^{-1}} \end{aligned} \tag{16}$$

The discretized equation for the nodal point in consideration is written as presented by Equation (17), where temperatures are considered in the Kelvin scale.

$$\boxed{26.310 T_\mathrm{P} = 6.416 T_\mathrm{W} + 6.737 T_\mathrm{E} + 9.584 T_\mathrm{S} + 3.570 T_\mathrm{N} + 9.255 \times 10^{-3} T_\mathrm{P}^{*2}} \tag{17}$$

(c) (1.0 v.) Considering the upwind differencing scheme, determine the discretized equation for the boundary node embraced by the control volume with faces coincident to $z = 0$ and $r = 0$. Present all intermediate calculations including the final expressions required to compute the center-point and neighboring node coefficients and constant term $b$.

**Solution:**

The discretized equation for such boundary grid node is obtained considering Equation (8) and taking into account the corresponding convective term discretization method (upwind differencing scheme) and the stated boundary conditions – see Equation (18).

$$\begin{aligned} \left[(\rho v_z T)_\mathrm{n} - (\rho v_z T)_\mathrm{s}\right] r_\mathrm{P} \Delta r = \left[\left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{e} - \left(\Gamma r \frac{\partial T}{\partial r}\right)_\mathrm{w}\right] \Delta z + \\ \left[\left(\Gamma \frac{\partial T}{\partial z}\right)_\mathrm{n} - \left(\Gamma \frac{\partial T}{\partial z}\right)_\mathrm{s}\right] r_\mathrm{P} \Delta r + \frac{\dot{q}}{c_p} r_\mathrm{P} \Delta r \Delta z \Leftrightarrow \rho v_z r_\mathrm{P} \Delta r \left(T_\mathrm{P} - T_\mathrm{in}\right) = \\ \left[\frac{\Gamma_\mathrm{e} r_\mathrm{e} \Delta z}{\Delta r}\left(T_\mathrm{E} - T_\mathrm{P}\right) - 0\right] + \left[\frac{\Gamma_\mathrm{n} r_\mathrm{P} \Delta r}{\Delta z}\left(T_\mathrm{N} - T_\mathrm{P}\right) - \frac{2\Gamma_\mathrm{in} r_\mathrm{P} \Delta r}{\Delta z}\left(T_\mathrm{P} - T_\mathrm{in}\right)\right] + \left(S_\mathrm{C} + S_\mathrm{P} T_\mathrm{P}\right) \Leftrightarrow \\ \Leftrightarrow F_z\left(T_\mathrm{P} - T_\mathrm{in}\right) = D_\mathrm{e}\left(T_\mathrm{E} - T_\mathrm{P}\right) + D_\mathrm{n}\left(T_\mathrm{N} - T_\mathrm{P}\right) - 2D_\mathrm{in}\left(T_\mathrm{P} - T_\mathrm{in}\right) + \left(S_\mathrm{C} + S_\mathrm{P} T_\mathrm{P}\right) \Leftrightarrow \\ \Leftrightarrow \underbrace{\left[D_\mathrm{e} + D_\mathrm{n} + 2D_\mathrm{in} + F_z - S_\mathrm{P}\right]}_{a_\mathrm{P}} T_\mathrm{P} = \\ = \underbrace{0}_{a_\mathrm{W}} T_\mathrm{W} + \underbrace{D_\mathrm{e}}_{a_\mathrm{E}} T_\mathrm{E} + \underbrace{0}_{a_\mathrm{S}} T_\mathrm{S} + \underbrace{D_\mathrm{n}}_{a_\mathrm{N}} T_\mathrm{N} + \underbrace{2D_\mathrm{in} T_\mathrm{in} + F_z T_\mathrm{in} + S_\mathrm{C}}_{b} \end{aligned} \tag{18}$$

The following table summarizes the expressions required to compute the coefficients and constant term for the discretized equation of any interior (bulk) node.

| $a_{\mathrm{W}}$ | $a_{\mathrm{E}}$ | $a_{\mathrm{S}}$ | $a_{\mathrm{N}}$ | $a_{\mathrm{P}}$ | $S_{\mathrm{P}}^{\mathrm{T}}$ | $b$ |
|---|---|---|---|---|---|---|
| 0 | $D_{\mathrm{e}} \equiv \frac{\Gamma_{\mathrm{e}} r_{\mathrm{e}} \Delta z}{\Delta r}$ | 0 | $D_{\mathrm{n}} \equiv \frac{\Gamma_{\mathrm{n}} r_{\mathrm{P}} \Delta r}{\Delta z}$ | $\sum a_{\mathrm{nb}} - S_{\mathrm{P}}^{\mathrm{T}}$ | $-2D_{\mathrm{in}} - F_z + S_{\mathrm{P}}$ | $S_{\mathrm{C}}^{\mathrm{T}} \equiv 2D_{\mathrm{in}} T_{\mathrm{in}} + F_z T_{\mathrm{in}} + S_{\mathrm{C}}$ |