Heat Transfer

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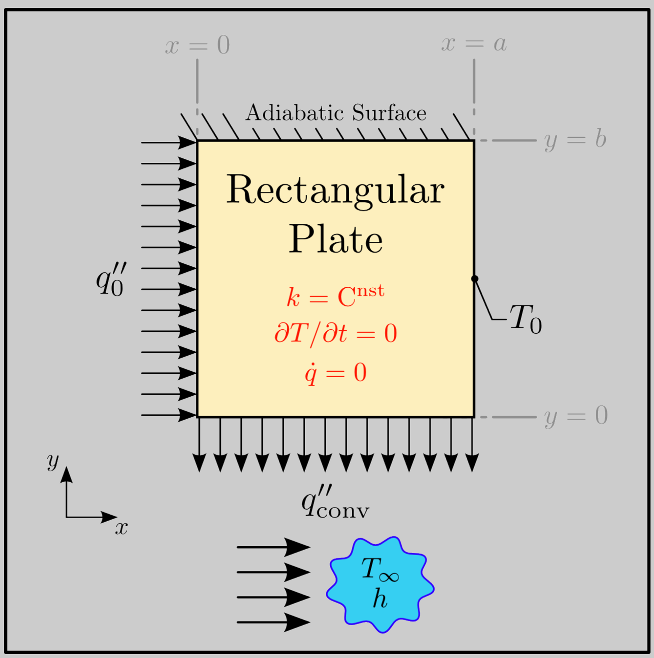
Practical Lecture 3 (Solved Problems)

6. Consider heat conduction on a rectangular plate in steady-state, the length (z direction) being infinite.. The surface $x = 0$ is electrically heated with a heat flux q''_0 [$\mathrm{W\,m^{-2}}$]. The surface $x = a$ is maintained at a constant temperature T_0. The surface $y = b$ is insulated. The surface $y = 0$ dissipates heat by convection to a medium at temperature T_∞ with a convection coefficient h. The thermal conductivity of the material is uniform and there is no internal generation of energy. Formulate the heat conduction problem, establishing the equation governing temperature distribution on the plate along with the associated boundary conditions.

Solution:

The following figure presents a schematic representation of the problem geometry and boundary conditions. In the plate domain, the heat transport occurs exclusively by diffusion (*i.e.*, conduction).



The general form of the heat diffusion equation is described by Equation (1). This equation governs the (spatial and temporal) temperature distribution in stationary homogeneous media (media without bulk fluid motion – advection of energy) in which the only mechanism of heat transport is diffusion. The heat diffusion equation is obtained by applying the conservation of energy requirement (first law of thermodynamics) to a differential (infinitesimal) control volume – $dx \cdot dy \cdot dz$, $dr \cdot rd\phi \cdot dz$, and $dr \cdot r\sin\theta d\phi \cdot rd\theta$ for the case of Cartesian, cylindrical, and spherical coordinates, respectively.

$$\nabla \cdot (k \nabla T) + \dot{q} = \rho c_p \frac{\partial T}{\partial t} \tag{1}$$

For the temperature field described in Cartesian coordinates, $T\,(x, y, z)$, the first term of the first member of Equation (1) can be written according to the first three terms of the first equation member of the following equation – Equation (2).

$$\underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)}_{\nabla\cdot(k\nabla T)} + \dot{q} = \rho c_p \frac{\partial T}{\partial t} \tag{2}$$

Equation (2) can be simplified taking into account the details of the problem as follows:

1. since the problem is bi-dimensional (rectangular plate defined in plane (x, y)) temperature gradients (and heat fluxes) in the coordinate direction orthogonal to the plane (x, y), *i.e.*, coordinate direction z, are negligible, and consequently, the term $\partial/\partial z\,(k\partial T/\partial z)$ vanishes;
2. since steady-state conditions are under consideration, the temperature distribution is not a function of time (*i.e.*, $\partial T/\partial t = 0$), and therefore the term of the second equation member – $\rho c_p \partial T/\partial t$, rate of change of thermal energy stored withing the plate – is neglected;
3. since there is no internal generation of thermal energy in the plate ($\dot{q} = 0$), the fourth term of the first equation member ($\dot{q}$) vanishes; and
4. since the thermal conductivity k is constant in the whole plate domain, $\partial k/\partial x = \partial k/\partial y = 0$, and consequently, the temperature governing equation does not feature a dependence with k.

Applying the stated simplifying assumptions, Equation (2) is written as follows – see Equation (3).

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0} \tag{3}$$

Equation (3) – which corresponds to the Laplace equation ($\nabla^2 T = 0$ or $\Delta T = 0$) – governs the temperature distribution $T\,(x, y)$ in the plate. However, the actual temperature value at each local position in the plate domain depends on the plate thermal interactions with its surroundings – through its physical boundaries ($x = 0$, $x = a$, $y = 0$, and $y = b$). These thermal interactions are considered mathematically in the problem formulation through the statement of boundary conditions. Since the governing equation – Equation (3) – is second order in each spatial coordinate, two boundary conditions must be specified for each spatial coordinate (x and y). According to the problem statement, the four physical boundaries of the plate domain are subjected to different thermal conditions, as described by the following boundary conditions:

x=0:

$$\boxed{-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q''_0} \tag{4}$$

Equation (4) corresponds to a boundary condition of the second kind (or Neumann boundary condition, or a prescribed heat flux boundary condition). This equation states that the conduction heat flux through the plate at $x = 0$ equals q''_0.

x=a:

$$\boxed{T(x = a, y) = T_0} \tag{5}$$

Equation (5) is a boundary condition of the first kind (or Dirichlet boundary condition, or a prescribed temperature boundary condition).

y=0:

$$\boxed{-k\frac{\partial T}{\partial y}\bigg|_{y=0} = h\left[T_\infty - T\left(x, y = 0\right)\right]} \tag{6}$$

Equation (6) corresponds to a boundary condition of the third kind (or convection boundary condition). This equation states that the conduction heat flux to the plate at $y = 0$ equals the convection heat flux from the adjoining fluid to the plate surface at $y = 0$. (This type of boundary condition is derived from a surface energy balance).

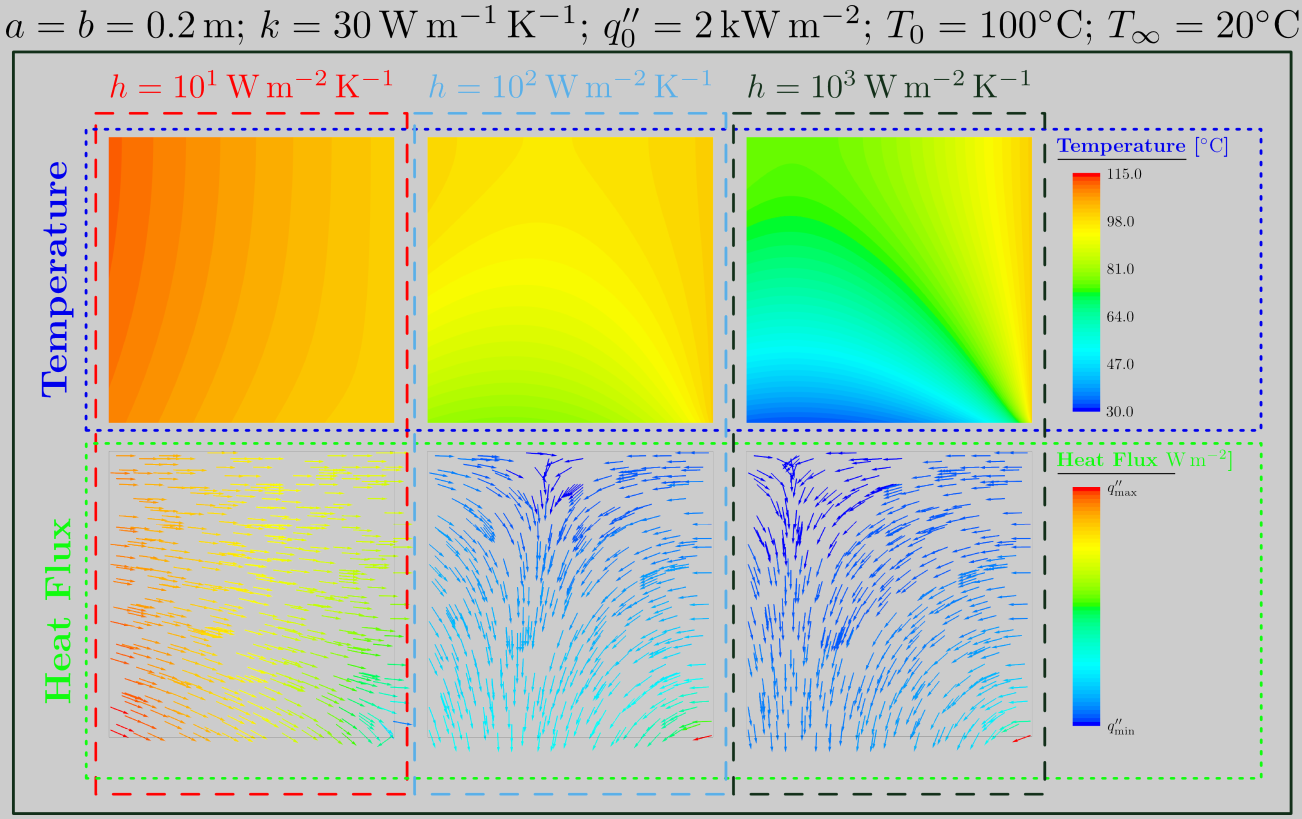
y=b:

$$\boxed{\frac{\partial T}{\partial y}\bigg|_{y=b} = 0} \tag{7}$$

Equation (7) represents a particular case of second kind boundary conditions (see Equation (4)) since it is equivalent to a prescribed zero heat flux value – there is no heat transfer across surface $y = b$ (adiabatic surface).

The following figure presents the temperature field (first row) and heat flux vectors, $\mathbf{q}''$ (second row) for the current problem – governing equation and boundary conditions – considering three different values for the convection heat transfer coefficient (h): 10, 100, and 1000 $\mathrm{W\,m^{-2}\,K^{-1}}$. (The actual convection heat transfer coefficient value affects the solution for the temperature distribution (and, consequently, heat flux vectors) through the convection boundary condition described by Equation (6).) The remaining geometric parameters (a and b), transport properties (k) and thermal conditions at the physical boundaries (T_0, T_∞, and q''_0) that are required to solve the problem are stated in the figure.

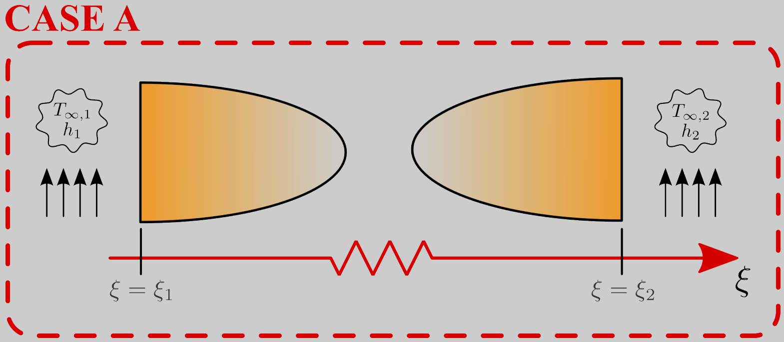
The figure shows that an increase in the convection heat transfer coefficient promotes an increase on the thermal energy (heat) extraction rate through the surface $y = 0$. As a consequence, the plate temperatures decrease, being such a decrease particularly striking near the vicinity of surface $y = 0$. Note that because surface $y = b$ is adiabatic, the isothermal surfaces (isotherms – lines of constant temperature) are perpendicular to this surface. (By the definition of the Fourier's law the heat flux vectors are perpendicular to the isotherms). Therefore, at $y = b$ the heat flux vectors have a negligible y component ($q''_y = 0$) which is in full agreement with the corresponding boundary condition (see Equation (7)). Considering the lowest convection heat transfer coefficient ($h = 10\,\mathrm{W\,m^{-2}\,K^{-1}}$), a preferential heat transport path – from the surface $x = 0$ to the surface $x = a$ – is established (see the isotherms and heat flux vectors).

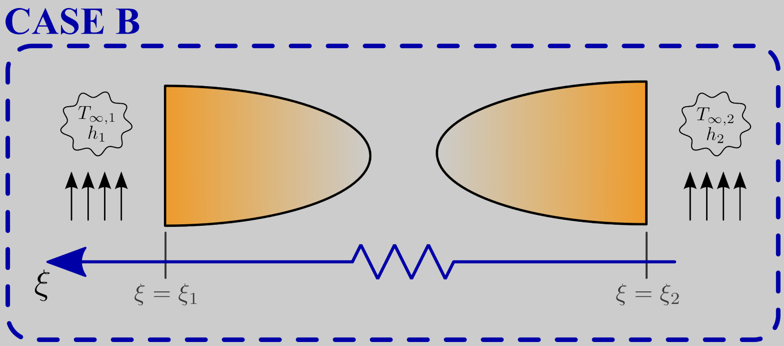


Convection Boundary Conditions – Final Remark

Bear in mind that a convection boundary condition is derived from a surface energy balance ($\dot{E}_{\mathrm{in}} - \dot{E}_{\mathrm{out}} = 0$) in which a convective heat transfer rate equals a conductive heat transfer rate at the surface of interest. A common mistake in formulating this kind of boundary condition is related to an incorrect signal affecting, for instance, the convective heat flux term (*i.e.*, $\pm h\,[T_\infty - T\,(\xi = \xi_1)]$, where ξ corresponds to the coordinate direction).

Consider the (plane, cylindrical, or spherical) wall presented in the following figures – Cases A and B. The coordinate direction ξ corresponds to the coordinate direction x (or y or z) in rectangular (Cartesian) coordinates (plane wall) or to r for radial systems (cylindrical and spherical walls). For each case, consider that both wall surfaces (located at $\xi = \xi_1$ and $\xi = \xi_2$) are exchanging heat by convection with and adjoining fluid. The only difference between both cases relies on the positive direction of the coordinate system – axis ξ. The mathematical formulation of the convection boundary condition at each surface depends on the chosen (positive) direction for the coordinate system as it can be concluded by comparing the formulation of the boundary conditions at the same surface for both cases – compare Equations (8) and (10) and Equations (9) and (11).





Case A

$\underline{\xi = \xi_1}$**:**

$$-k\frac{\partial T}{\partial \xi}\bigg|_{\xi=\xi_1} = h_1\left[T_{\infty,1} - T\left(\xi = \xi_1\right)\right] \tag{8}$$

$\underline{\xi = \xi_2}$**:**

$$-k\frac{\partial T}{\partial \xi}\bigg|_{\xi=\xi_2} = h_2\left[T\left(\xi = \xi_2\right) - T_{\infty,2}\right] \tag{9}$$

Case B

$\underline{\xi = \xi_1}$**:**

$$-k\frac{\partial T}{\partial \xi}\bigg|_{\xi=\xi_1} = h_1\left[T\left(\xi = \xi_1\right) - T_{\infty,1}\right] \tag{10}$$

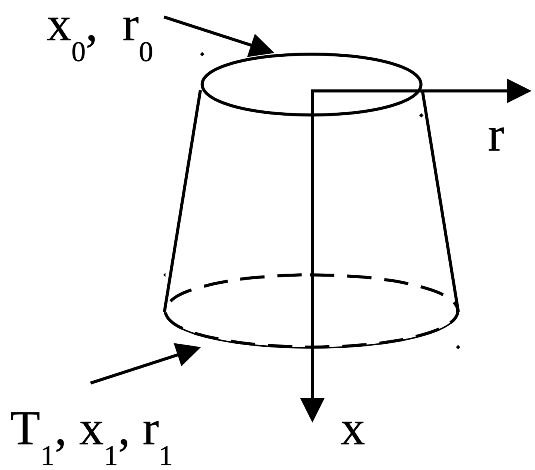
$\underline{\xi = \xi_2}$**:**

$$-k\frac{\partial T}{\partial \xi}\bigg|_{\xi=\xi_2} = h_2\left[T_{\infty,2} - T\left(\xi = \xi_2\right)\right] \tag{11}$$

- Note that Equation (6) resembles to Equation (8) (convection boundary condition for Case A at the surface $\xi = \xi_1$) and Equation (11) (convection boundary condition for Case B at the surface $\xi = \xi_2$).
- If a boundary condition of the third type were considered for surface $y = b$ – instead of being a perfectly insulated (adiabatic) surface – the corresponding mathematical formulation would be similar to that described by Equations (9) and (10) – *i.e.*, it would be described by Equation (12).

$$-k\frac{\partial T}{\partial y}\bigg|_{y=b} = h\left[T\left(x, y = b\right) - T_{\infty}\right] \tag{12}$$

9. Consider a truncated cone as shown in the figure. The coordinates x_0 and x_1 indicate the positions of the faces where the cone was truncated relatively to the apex of the cone. Consider that the lateral surface is insulated and the temperature in each section $x =$ constant is uniform. In the larger face, the temperature T_1 is known and, in the smaller face, the heat flux q_0'' is imposed. Determine the temperature distribution along x.



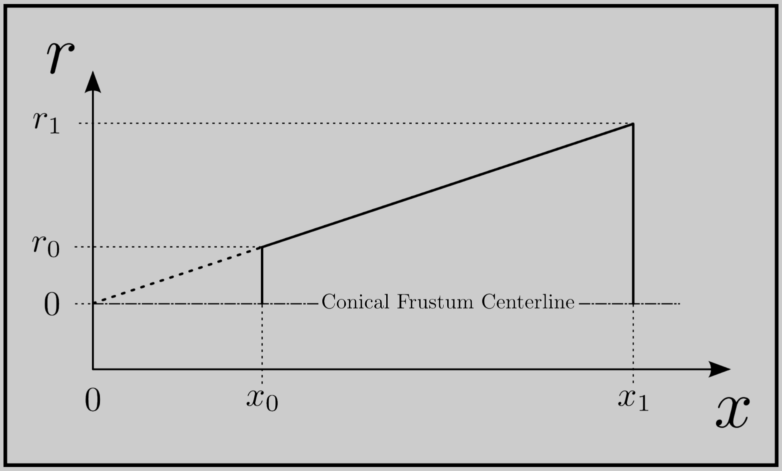
Solution:

Considering steady-state conditions, with no internal generation of thermal energy and no heat losses from the sides of the truncated cone (conical frustum), as a result of the conservation of energy principle, the heat rate is constant along x. Furthermore, if radial temperature gradients were neglected the conduction heat transfer problem becomes one-dimensional (along x). Under these circumstances an alternative approach for conduction analysis – in relation to the standard approach that consists in the integration of a proper form of the heat diffusion equation and taking the boundary conditions to obtain the corresponding integration constants in order to evaluate the temperature distribution – can be applied. This alternative conduction analysis consists in applying the integral form of the Fourier's law which considering that at section $x = x_1$ the corresponding temperature is T_1 (in accordance to the problem statement) reads as follows – see Equation (13).

$$q_x \int_{x}^{x_1} \frac{1}{A(x)} dx = - \int_{T(x)}^{T_1} k(T)\, dT \tag{13}$$

The functional form of the temperature distribution, $T(x)$, – the solution for this problem – can be obtained by performing the integration from x at which the temperature is $T(x)$ to x_1 at which the temperature is equal to T_1. Therefore, it is mandatory to evaluate the cross-sectional area of the conical frustum as a function of the cross section axial position – $A(x)$.

The radius of the conical frustum is calculated in accordance to Equation (14), where r_0 corresponds to the conical frustum radius at x_0 (smaller face section) – see figure below that presents the conical frustum radius, r, as a function of the axial position, x.



$$r\left(x\right)=r_0\left(\frac{x}{x_0}\right) \tag{14}$$

Equation (14) is considered in Equation (15) in order to obtain the value of the conical frustum cross-sectional area as a function of the axial position.

$$A\left(x\right)=\pi\left[r\left(x\right)\right]^2\Leftrightarrow A\left(x\right)=\pi r_0^2\left(\frac{x}{x_0}\right)^2 \tag{15}$$

The heat transfer rate, q_x, (constant along x) can be calculated taking into account the heat flux imposed at the smaller surface, $x=x_0$, that is equal to q_0' – see Equation (16).

$$q_x=q_0''A\left(x=x_0\right)\Leftrightarrow q_x=q_0''\pi r_0^2 \tag{16}$$

The temperature distribution is obtained considering the thermal conductivity constant (independent of temperature) and replacing Equations (15) and (16) in Equation (13) and performing the integration – see Equation (17).

$$\begin{aligned}&q_x\int_x^{x_1}\frac{1}{A\left(x\right)}dx=-\int_{T(x)}^{T_1}k\left(T\right)dT\Leftrightarrow q_0''x_0^2\int_x^{x_1}\frac{1}{x^2}dx=-k\int_{T(x)}^{T_1}dT\Leftrightarrow\\&\Leftrightarrow q_0''x_0^2\left(-\frac{1}{x}\right)\Bigg|_x^{x_1}=-k\left.\left(T\right)\right|_{T(x)}^{T_1}\Leftrightarrow q_0''x_0^2\left(\frac{1}{x}-\frac{1}{x_1}\right)=k\left[T\left(x\right)-T_1\right]\Leftrightarrow\\&\Leftrightarrow\boxed{T\left(x\right)=T_1+\frac{q_0''}{k}x_0^2\left(\frac{1}{x}-\frac{1}{x_1}\right)}\end{aligned} \tag{17}$$

Equation (17) can also be written according to Equation (18) taking into consideration Equation (19).

$$T\left(x\right)=T_1+\frac{q_0''}{k}\underbrace{\left(\frac{r_0}{r_1}\right)^2x_1^2}_{x_0^2}\left(\frac{1}{x}-\frac{1}{x_1}\right) \tag{18}$$

(Note that Equation (19) simply states that since the slope of the conical frustum radius with the axial position (dr/dx) is constant it can be evaluated with the radial and axial coordinates (r, x) of either the smaller or the larger conical frustum face.)

$$\frac{r_0}{x_0} = \frac{r_1}{x_1} \Leftrightarrow x_0 = \left(\frac{r_0}{r_1}\right) x_1 \tag{19}$$

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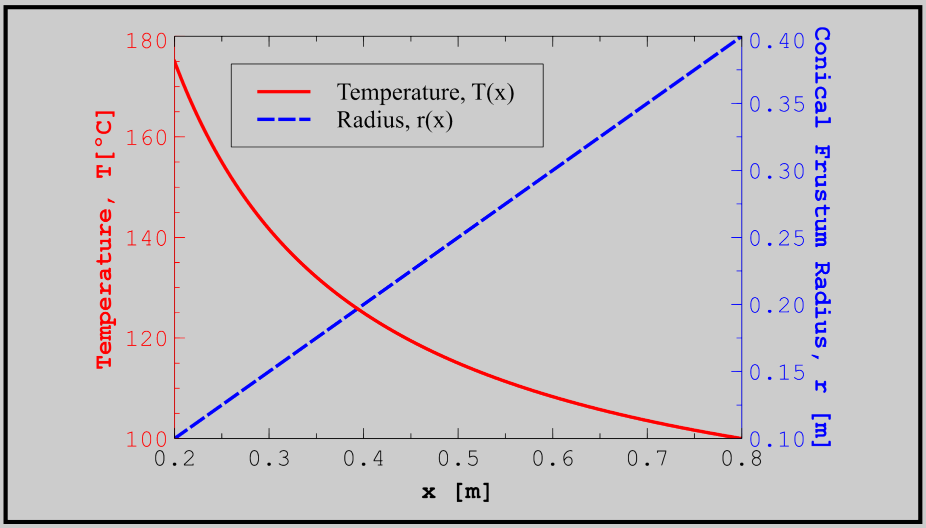
① The figure below presents the temperature profile – computed according to Equation (17) – for the geometrical parameters (x_0 and x_1), thermophysical properties (k), and operating conditions (q''_0 and T_1) therein stated.

$x_0 = 0.2\,\mathrm{m}$, $x_1 = 0.8\,\mathrm{m}$, $r_0 = 0.1\,\mathrm{m}$, $k = 10\,\mathrm{W\,m^{-1}\,K^{-1}}$, $q''_0 = 5\,\mathrm{kW\,m^{-2}}$, and $T_1 = 100°\mathrm{C}$



② The following figure presents the solution of the heat diffusion equation – 2D axisymmetric heat diffusion equation (see Equation (20)) – and the adequate boundary conditions (see Equations (21) to (24)) for the temperature field. (Equation (23) corresponds to the boundary conditions for the symmetry axis (conical frustum centerline) and in Equation (24), $\partial T/\partial n$ corresponds to the temperature gradient in a direction perpendicular to the lateral surface of the conical frustum.)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = 0 \tag{20}$$

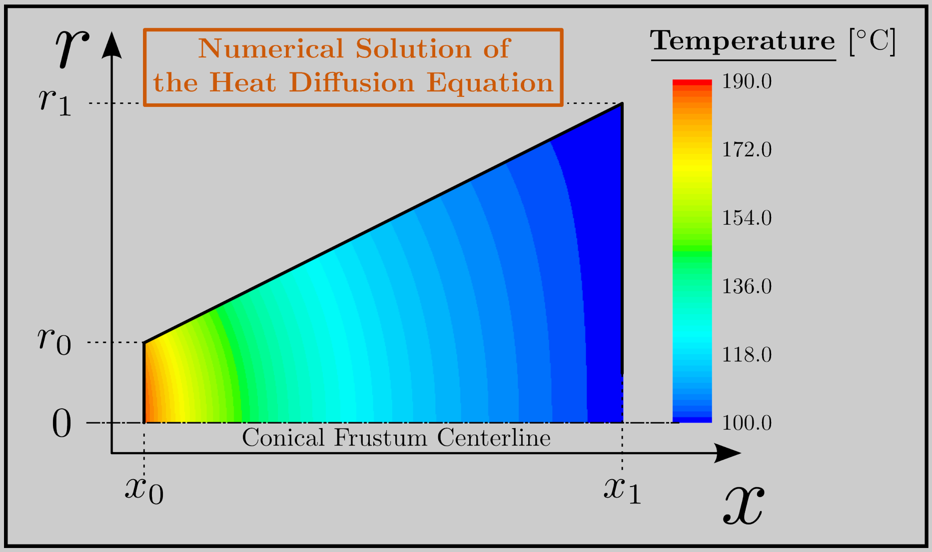
$$-k\left.\frac{\partial T}{\partial x}\right|_{x=x_0} = q''_0 \tag{21}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0 \tag{23}$$

$$T\left(r, x = x_1\right) = T_1 \tag{22}$$

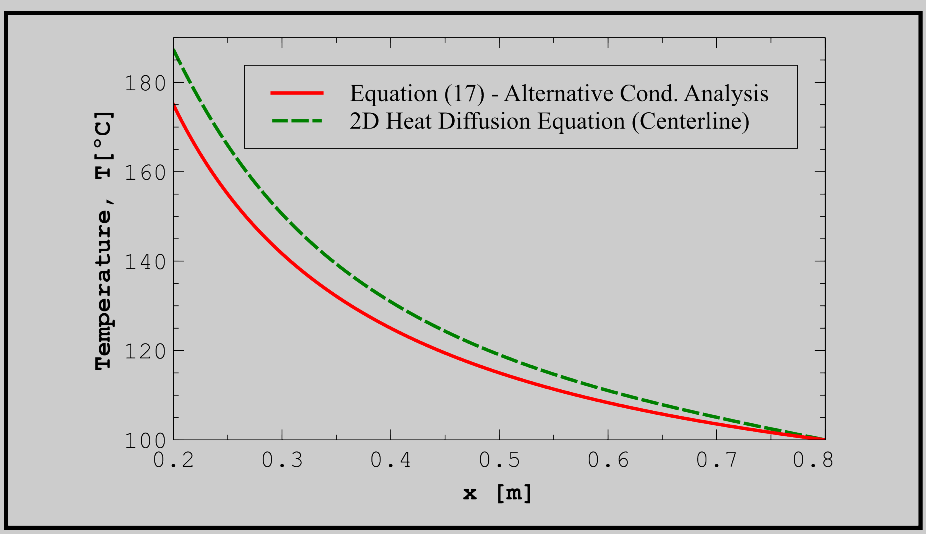
$$\left.\frac{\partial T}{\partial n}\right|_{r=r_0(x/x_0)} = 0 \tag{24}$$

$x_0 = 0.2\,\mathrm{m}$, $x_1 = 0.8\,\mathrm{m}$, $r_0 = 0.1\,\mathrm{m}$, $k = 10\,\mathrm{W\,m^{-1}\,K^{-1}}$, $q_0'' = 5\,\mathrm{kW\,m^{-2}}$, and $T_1 = 100\,^{\circ}\mathrm{C}$



The figure below shows the comparison between the temperature distribution obtained with Equation (17) – alternative conduction analysis – and the temperature profile at the conical frustum centerline computed with the 2D axisymmetric heat diffusion equation (and corresponding boundary conditions) – standard approach – for the same properties and conditions. Slightly lower temperatures are obtained with the alternative approach in relation to the standard procedure. The temperature difference between both approaches increases as the distance to the section $x = x_0$ decreases. The differences observed between both approaches are mainly derived by the one-dimensional heat transfer assumption (no radial temperature gradients) considered behind the formulation of the alternative approach. In fact, the 2D results show that for the current geometry (slope of the lateral surface) significant temperature differences are observed at each conical frustum cross-section (sections $x =$ constant).

$x_0 = 0.2\,\mathrm{m}$, $x_1 = 0.8\,\mathrm{m}$, $r_0 = 0.1\,\mathrm{m}$, $k = 10\,\mathrm{W\,m^{-1}\,K^{-1}}$, $q_0'' = 5\,\mathrm{kW\,m^{-2}}$, and $T_1 = 100\,^{\circ}\mathrm{C}$



The next table compares the results – temperatures at $x = x_0$ and $r = 0$ – obtained with the alternative conduction analysis with the standard approach as the slope of the lateral surface of the conical frustum decreases, *i.e.*, as r_1 tends to r_0. (r_0 and $x_1 - x_0$ was kept constant and equal to 0.1 m and 0.6 m, respectively.) As r_1 approaches r_0 the radial temperature gradients become less relevant and the results computed with the alternative conduction analysis and the 2D axisymmetric (Ax.) heat equation become similar – the (relative) error of the results obtained with alternative conduction analysis in relation to the 2D-Ax. heat equation results decreases. For the limiting case of $r_1 = r_0$ (conical frustum degenerates into a cylinder) the alternative conduction analysis provides the same solution as the (exact) solution calculated with heat diffusion equation. This is observed because in the case of a cylinder there are no temperature gradients in the radial direction (heat transfer is exclusively one-dimensional) and the assumption of isothermal cross-sections – underlying assumption for application of the integral form of the Fourier's law to evaluate the temperature distribution – is fully observed. (The 2D-Ax. results were obtained numerically and for that reason the corresponding solution for the cylinder case is not (exactly) equal to the analytical solution – numerical solution contains numerical errors.)

		Temperature at $x = x_0$ and $r = 0$, $T(r = 0, x = x_0)$ [°C]		
r_1 [m]	Slope, m [−]	Alter. Cond. Analys.	Standard Approach – Heat Equation	
		Eq. (17)	2D-Ax.	1D
0.400	0.500	175.0	187.1	400.0
0.250	0.250	220.0	225.8	
0.175	0.125	271.4	273.9	
0.100	0.000	400.0	399.8	