Heat Transfer

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Practical Lecture 4 (Solved Problems)

17. Consider a conductive wire (electric conductivity equal to $5.1 \times 10^6\,\Omega^{-1}\,\mathrm{m}^{-1}$), of circular cross-section (diameter equal to 0.04 m), unshielded, where direct current I equal to 450 A is passing. The generated power (by Joule effect) per unit length of the wire is $31.6\,\mathrm{W\,m^{-1}}$ and the thermal conductivity of the material is $300\,\mathrm{W\,m^{-1}\,k^{-1}}$. Assume steady-state.

    (a) Assuming a maximum conductor temperature of 80°C and an ambient temperature of 20°C, calculate the convection heat transfer coefficient.

Solution:

Equation (1) provides the temperature distribution along the radial direction, r, of the conductive wire (cylindrical rod) whose external radius is r_{wire}, under steady-state conditions, one-dimensional conduction, constant thermal conductivity, k, and with a uniform volumetric rate of thermal energy generation, $\dot{q}$.

$$T(r) = \frac{\dot{q} r_{\text{wire}}^2}{4k}\left(1 - \frac{r^2}{r_{\text{wire}}^2}\right) + T_s \tag{1}$$

The volumetric rate of thermal energy generation – due to Joule heating (conversion of electrical energy to thermal energy) – can be evaluated with the provided heat transfer rate per unit length of the wire, q', through Equation (2).

$$\dot{q} = \frac{\dot{E}_g}{V} \Leftrightarrow \dot{q} = \frac{q' L}{\pi r_{\text{wire}}^2 L} \Leftrightarrow \dot{q} = \frac{q'}{\pi r_{\text{wire}}^2} \tag{2}$$

To calculate the convection heat transfer coefficient, the wire surface temperature must be known. When the maximum temperature in the conductor – temperature observed at the wire centerline ($r = 0$) – is equal to 80°C, the surface temperature of the wire is computed by replacing Equation (2) in Equation (1) and considering $r = 0$ and $T(r = 0) = 80$°C – see Equation (3).

$$T_s = T(r) - \frac{q'}{4\pi k}\left(1 - \frac{r^2}{r_{\text{wire}}^2}\right) \Rightarrow T_s = T(0) - \frac{31.6}{4 \times 300\pi}\left(1 - \frac{0^2}{0.02^2}\right) \Leftrightarrow \Leftrightarrow T_s \approx T(0) = 80\,^{\circ}\mathrm{C} \tag{3}$$

Equation (3) shows that negligible temperature gradients are observed along the wire radial direction. Therefore, the wire surface temperature is about the same as the temperature observed at the wire centerline.

Once the wire surface temperature is known ($T_s = 80\,^{\circ}\mathrm{C}$), the convection heat transfer coefficient can be computed through the application of the conservation of energy requirement (energy balance) to a control volume embracing the conductive wire – see Equation (4).

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \dot{E}_g = \dot{E}_{\text{st}} \Leftrightarrow \cancel{\dot{E}_{\text{in}}} - \dot{E}_{\text{out}} + \dot{E}_{\text{g}} = \cancel{\dot{E}_{\text{st}}} \Rightarrow \dot{E}_{\text{g}} = \dot{E}_{\text{out}} \Leftrightarrow$$
$$\Leftrightarrow q'L = 2\pi r_{\text{wire}} L h \left(T_s - T_\infty\right) \Leftrightarrow h = \frac{q'}{2\pi r_{\text{wire}} \left(T_s - T_\infty\right)} \Leftrightarrow$$
$$\Leftrightarrow h = \frac{31.6}{2\pi \times (0.04/2) \times (80 - 20)} \Leftrightarrow \boxed{h \approx 4.191\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}} \tag{4}$$

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① The problem statement provides more data than that required to solve it. The provided values for the electric conductivity and current (or, alternatively, the provided value for the heat transfer rater per unit length) would be sufficient to solve the problem. Note that the heat transfer rate from the external wire surface per unit length of the wire – due to Joule (Ohmic or resistance) heating – can be calculated with the electric conductivity (σ_e), electric current (I), and radius of the conductive wire (r_{wire}) – see Equation (5).

$$q' = \frac{\dot{E}_g}{L} = \frac{R_e}{L} I^2 \Leftrightarrow q' = \frac{L}{L\sigma_e A_c} I^2 \Leftrightarrow q' = \frac{1}{\sigma_e \pi r_{\text{wire}}^2} I^2 \Leftrightarrow$$
$$\Leftrightarrow q' = \frac{1}{5.1 \times 10^6 \times (0.04/2)^2 \times \pi} \times 450^2 \Leftrightarrow q' \approx 31.597\,\text{W}\,\text{m}^{-1} \tag{5}$$

② Equation (1) is derived from the integration of the proper form of the heat diffusion equation (in cylindrical coordinates) followed by the application of suitable boundary conditions for the evaluation of the integration constants. The appropriate form of the heat diffusion equation under one-dimensional heat conduction along the radial direction of a solid cylinder (cylindrical rod), steady-state conditions, with constant thermal conductivity and uniform internal generation of thermal energy is given by Equation (6).

$$\frac{1}{r}\frac{\partial T}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial T}{\partial \phi}\left(k\cancel{\frac{\partial T}{\partial \phi}}\right) + \frac{\partial T}{\partial z}\left(k\cancel{\frac{\partial T}{\partial z}}\right) + \dot{q} = \rho c_p \cancel{\frac{\partial T}{\partial t}} \Rightarrow$$
$$\Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}}{k} = 0 \tag{6}$$

The general solution of Equation (6) is obtained by integrating twice Equation (6) – see Equations (7) and (8).

$$\int d\left(r\frac{dT}{dr}\right) = -\frac{\dot{q}}{k}\int r dr \Leftrightarrow r\frac{dT}{dr} = -\frac{\dot{q}r^2}{2k} + C_1 \tag{7}$$

$$\int dT = -\frac{\dot{q}}{2k}\int r dr + C_1 \int \frac{1}{r} dr \Leftrightarrow T(r) = -\frac{\dot{q}}{4k} r^2 + C_1 \ln(r) + C_2 \quad (8)$$

The constants of integration C_1 and C_2 are obtained by applying the adequate boundary conditions. At the cylindrical rod centerline ($r = 0$), a zero-Neumann boundary condition stating this radial location as an axis of symmetry is considered – see Equation (9). At the external rod surface ($r = r_s$), a prescribed temperature value is applied (first kind boundary condition) – see Equation (10).

$$\left.\frac{dT}{dr}\right|_{r=0} = 0 \quad (9)$$

$$T(r = r_s) = T_s \quad (10)$$

The integration constant C_1 is obtained replacing Equation (7) in Equation (9) – see Equation (11).

$$\left.\frac{dT}{dr}\right|_{r=0} = 0 \Leftrightarrow r\left.\frac{dT}{dr}\right|_{r=0} = 0 \Leftrightarrow -\frac{\dot{q} \times 0^2}{2k} + C_1 = 0 \Leftrightarrow C_1 = 0 \quad (11)$$

The integration constant C_2 is obtained replacing Equation (8) in Equation (10) and considering $C_1 = 0$ – see Equation (12).

$$T(r = r_s) = T_s \Leftrightarrow -\frac{\dot{q} r_s^2}{4k} + 0 \times \ln(r_s) + C_2 = T_s \Leftrightarrow C_2 = T_s + \frac{\dot{q} r_s^2}{4k} \quad (12)$$

Finally, the governing equation for the temperature distribution under the stated conditions is obtained by replacing the integration constants C_1 and C_2 (Equations (11) and (12)) in Equation (8) – see Equation (13) which is equal to Equation (1). (Note that r_s (radius of the cylindrical rod) in Equation (13) corresponds to r_wire in Equation (1).)

$$T(r) = \frac{\dot{q} r_\mathrm{s}^2}{4k}\left(1 - \frac{r^2}{r_\mathrm{s}^2}\right) + T_s \quad (13)$$

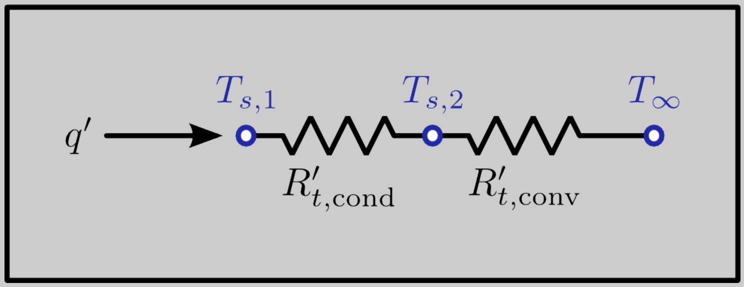
(b) Consider two insulators A and B, with 0.005 m of thickness and thermal conductivities $k_\mathrm{A} = 0.3\,\mathrm{W\,m^{-1}\,K^{-1}}$ and $k_\mathrm{B} = 4\,\mathrm{W\,m^{-1}\,K^{-1}}$. In order to protect the outer surface of the conductor wire, which of the two insulators would you choose in order to prevent the maximum conductor temperature of reaching more than 10% of the value previously considered? Consider for this question $h = 3\,\mathrm{W\,m^{-2}\,K^{-1}}$.

Solution:

In the previous question it was concluded that the temperature gradients along the radial direction of the conductive wire are negligible, and consequently, the wire surface temperature and the wire centerline (maximum) temperature are similar.

The equivalent thermal circuit for the conduction along the insulation thickness and convection from the external insulation surface to the surrounding environment is presented in the figure below. In this figure, $T_{s,1}$ is the temperature at the interface of the conductive wire with the insulation layer – $T_{s,1} \approx T(r=0) = T_{\text{max,wire}}$.

Equivalent Thermal Circuit



The radial position at the external insulation surface is given in Equation (1), where t_{ins} corresponds to the insulation thickness.

$$r_{\text{ins}} = r_{\text{wire}} + t_{\text{ins}} \Leftrightarrow r_{\text{ins}} = (0.04/2) + 0.005 \Leftrightarrow r_{\text{ins}} = 0.025\,\text{m} \tag{14}$$

Equation (15) allows to evaluate the conductive wire surface temperature (similar to the maximum wire temperature) for materials with different thermal conductivity values.

$$q' = \frac{T_{s,1} - T_\infty}{R'_{t,\text{tot}}} \Leftrightarrow T_{s,1} = q' R'_{t,\text{tot}} + T_\infty \Leftrightarrow T_{s,1} = q'\left(R'_{t,\text{cond}} + R'_{t,\text{conv}}\right) + T_\infty \Leftrightarrow$$
$$\Leftrightarrow T_{s,1} = q'\left[\frac{\ln(r_{\text{ins}}/r_{\text{wire}})}{2\pi k} + \frac{1}{2\pi r_{\text{ins}} h}\right] + T_\infty \tag{15}$$

The maximum temperatures in the conductive wire considering insulation materials A and B are calculated in Equations (16) and (17), respectively.

$$T_{s,1} = q'\left[\frac{\ln(r_{\text{ins}}/r_{\text{wire}})}{2\pi k_{\text{A}}} + \frac{1}{2\pi r_{\text{ins}} h}\right] + T_\infty \Leftrightarrow$$
$$\Leftrightarrow T_{s,1} = 31.6 \times \left[\frac{\ln(0.025/0.02)}{2 \times 0.3\pi} + \frac{1}{2 \times 0.025 \times 3\pi}\right] + 20 \Leftrightarrow$$
$$\Leftrightarrow T_{s,1} \approx 90.798°\text{C} \ \left((\text{T}_{\text{s},1}[°\text{C}] - 80)/80 \approx 13.498\%\right) \tag{16}$$

$$T_{s,1} = q'\left[\frac{\ln(r_{\text{ins}}/r_{\text{wire}})}{2\pi k_{\text{B}}} + \frac{1}{2\pi r_{\text{ins}} h}\right] + T_\infty \Leftrightarrow$$
$$\Leftrightarrow T_{s,1} = 31.6 \times \left[\frac{\ln(0.025/0.02)}{2 \times 4\pi} + \frac{1}{2 \times 0.025 \times 3\pi}\right] + 20 \Leftrightarrow \tag{17}$$
$$\Leftrightarrow \boxed{T_{s,1} \approx 87.338°\text{C}\ (9.173\%) \Rightarrow \text{Insulation material B respects the requirements!}}$$

The insulation material A leads to an increase in the conductive wire maximum temperature of about 13.5% in relation to the previous considered maximum temperature value (80°C) – see Equation (16). On the other hand, the application of the insulation material B leads to an increase of the maximum temperature below 10% – see Equation (17). Therefore, the insulation material B should be selected.

(c) What is the outside temperature of the insulation under the conditions of (b)?

Solution:

Considering the equivalent thermal circuit presented in the figure of the solution of question (b), $T_{s,2}$ corresponds to the outside temperature of the insulation that can be computed according to Equation (18).

$$\begin{aligned} q' = \frac{T_{s,2} - T_\infty}{R'_{\text{conv}}} \Leftrightarrow T_{s,2} = q' R'_{\text{conv}} + T_\infty \Leftrightarrow T_{s,2} = q' \frac{1}{2\pi r_{\text{ins}} h} + T_\infty \Leftrightarrow \\ \Leftrightarrow T_{s,2} = 31.6 \times \frac{1}{0.025 \times 3 \times 2\pi} + 20 \Leftrightarrow T_{s,2} \approx 87.057\,^\circ\text{C} \end{aligned} \tag{18}$$

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① In alternative to Equation (18), the outside temperature of the insulation can also be computed according to Equation (19) considering the thermal conductivity of the insulation material B ($k_{\text{B}} = 4\,\text{W}\,\text{m}^{-1}\,\text{K}^{-1}$) and the corresponding conductive wire surface temperature computed in the previous question ($T_{s,1} = 87.338$°C).

$$\begin{aligned} q' = \frac{T_{s,1} - T_{s,2}}{R'_{\text{cond}}} \Leftrightarrow T_{s,2} = T_{s,1} - q' R'_{\text{cond}} \Leftrightarrow T_{s,2} = T_{s,1} - q' \frac{\ln(r_{\text{ins}}/r_{\text{wire}})}{2\pi k_{\text{B}}} \Leftrightarrow \\ \Leftrightarrow T_{s,2} = 87.338 - 31.6 \times \frac{\ln(0.025/0.02)}{4 \times 2\pi} \Leftrightarrow T_{s,2} \approx 87.057^\circ\text{C} \end{aligned} \tag{19}$$

18. (Homework) A cable of circular cross-section, with radius R, drives an electrical current and lays in a medium at temperature T_∞. Consider that the power released by Joule effect, per unit volume, is uniform in a cross-section of the cable.

(a) Explain, justifying, how the thermal conductivity of the material influences the temperature at $r = 0$ and $r = R$. Sketch the radial temperature profile $T(r)$ in a graph.

Solution:

The Final Comment (2) of Question 17(a) provides the detailed procedure to obtain the temperature distribution $T(r)$ from the integration of the appropriate form of the heat diffusion and application of the boundary conditions for the same conditions as those herein considered – one-dimensional heat conduction along the radial direction of a cylindrical rod, steady-state conditions, constant thermal conductivity, and uniform internal generation of thermal energy. The resulting equation for the radial temperature distribution is reproduced in Equation (20) where T_s corresponds to the temperature at the (outer) surface of the cable, *i.e.*, the temperature at the radial position $r = R$.

$$T\left(r\right) = \frac{\dot{q}R^2}{4k}\left(1 - \frac{r^2}{R^2}\right) + T_s \tag{20}$$

T_s can be readily obtained by applying the energy conservation requirement to a control volume embracing the cable and considering that convection heat transfer is the sole mechanism for thermal energy extraction from the cable – see Equation (21). (Alternatively, T_s could be obtained by the application of an energy balance to the outer cable surface (surface $r = R$) as shown in Final Comment (1) below.)

$$\begin{aligned}\dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \dot{E}_g = \dot{E}_{\text{st}} \Leftrightarrow \cancel{\dot{E}_{\text{in}}} - \dot{E}_{\text{out}} + \dot{E}_{\text{g}} = \cancel{\dot{E}_{\text{st}}} \Rightarrow \dot{E}_{\text{g}} = \dot{E}_{\text{out}} \Leftrightarrow \\ \Leftrightarrow \dot{q}V = Ah\left(T_s - T_\infty\right) \Leftrightarrow \dot{q}\pi R^2 L = 2\pi RLh\left(T_s - T_\infty\right) \Leftrightarrow \\ \Leftrightarrow \boxed{T_s = T(r = R) = \frac{\dot{q}R}{2h} + T_\infty}\end{aligned} \tag{21}$$

Equation (21) shows that <u>the thermal conductivity does not play any role on the cable surface temperature $T_s = T(r = R)$</u>. On the other hand, Equation (21) shows that increasing the convection heat transfer coefficient (h) or decreasing the cable radius (R) – *i.e.*, decreasing the thermal resistance for convection (R_{conv})– the cable surface temperature (T_s) approaches to fluid temperature (T_∞).

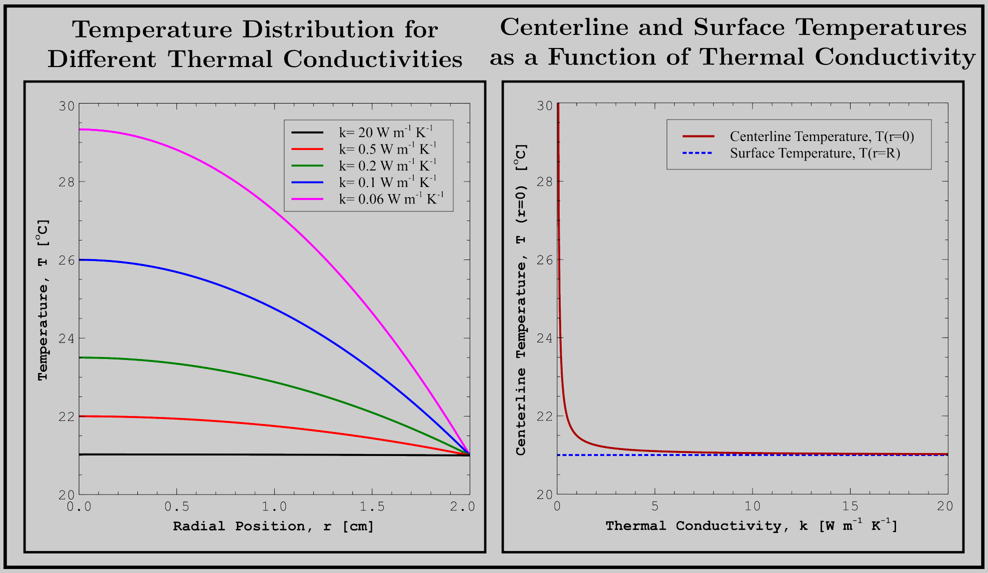
The temperature at the cable centerline ($T(r = 0)$) can be calculated considering $r = 0$ in Equation (20) and T_s given by Equation (21) – see Equation (22).

$$T\left(r = 0\right) = \frac{\dot{q}R^2}{4k}\left(1 - \frac{0^2}{R^2}\right) + T_s \Leftrightarrow \boxed{T\left(r = 0\right) = \frac{\dot{q}R^2}{4k} + \underbrace{\frac{\dot{q}R}{2h} + T_\infty}_{T_s}} \tag{22}$$

Equation (22) shows that <u>decreasing (increasing) the thermal conductivity the temperature at the cable centerline increases (decreases) in relation to the cable surface temperature and fluid temperature.</u>

The following figure presents the radial temperature distribution for five different thermal conductivity values (left) and the temperatures at the cable centerline and surface as a function of the material thermal conductivity (right). Note that as the thermal conductivity increases the temperature gradients within the cable become negligible with no effect on the cable surface temperature that remains constant (and equal to 21°C).

$R = 2\,\mathrm{cm}$, $\dot{q} = 5\,\mathrm{kW\,m^{-3}}$, $h = 50\,\mathrm{W\,m^{-2}\,K^{-1}}$, and $T_\infty = 20\,°\mathrm{C}$



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① A surface energy balance applied to the cable cylindrical surface $r = R$ can be considered to obtain the cable surface temperature, T_s, in alternative to the application of an energy balance to the overall cable – as considered in Equation (21). Note that the temperature gradient (dT/dr) in Equation (23) was evaluated with the temperature distribution provided by Equation (20).

$$\begin{aligned}\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0 \Leftrightarrow A\,(r = R)\,q''_{\text{cond}}\,(r = R) = A\,(r = R)\,q''_{\text{conv}} \Leftrightarrow \\ \Leftrightarrow q''_{\text{cond}}\,(r = R) = q''_{\text{conv}} \Leftrightarrow -k\,\frac{dT}{dr}\bigg|_{r=R} = h\,(T_s - T_\infty) \Leftrightarrow \\ \Leftrightarrow -k\left(-\frac{\dot{q}R}{2k}\right) = h\,(T_s - T_\infty) \Leftrightarrow T_s = \frac{\dot{q}R}{2h} + T_\infty\end{aligned} \tag{23}$$

(b) Suddenly the electrical current is cut-off. On the same graph, plot the temperature profile at the instant the current is cut-off and at a later time before the steady-state has been restored. Justify the shape of the plotted profiles and compare, for these two time instants, the temperature and temperature gradients at $r = 0$ and $r = R$.

Solution:

The following equation (after applying the simplifying assumptions) governs the spatial and temporal temperature distribution after the current cut-off – see Equation (24). In this equation, α corresponds to the thermal diffusivity of the cable ($\alpha = k/(\rho c_p)$).

$$\frac{1}{r}\frac{\partial T}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial T}{\partial \phi}\left(k\cancel{\frac{\partial T}{\partial \phi}}\right)+\frac{\partial T}{\partial z}\left(k\cancel{\frac{\partial T}{\partial z}}\right)+\cancel{\dot{q}}=\rho c_p\frac{\partial T}{\partial t}\Rightarrow$$
$$\Rightarrow\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{24}$$

The governing equation is subjected to the boundary conditions given by Equations (25) and (26) and to the initial condition provided by Equation (27).

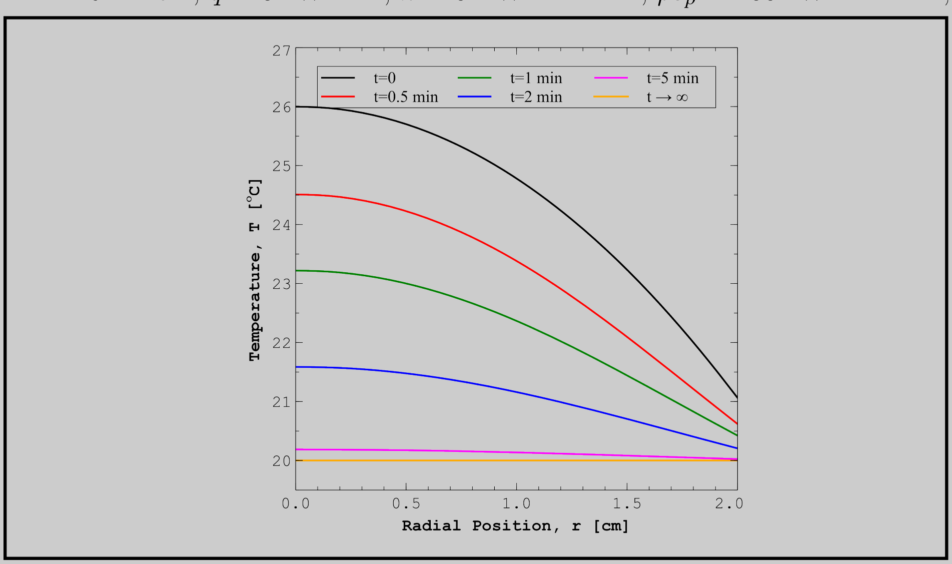
$$\left.\frac{dT}{dr}\right|_{r=0}=0 \tag{25}$$

$$-k\left.\frac{dT}{dr}\right|_{r=R}=h\left[T\left(r=R,t\right)-T_{\infty}\right] \tag{26}$$

$$T\left(r,t=0\right)=\frac{\dot{q}R^2}{4k}\left(1-\frac{r^2}{R^2}\right)+\frac{\dot{q}R}{2h}+T_{\infty} \tag{27}$$

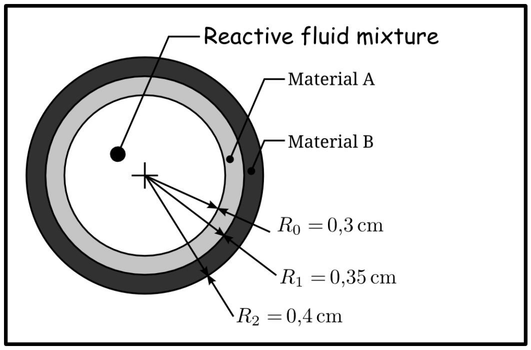
The stated problem was solved numerically and the temperature distribution profile for different time instants after the current cut-off is presented in the next figure. The actual properties and conditions considered are listed in the figure. As the time evolves the temperatures within the cable tend towards the fluid temperature. Notice that for all time instants the gradient dT/dr at $r=0$ (cable centerline) is zero in full accordance to the boundary condition defined in Equation (25). On the other hand, at the cable surface ($r=R$) the magnitude of the temperature gradient $|dT/dr|$ decreases as the elapsed time increases because the convective heat flux from the surface decreases as the surface temperature approaches the fluid temperature which is in full agreement with the boundary condition defined in Equation (27).

$R=2\,\mathrm{cm}$, $\dot{q}=5\,\mathrm{kW\,m^{-3}}$, $k=0.1\,\mathrm{W\,m^{-1}\,K^{-1}}$, $\rho c_p=100\,\mathrm{kW\,m^{-3}\,K^{-1}}$,



$h=50\,\mathrm{W\,m^{-2}\,K^{-1}}$, and $T_{\infty}=20\,^{\circ}\mathrm{C}$

19. (Homework) Consider a spherical reservoir where a mixture of fluids undergoes an exothermic reaction. As shown in the figure (below), the reservoir is formed by two layers where the thermal conductivity of layer A is $k_A = 19\,\mathrm{W\,m^{-1}\,K^{-1}}$ and that of material B is $k_B = 0.21\,\mathrm{W\,m^{-1}\,K^{-1}}$. The dimensions of the reservoir are $R_0 = 0.3\,\mathrm{m}$, $R_1 = 0.35\,\mathrm{m}$, and $R_2 = 0.4\,\mathrm{m}$. For the sake of the materials resistance, the temperatures within materials A and B should not be higher than 450°C and 400°C, respectively. The reactor lays in an environment at temperature $T_{\mathrm{amb}} = 35°\mathrm{C}$ where the convection coefficient at the outer surface of the reactor is $h_{\mathrm{ext}} = 8\,\mathrm{W\,m^{-2}\,K^{-1}}$. The convection coefficient at the inner surface of the reactor is $h_{\mathrm{int}} = 200\,\mathrm{W\,m^{-2}\,K^{-1}}$ and the mixture of reactants is homogeneous and is at a uniform temperature. Neglect the thermal contact resistance between materials A and B.

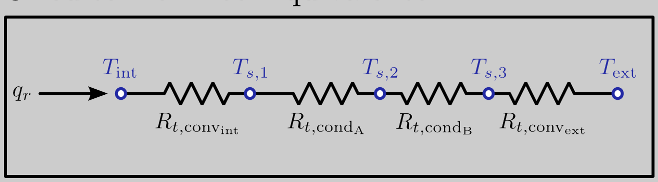


(a) Calculate the maximum power that can be released within the reactor.

Solution:

The equivalent thermal circuit for the problem under consideration is presented in the figure below. In this figure, the heat transfer direction is identified. Since the reaction is exothermic (there is conversion of chemical energy to thermal energy), the heat flow direction is observed from the reactive fluid mixture towards the external environment. Note that the heat transfer is one-dimensional along the radial direction of the spherical shell – this is an outcome of the following facts: the convection coefficients and fluid temperatures at the inner and outer reservoir surfaces are constant and independent of the angular (ϕ and θ) coordinates as well as the thermal conductivity which only varies in the radial direction.

Circuito Térmico Equivalente



The maximum temperature limits in the spherical layers A and B must be respected for the calculation of the maximum power ($q_{r,\max}$) that can be released from the reactive fluid mixture. Due to the heat flow direction, the maximum temperature in layer A is observed at $r = R_0$ ($T_{s,1}$), while the maximum temperature in layer B is registered at $r = R_1$ ($T_{s,2}$). The procedure herein considered to evaluate the maximum power is divided into two stages: (1) fix $T_{s,1} = T_{\max_A}$ in order to determine if $T_{s,2} \leq T_{\max_B}$; and (2) fix $T_{s,2} = T_{\max_B}$ in order to evaluate if $T_{s,1} \leq T_{\max_A}$. The stage assuring that the maximum temperatures in both layers are respected corresponds to the stage that defines the temperatures $T_{s,1}$ (or $T_{s,2}$)

for the computation of the maximum power.

Stage (1): Fixing $T_{s,1} = T_{\max_\mathrm{A}}$, verify if $T_{s,2} \leq T_{\max_\mathrm{B}}$.

$$\begin{aligned} q_r = \frac{(T_{\max_\mathrm{A}} - T_\mathrm{ext})}{R_{t,\mathrm{cond_A}} + R_{t,\mathrm{cond_B}} + R_{t,\mathrm{conv_{ext}}}} = \frac{T_{\max_\mathrm{A}} - T_{s,2}}{R_{t,\mathrm{cond_A}}} \Leftrightarrow \\ \Leftrightarrow T_{s,2} = T_{\max_\mathrm{A}} + \frac{R_{t,\mathrm{cond_A}}}{R_{t,\mathrm{cond_A}} + R_{t,\mathrm{cond_B}} + R_{t,\mathrm{conv_{ext}}}} (T_\mathrm{ext} - T_{\max_\mathrm{A}}) \Leftrightarrow \\ \Leftrightarrow T_{s,2} = T_{\max_\mathrm{A}} + \frac{\frac{(1/R_0)-(1/R_1)}{4\pi k_\mathrm{A}}}{\frac{(1/R_0)-(1/R_1)}{4\pi k_\mathrm{A}} + \frac{(1/R_1)-(1/R_2)}{4\pi k_\mathrm{B}} + \frac{1}{4\pi R_2^2 h_\mathrm{ext}}} (T_\mathrm{ext} - T_{\max_\mathrm{A}}) \Leftrightarrow \\ \Leftrightarrow T_{s,2} = 450 + \frac{\frac{(1/0.3)-(1/0.35)}{4\pi\times 19}}{\frac{(1/0.3)-(1/0.35)}{4\pi\times 19} + \frac{(1/0.35)-(1/0.4)}{4\pi\times 0.21} + \frac{1}{4\pi\times 0.4^2\times 8}} (35 - 450) \Leftrightarrow \\ \Leftrightarrow T_{s,2} \approx 445.85^\circ\mathrm{C} > 400^\circ\mathrm{C}\,(= \mathrm{T}_{\max_\mathrm{B}}) \end{aligned} \tag{28}$$

Considering $T_{s,1} = T_{\max_\mathrm{A}}$, the maximum temperature of Material B is not respected.

Stage (2): Fixing $T_{s,2} = T_{\max_\mathrm{B}}$, verify if $T_{s,1} \leq T_{\max_\mathrm{A}}$.

$$\begin{aligned} q_r = \frac{(T_{\max_\mathrm{B}} - T_\mathrm{ext})}{R_{t,\mathrm{cond_B}} + R_{t,\mathrm{conv_{ext}}}} = \frac{T_{s,1} - T_{\max_\mathrm{B}}}{R_{t,\mathrm{cond_A}}} \Leftrightarrow \\ \Leftrightarrow T_{s,1} = T_{\max_\mathrm{B}} + \frac{R_{t,\mathrm{cond_A}}}{R_{t,\mathrm{cond_B}} + R_{t,\mathrm{conv_{ext}}}} (T_{\max_\mathrm{B}} - T_\mathrm{ext}) \Leftrightarrow \\ \Leftrightarrow T_{s,1} = T_{\max_\mathrm{B}} + \frac{\frac{(1/R_0)-(1/R_1)}{4\pi k_\mathrm{A}}}{\frac{(1/R_1)-(1/R_2)}{4\pi k_\mathrm{B}} + \frac{1}{4\pi R_2^2 h_\mathrm{ext}}} (T_{\max_\mathrm{B}} - T_\mathrm{ext}) \Leftrightarrow \\ \Leftrightarrow T_{s,1} = 400 + \frac{\frac{(1/0.3)-(1/0.35)}{4\pi\times 19}}{\frac{(1/0.35)-(1/0.4)}{4\pi\times 0.21} + \frac{1}{4\pi\times 0.4^2\times 8}} (400 - 35) \Leftrightarrow \\ \Leftrightarrow T_{s,1} \approx 403.49^\circ\mathrm{C} < \mathrm{T}_{\max_\mathrm{A}} \end{aligned} \tag{29}$$

Fixing $T_{s,2} = T_{\max_\mathrm{B}}$, the maximum temperature limits of Material A and B are respected. Considering this condition, the maximum power that can be released from the reactor is calculated through the following equation – see Equation (28).

$$\begin{aligned} q_{r,\max} = \frac{(T_{\max_\mathrm{B}} - T_\mathrm{ext})}{R_{t,\mathrm{cond_B}} + R_{t,\mathrm{conv_{ext}}}} \Leftrightarrow q_{r,\max} = \frac{(T_{\max_\mathrm{B}} - T_\mathrm{ext})}{\frac{(1/R_1)-(1/R_2)}{4\pi k_\mathrm{B}} + \frac{1}{4\pi R_2^2 h_\mathrm{ext}}} \Leftrightarrow \\ \Leftrightarrow q_{r,\max} = \frac{(400 - 35)}{\frac{(1/0.35)-(1/0.4)}{4\pi\times 0.21} + \frac{1}{4\pi\times 0.4^2\times 8}} \Leftrightarrow \boxed{q_{r,\max} = 1848.048\,\mathrm{W}} \end{aligned} \tag{30}$$

(b) Under these circumstances, what is the temperature inside the reactor?

Solution:

Since the maximum power that can be released from the reactor ($q_{r,\max}$) was previously computed (see last question), this value can be considered in Equation (31) to calculate the temperature inside the reactor, T_{int}, considering the thermal resistance for convection by the inner reservoir surface side ($R_{t,\text{conv}_{\text{int}}}$) and the temperature $T_{s,1}$ ($\approx 403.49°\text{C}$) calculate under the conditions considered to compute $q_{r,\max}$ – see Stage 2 of last question.

$$\begin{aligned} q_r = \frac{(T_{\text{int}} - T_{s,1})}{R_{t,\text{conv}_{\text{int}}}} \Leftrightarrow T_{\text{int}} = q_r R_{t,\text{conv}_{\text{int}}} + T_{s,1} \Leftrightarrow \\ \Leftrightarrow q_r \frac{1}{4\pi R_0^2 h_{\text{int}}} + T_{s,1} \Leftrightarrow T_{\text{int}} = 1848.048 \times \frac{1}{4\pi \times 0.3^2 \times 200} + 403.49 \Leftrightarrow \\ \Leftrightarrow \boxed{T_{\text{int}} \approx 411.660°\text{C}} \end{aligned} \tag{31}$$

(c) If the rate of heat release were increased by 50%, what should be the new value of the outer radius, R_2, in order to ensure a proper operation of the system? Suppose that all parameters keep their values.

Solution:

In question (a), it was concluded that considering $T_{s,2} = T_{\max_{\text{B}}}$ the temperatures in both layers do not exceed the maximum recommended values ($T_{\max_{\text{A}}}$ and $T_{\max_{\text{B}}}$). Therefore, considering a different value for the heat transfer rate ($q_r = q_{r,\max}(1 + 0.5)$) and $T_{s,2} = T_{\max_{\text{B}}}$, an updated value for R_2 – that ensures a proper operation of the system – can be calculated through the following equation – see Equation (32).

$$\begin{aligned} q_r = \frac{(T_{s,2} - T_{\text{ext}})}{R_{t,\text{cond}_{\text{B}}} + R_{t,\text{conv}_{\text{ext}}}} \Leftrightarrow q_{r,\max}(1 + 0.5) = \frac{(T_{\max_{\text{B}}} - T_{\text{ext}})}{\frac{(1/R_1)-(1/R_2)}{4\pi k_{\text{B}}} + \frac{1}{4\pi R_2^2 h_{\text{ext}}}} \Leftrightarrow \\ \Leftrightarrow 1848.048 \times 1.5 = \frac{(400 - 35)}{\frac{(1/0.35)-(1/R_2)}{4\pi \times 0.21} + \frac{1}{4\pi R_2^2 \times 8}} \Rightarrow \boxed{R_2 \approx 0.37\text{m}} \end{aligned} \tag{32}$$

21. Calculate the steady-state temperature distribution in a compact sphere subjected to a uniform internal heat generation $\dot{q}$ [$\mathrm{W\,m^{-3}}$], with the surface behaving as a blackbody and exchanging heat exclusively by radiation with the environment at a temperature T_{sur}.

Solution:

The general form of the heat diffusion equation in spherical coordinates is given by Equation (33).

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \tag{33}$$

The appropriate form of the heat equation for the problem under consideration is obtained in Equation (34) after applying the adequate simplifying assumptions – negligible angular temperature gradients, steady-state conditions, and constant thermal conductivity.

$$\begin{gathered}\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)}+\cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)}+\dot{q}=\cancel{\rho c_p\frac{\partial T}{\partial t}}\Rightarrow\\ \Rightarrow\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0\end{gathered} \tag{34}$$

The general solution of Equation (34) is obtained by integrating twice Equation (34) – see Equations (35) and (36). (Note that the volumetric rate of thermal energy generation, $\dot{q}$, is uniform in the sphere volume, and consequently, independent of the sphere radius.)

$$\int d\left(r^2\frac{dT}{dr}\right)=-\frac{\dot{q}}{k}\int r^2dr\Leftrightarrow r^2\frac{dT}{dr}=-\frac{\dot{q}r^3}{3k}+C_1 \tag{35}$$

$$\int dT=-\frac{\dot{q}}{3k}\int rdr+C_1\int\frac{1}{r^2}dr\Leftrightarrow T\left(r\right)=-\frac{\dot{q}r^2}{6k}-\frac{C_1}{r}+C_2 \tag{36}$$

To obtain the particular solution of Equation (34), the constants of integration C_1 and C_2 must be calculated through the application of suitable boundary conditions. At the sphere centerpoint ($r=0$), a zero-Neumann boundary condition stating this radial location as a point of symmetry is considered – see Equation (37). At the external sphere surface ($r=r_s$), a prescribed temperature value is applied (first kind boundary condition) – see Equation (38). The prescribed temperature value at the external sphere surface is derived from the application of an overall energy balance to the sphere considering that the sphere is losing heat by radiation to the surrounding surfaces at temperature T_{sur}.

$$\left.\frac{dT}{dr}\right|_{r=0}=0 \tag{37}$$

$$
\begin{aligned}
\dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \dot{E}_g = \dot{E}_{\text{st}} \Leftrightarrow \cancel{\dot{E}_{\text{in}}} - \dot{E}_{\text{out}} + \dot{E}_{\text{g}} = \cancel{\dot{E}_{\text{st}}} \Rightarrow \dot{E}_{\text{g}} = \dot{E}_{\text{out}} \Leftrightarrow \\
\Leftrightarrow \dot{q}V = A\sigma\left(T_s^4 - T_{\text{sur}}^4\right) \Leftrightarrow \dot{q}\frac{4}{3}\pi r_s^3 = 4\pi r_s^2\sigma\left(T_s^4 - T_{\text{sur}}^4\right) \Leftrightarrow \\
\Leftrightarrow T_s = \left(\frac{\dot{q}r_s}{3\sigma} + T_{\text{sur}}^4\right)^{1/4}
\end{aligned}
\tag{38}
$$

The integration constant C_1 is obtained replacing Equation (35) in Equation (37) – see Equation (39).

$$
\left.\frac{dT}{dr}\right|_{r=0} = 0 \Leftrightarrow r^2\left.\frac{dT}{dr}\right|_{r=0} = 0 \Leftrightarrow -\frac{\dot{q} \times 0^3}{3k} + C_1 = 0 \Leftrightarrow C_1 = 0 \tag{39}
$$

The integration constant C_2 is obtained replacing Equation (36) in Equation (38) and considering $C_1 = 0$ – see Equation (40).

$$
\begin{aligned}
T\left(r = r_s\right) = \left(\frac{\dot{q}r_s}{3\sigma} + T_{\text{sur}}^4\right)^{1/4} \Leftrightarrow -\frac{\dot{q}r_s^2}{6k} - \frac{0}{r_s} + C_2 = \left(\frac{\dot{q}r_s}{3\sigma} + T_{\text{sur}}^4\right)^{1/4} \Leftrightarrow \\
\Leftrightarrow C_2 = \left(\frac{\dot{q}r_s}{3\sigma} + T_{\text{sur}}^4\right)^{1/4} + \frac{\dot{q}r_s^2}{6k}
\end{aligned}
\tag{40}
$$

Finally, the governing equation for the temperature distribution under the stated conditions is obtained by replacing the integration constants C_1 and C_2 (Equations (39) and (40)) in Equation (36) – see Equation (41).

$$
\begin{aligned}
T\left(r\right) = -\frac{\dot{q}r^2}{6k} - \frac{C_1}{r} + C_2 \Leftrightarrow T\left(r\right) = -\frac{\dot{q}r^2}{6k} - \frac{0}{r} + \left(\frac{\dot{q}r_s}{3\sigma} + T_{\text{sur}}^4\right)^{1/4} + \frac{\dot{q}r_s^2}{6k} \Leftrightarrow \\
\Leftrightarrow \boxed{T\left(r\right) = \frac{\dot{q}r_s^2}{6k}\left(1 - \frac{r^2}{r_s^2}\right) + \left(\frac{\dot{q}r_s}{3\sigma} + T_{\text{sur}}^4\right)^{1/4}}
\end{aligned}
\tag{41}
$$