Advanced Heat Transfer

Part IV: Numerical Heat Transfer Methods

4. Convective Heat Transfer

(Application of the Finite Volume Method)

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Convective Heat Transfer — Outline

1. Fluid Flow and Heat Transfer

Introduction – Motivation

- If the flow field (velocity components and density) is known (prescribed or calculated), convective heat transport (fluid temperature distribution and heat transfer rates) can be readily evaluated with the discretization practices introduced in Section 3 taking into account the corresponding formulation for the energy governing equation and auxiliary (boundary and initial) conditions.
- If the flow field is unknown, convective heat transport rates cannot be accurately predicted.
- The flow field solution is obtained by solving the continuity and momentum equations (Navier-Stokes equations). These coupled set of equations must be solved in combination with other transport equations if the corresponding transported properties influence the fluid properties – for instance, fluid properties (such as ρ and μ) may depend strongly on temperature or fluid composition (reactive flows). Otherwise, the flow field should be determined prior to the solution of other transported properties.

1. Fluid Flow and Heat Transfer

Governing Equations – Convection Transfer Equations

The governing equations for a two-dimensional, steady-state, and non-isothermal laminar flow of a Newtonian fluid are given as follows:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

$$\frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho v u) = \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{\partial p}{\partial x} + S_u$$

$$\frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v v) = \frac{\partial}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial v}{\partial y}\right) - \frac{\partial p}{\partial y} + S_v$$

$$\frac{\partial}{\partial x}(\rho u u_{\mathrm{t}}) + \frac{\partial}{\partial y}(\rho v u_{\mathrm{t}}) = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) - p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \mu\Phi + \dot{q}$$

1. Fluid Flow and Heat Transfer

Solution Procedure Details and Challenges (1/3)

- The solution for the velocity components (u, v, and w) is given by the ($x-$, $y-$, and $z-$direction) momentum equations. Momentum equations for each coordinate direction (as well as the continuity and energy equations) are particular cases of the general transport equation for ϕ – See Section 1.

- The convective terms of the momentum equations contain nonlinear quantities – notice, for instance, in the $x-$momentum equation the derivative of ρu^2 ($\partial(\rho uu)/\partial x$). A similar iterative approach as that applied to handle a temperature dependent thermal conductivity (see nonlinearities in diffusion problems – Section 2) can be herein applied – *i.e.*, in each iteration, the convective coefficient ρu (that depends on the dependent variable u) is held constant and equal to the value computed with the solution of the previous iteration.

1. Fluid Flow and Heat Transfer

Solution Procedure Details and Challenges (2/3)

- The solution of the discretized momentum equations – applying indirect (iterative) solution techniques for each discretized momentum equation (inner iterations) in a segregated fashion – requires an iterative procedure (outer iterations) at least because the momentum equations are coupled (notice, for instance, that the solution of the $x-$momentum equation, u, is required for the solution of the $y-$momentum equation.)
- The main difficulty in obtaining the velocity field solution lies in the unknown pressure field – spatial derivatives of the pressure field are required in the momentum equations. (If the pressure field is known, the solution is straightforward using the procedure previously introduced for convection-diffusion problems of a scalar property ϕ – note that the momentum equations are convection-diffusion equations only with an additional term that corresponds to the pressure gradient. The solution for the velocity components obtained with the correct pressure field in the momentum equations satisfies the continuity equation.)

1. Fluid Flow and Heat Transfer

Solution Procedure Details and Challenges (3/3)

- For incompressible flows, there is no obvious equation to evaluate the pressure – in such conditions, the continuity equation represents a kinematic constrain on the velocity field rather than a dynamic equation from which pressure can be calculated. The pressure-velocity coupling is not straightforward, however it is known that the correct pressure provided to compute the velocity field satisfies the continuity equation. (For compressible flows, the density can be evaluated with the continuity equation – density is the dominant variable in the mass conservation equation under such conditions – and with the addition of an energy equation to compute the temperature field, the pressure is obtained through an equation of state $p = p(\rho, T)$).
- The nonlinearities in momentum equations, the strong coupling between momentum equations, and the pressure-velocity coupling (required for consistent flow field solutions) can be handled with an iterative solution approach known as the SIMPLE algorithm.

1. Fluid Flow and Heat Transfer

Anticipated Issues (1/2)

If the grid (nodal positioning) and interpolation procedures considered so far were employed to obtain the solution of momentum equations, an irregular pressure distribution, such as checkerboard or zigzag (wavy) pressure distributions can be taken as a uniform pressure field since a negligible (erroneous) pressure gradient can be obtained. In such conditions, inconsistent (uncoupled) solutions for the velocity and pressure fields are obtained.

To demonstrate this serious issue, consider the pressure gradients required by the momentum equations which for a 2D uniform mesh and assuming a piecewise-linear profile for pressure (interpolation) are given as follows:

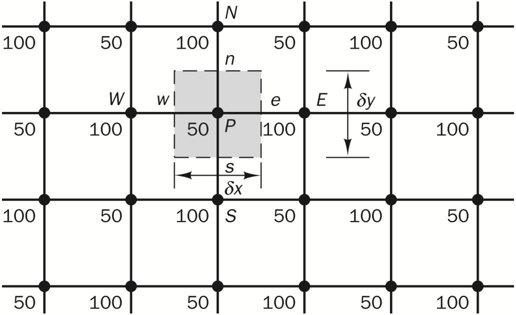
$$\frac{\partial p}{\partial x} = \frac{p_{\mathrm{e}} - p_{\mathrm{w}}}{\Delta x} = \frac{\frac{p_{\mathrm{E}}+p_{\mathrm{P}}}{2} - \frac{p_{\mathrm{P}}+p_{\mathrm{W}}}{2}}{\Delta x} \Leftrightarrow \frac{\partial p}{\partial x} = \frac{p_{\mathrm{E}} - p_{\mathrm{W}}}{2\Delta x}$$

$$\frac{\partial p}{\partial y} = \frac{p_{\mathrm{n}} - p_{\mathrm{s}}}{\Delta y} = \frac{\frac{p_{\mathrm{N}}+p_{\mathrm{P}}}{2} - \frac{p_{\mathrm{P}}+p_{\mathrm{S}}}{2}}{\Delta y} \Leftrightarrow \frac{\partial p}{\partial y} = \frac{p_{\mathrm{N}} - p_{\mathrm{S}}}{2\Delta y}$$

Note that the pressure gradients do not take into account the pressure at center-point P.

Anticipated Issues (2/2)

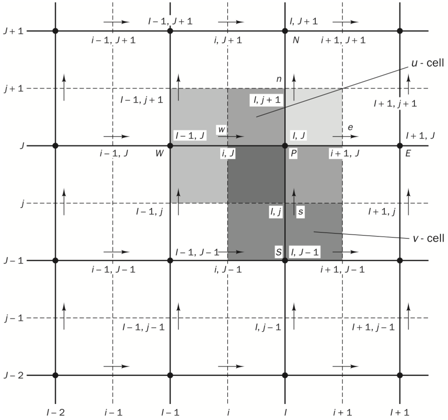
The application of the previous expressions for the pressure gradients to the following checkerboard pressure distribution (see figure below) leads to a zero pressure gradient, and consequently, to a negligible source contribution to the $x-$ and $y-$momentum equations. Under such conditions, the velocity field solution will not agree with the pressure distribution. This issue must be prevented when a flow field solution is sought.



1. Fluid Flow and Heat Transfer

The Staggered Grid Arrangement (1/3)

The staggered grid arrangement is one remedy available to avoid such issue from occurring. A 2D staggered grid illustrating generic control volumes (shaded regions) for the calculation of velocity components and scalars is shown below.



The Staggered Grid Arrangement (2/3)

In a staggered grid, the velocity components are not computed (stored) at the same locations where the pressure (and other scalar variables) are calculated. (Pressure (scalar or main) nodes correspond to the solid dots presented in the 2D staggered grid – see previous slide.) In a staggered grid, the $x-(y-)$velocity components are calculated at the center of the west and east (south and north) faces of the control volumes used to calculate the pressure – see the horizontal (vertical) arrows. Similarly, the pressure nodes are located at the center of faces w and e (s and n) for the u (v) control volumes.

With the staggered grid arrangement, the pressure gradients required by the momentum equations are computed as follows:

$$\frac{\partial p}{\partial x} = \frac{p_{\mathrm{P}} - p_{\mathrm{W}}}{\Delta x} \qquad\qquad \frac{\partial p}{\partial y} = \frac{p_{\mathrm{P}} - p_{\mathrm{S}}}{\Delta x}$$

With the current grid arrangement, pressure gradients are calculated with the values of two adjacent nodes (nodes P and W or P and W) rather than resorting to the values of two alternate nodes (nodes E and W or N and S).

1. Fluid Flow and Heat Transfer

The Staggered Grid Arrangement (3/3)

- The staggered grid arrangement prevents the development of negligible pressure differences, and consequently, unrealistic velocity solutions – solutions decoupled from the pressure field – in the presence of a checkerboard pressure distribution.
- Furthermore, the staggered grid arrangement allows to avoid interpolations to calculate velocities at scalar control volume faces – for instance, to evaluate convective mass fluxes required for the solution of transported scalar(s) governed by convection-diffusion equations – because the velocity values are readily available at such locations.

1. Fluid Flow and Heat Transfer

Discretized Momentum Equations (Staggered Grids)

The discretized form of the $x-$momentum equation for the control volume centered at the center of the west face of a pressure node P reads as follows:

$$a_{\mathrm{w}} u_{\mathrm{w}} = \sum a_{\mathrm{nb}} u_{\mathrm{nb}} + (p_{\mathrm{W}} - p_{\mathrm{P}}) A_{\mathrm{w}} + b$$

For a uniform 2D grid, $A_{\mathrm{w}} = \Delta y \cdot 1$ and $b = \overline{S_u} \Delta x \Delta y \cdot 1$. Considering the notation system presented in the 2D grid schematic representation (see Slide 10), the previous equations is written as follows:

$$a_{i,J} u_{i,J} = \sum a_{\mathrm{nb}} u_{\mathrm{nb}} + (p_{I-1,J} - p_{I,J}) A_{i,J} + b_{i,J}$$

where $\sum a_{\mathrm{nb}} u_{\mathrm{nb}} = a_{i-1,J} + a_{i+1,J} + a_{i,J-1} + a_{i,J+1}$. Similarly, for the discretized $y-$momentum equation, one obtains:

$$a_{I,j} v_{I,j} = \sum a_{\mathrm{nb}} v_{\mathrm{nb}} + (p_{I,J-1} - p_{I,J}) A_{I,j} + b_{I,j}$$

1. Fluid Flow and Heat Transfer

Discretized Momentum Equations (Staggered Grids)

The coefficients of the discretized momentum equations are calculated as presented before for convection-diffusion equations – applying for instance the upwind or hybrid discretization schemes. (Note that each momentum equation is a convection-diffusion equation.) The coefficients of the discretized momentum equations (a_{P} and a_{nb}) contain the values of the dependent variable at a previous iteration (or guessed values in the first iteration) due to nonlinearities and the intricately coupled nature of the set of equations.

2. SIMPLE Algorithm

Momentum Equations (Staggered Grids)

- When the pressure field is not known, the SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) algorithm can be applied for a consistent pressure-velocity coupling.
- The first step of this algorithm corresponds to the definition of an initial guess for pressure and velocity components (p^*, u^*, and v^*). Taking into account these values, the discretized momentum equations are solved yielding new estimates for u^* and v^*.

$$\boxed{a_{i,J} u^*_{i,J} = \sum a_{\mathrm{nb}} u^*_{\mathrm{nb}} + \left(p^*_{I-1,J} - p^*_{I,J}\right) A_{i,J} + b_{i,J}}$$

$$\boxed{a_{I,j} v^*_{I,j} = \sum a_{\mathrm{nb}} v^*_{\mathrm{nb}} + \left(p^*_{I,J-1} - p^*_{I,J}\right) A_{I,j} + b_{I,j}}$$

- Obtained (guessed) velocity values will be corrected (at each iteration) taking into account the procedure described in the following slides.

2. SIMPLE Algorithm

Pressure and Velocity Corrections (Staggered Grids)

A pressure correction (p') and velocity corrections (u' and v') are defined as the difference between the correct values (p, u, and v) and the values previously guessed or obtained from the previous iteration (p^*, u^*, and v^*):

$$p = p^* + p' \qquad u = u^* + u' \qquad v = v^* + v'$$

The following equations are obtained subtracting the discretized equation of $u^*_{i,J}$ ($v^*_{i,J}$) from the discretized equation of $u_{i,J}$ ($v_{i,J}$)

$$a_{i,J} u'_{i,J} = \cancel{\sum a_{\mathrm{nb}} u'_{\mathrm{nb}}} + \left(p'_{I-1,J} - p'_{I,J}\right) A_{i,J} \qquad a_{I,j} v'_{I,j} = \cancel{\sum a_{\mathrm{nb}} v'_{\mathrm{nb}}} + \left(p'_{I,J-1} - p'_{I,J}\right) A_{I,j}$$

Neglecting the summation term (SIMPLE approximation), one obtains:

$$u'_{i,J} = \underbrace{\frac{A_{i,J}}{a_{i,J}}}_{d_{i,J}} \left(p'_{I-1,J} - p'_{I,J}\right) \qquad v'_{I,j} = \underbrace{\frac{A_{I,j}}{a_{I,j}}}_{d_{I,j}} \left(p'_{I,J-1} - p'_{I,J}\right)$$

The SIMPLE approximation implies that the velocity corrections are calculated without taking into account the neighboring velocity corrections.

2. SIMPLE Algorithm

Velocity Corrections (Staggered Grids)

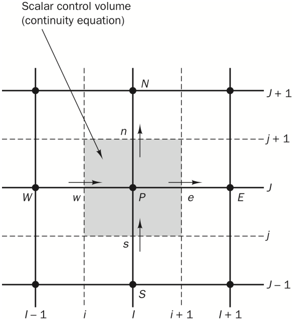
Velocity components are corrected from the corresponding guessed values (solution of the momentum equations) in each iteration considering:

$$\boxed{u_{i,J} = u^*_{i,J} + d_{i,J}\left(p'_{I-1,J} - p'_{I,J}\right)} \qquad \boxed{v_{I,j} = v^*_{I,j} + d_{I,j}\left(p'_{I,J-1} - p'_{I,J}\right)}$$

Pressure-Correction Equation (Staggered Grids) (1/3)

Since the velocity field must satisfy the continuity equation, the corrected velocity values ($u_{i,J}$, $u_{i+1,J}$, $v_{I,j}$, and $v_{I,j+1}$) must satisfy the discretized continuity equation for a general pressure node (scalar node) P located at (I, J) as follows:

$$\left[(\rho u A)_{i+1,J} - (\rho u A)_{i,J}\right] + \\ + \left[(\rho v A)_{I,j+1} - (\rho v A)_{I,j}\right] = 0$$



2. SIMPLE Algorithm

Pressure-Correction Equation (Staggered Grids) (2/3)

Substituting in the last equation (discretized continuity equation for a general scalar control volume) the expressions for the corrected velocities, the following discretized equation for pressure correction, p', (pressure-correction equation) is obtained

$$\boxed{a_{I,J}p'_{I,J} = a_{I-1,J}p'_{I-1,J} + a_{I+1,J}p'_{I+1,J} + a_{I,J-1}p'_{I,J-1} + a_{I,J+1}p'_{I,J+1} + b}$$

where,

$$a_{I-1,J} = (\rho dA)_{i,J} \qquad a_{I,J-1} = (\rho dA)_{I,j}$$

$$a_{I+1,J} = (\rho dA)_{i+1,J} \qquad a_{I,J+1} = (\rho dA)_{I,j+1}$$

$$a_{I,J} = a_{I-1,J} + a_{I+1,J} + a_{I,J-1} + a_{I,J+1}$$

$$b = -\left\{ \left[(\rho u^* A)_{i+1,J} - (\rho u^* A)_{i,J}\right] + \left[(\rho v^* A)_{I,j+1} - (\rho v^* A)_{I,j}\right] \right\}$$

2. SIMPLE Algorithm

Pressure-Correction Equation (Staggered Grids) (3/3)

- The source term of the pressure-correction equation (b) will eventually become negligible (during the iterative process) when the values u^* and v^* do satisfy the continuity equation – notice that the LHS of the discretized continuity equation (last equation of Slide 17) is equal to $-b$.
- If the problem is time-dependent and the fully implicit scheme were considered the term b includes also the term $(\rho^0_{I,J} - \rho_{I,J})V_{I,J}/\Delta t$, where $V_{I,J}$ for a uniform 2D mesh is equal to $\Delta x \Delta y \cdot 1$.
- Since the values of ρ are required at the faces of scalar control volumes – and they are calculated (and stored) at the nodes of scalar control volumes – a consistent interpolation procedure should be applied. (Otherwise, the conservativeness property is not respected).
- After the solution for the pressure-correction equation, the corrected pressure is obtained as follows:

$$\boxed{p_{I,J} = p^*_{I,J} + p'_{I,J}}$$

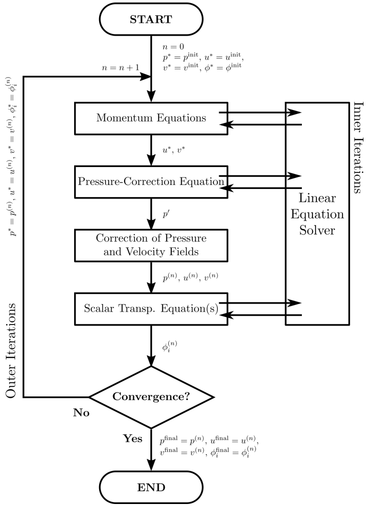
2. SIMPLE Algorithm

SIMPLE Algorithm – Full Description of Steps

1. Provide initial estimates for p^*, u^*, v^*, and ϕ_i^*. (ϕ_i corresponds to a property – *viz.* temperature, concentration, or turbulence quantities – with a known influence on the flow field through fluid properties, source terms, *etc.*);
2. Solve the discretized momentum equations (boxed equations in Slide 15) to obtain u^* and v^*;
3. Solve the pressure-correction equation (boxed equation in Slide 18) to obtain the pressure correction, p';
4. Correct pressure and velocity fields – *i.e.*, obtain new values for p, u, and v from the old/guessed values (p^*, u^*, and v^*) and the solution for p' (boxed equations of Slides 17 and 19.)
5. Solve the discretized equation for ϕ_i (if required);
6. Check convergence criteria:
    - if convergence is not achieved, assign the latest values of p, u, and v (obtained in Step 4), and ϕ_i (Step 5) to p^*, u^*, v^*, and ϕ_i^*, respectively, and return to Step 2;
    - if convergence is achieved the iterative procedure is concluded.

2. SIMPLE Algorithm

SIMPLE Algorithm – Flowchart



2. SIMPLE Algorithm

SIMPLE Algorithm – Recommendations for Fast and Robust Calculations

- If a particular property, ϕ, (whose solution is sought) does not influence the flow field solution, it is convenient to obtain the corresponding solution only after a converged flow field solution is achieved – *i.e.*, after the application of SIMPLE algorithm.
- The application of under-relaxation techniques to evaluate the final values of p, u, and v in each iteration is strongly acknowledge to improve the stability of the iterative procedure – avoid solution divergence. For a generic iteration n, the under-relaxation procedure is given as follows

$$p^{(n)} = p^{*} + \alpha_p p'$$

$$u^{(n)} = \alpha_u u + (1 - \alpha_u)\, u^{(n-1)} \qquad v^{(n)} = \alpha_v v + (1 - \alpha_v)\, v^{(n-1)}$$

where, α_p, α_u, and α_v correspond to the under-relaxation factors for the pressure, p, and $x-$ and $y-$corrected velocity values – u and v, respectively. $u^{(n-1)}$ and $v^{(n-1)}$ are the $x-$ and $y-$velocity values obtained at the previous iteration.

2. SIMPLE Algorithm

SIMPLE Algorithm – Recommendations and Improved Algorithms

- The (intermediate) solution of discretized (momentum, continuity, and scalar(s)) equations (inner iterations) does not need to be highly accurate – this would represent a waste of computational resources – since such solutions are obtained from approximate guesses. Only slight solution improvements are required. For instance, if the pressure field is far from the correct one (incorrect pressure field), solving the momentum equations with such pressure field through a large number of iterations (very accurate velocity field solution) is worthless because the obtained velocity values were derived from an incorrect pressure distribution.
- The presented flow field solution procedure can be easily extended to transient and 3D problems.
- Revised versions (refinements) and variants of the SIMPLE algorithm were proposed (such as, the SIMPLER, SIMPLEC, and PISO algorithms) to improve computational savings (lowering computational costs) and enhance iterative stability (convergence robustness).

2. SIMPLE Algorithm

The Collocated Grid Arrangement (1/2)

- An alternative strategy to the staggered grid arrangement – that was the initial remedy applied to avoid pressure oscillations and inconsistent (uncoupled) velocity-pressure fields – corresponds to the collocated (non-staggered) grids.
- Collocated grids can be applied for complex geometries (and unstructured meshes). Collocated grids do not offer significant advantages for Cartesian geometries (over the staggered arrangement) besides lower computer storage (memory) requirements.
- In collocated grids, there is only one group of nodes – pressure, scalars, and velocity components are all calculated with the same set of nodes.
- Pressure values at cell faces for the momentum equation (p_{e}, p_{w}, p_{s}, and p_{n}) are evaluated by (linear) interpolation.
- For consistent flow field solutions (and to prevent unphysical checkerboarding of pressure), the application of collocated grids demands the calculation of cell face velocities as a function of the pressure difference between adjacent nodes (nodes P and W or nodes P and S) through specific interpolation procedures, such as the Rhie-Chow interpolation.

2. SIMPLE Algorithm

The Collocated Grid Arrangement (2/2)

The $x-$direction velocity at the cell face w is approximated through a custom interpolation procedure and taking into account the pressure difference between adjacent nodes as shown below (overlined terms are obtained by interpolation).

$$u_\mathrm{w} = \overline{\left(\frac{\sum a^u_\mathrm{nb} u_\mathrm{nb} + b^u}{a^u_\mathrm{P}}\right)_\mathrm{w}} - (p_\mathrm{P} - p_\mathrm{W}) \underbrace{\overline{\left(\frac{A_\mathrm{w}}{a^u_\mathrm{P}}\right)_\mathrm{w}}}_{d_\mathrm{w}}$$

From the solution of the $x-$momentum equation, the new estimates (before correction) for u at the cell face w is approximated as follows:

$$u^*_\mathrm{w} = \overline{\left(\frac{\sum a^u_\mathrm{nb} u^*_\mathrm{nb} + b^u}{a^u_\mathrm{P}}\right)_\mathrm{w}} - (p^*_\mathrm{P} - p^*_\mathrm{W})\, d_\mathrm{w}$$

Subtracting the two equations and applying the SIMPLE approximation yields,

$$u'_\mathrm{w} = \cancel{\overline{\left(\frac{\sum a^u_\mathrm{nb} u'_\mathrm{nb} + b^u}{a^u_\mathrm{P}}\right)_\mathrm{w}}} - (p'_\mathrm{P} - p'_\mathrm{W})\, d_\mathrm{w}$$

2. SIMPLE Algorithm

Velocity Corrections (Collocated Grid Arrangement)

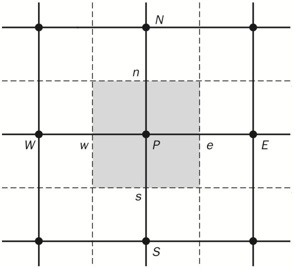
Velocity components are corrected from the corresponding guessed values (solution of the momentum equations) in each iteration considering:

$$\boxed{u_\mathrm{w} = u_\mathrm{w}^* + d_w\left(p'_W - p'_P\right)} \qquad \boxed{v_\mathrm{s} = v_\mathrm{s}^* + d_s\left(p'_S - p'_P\right)}$$

Pressure-Correction Equation (Collocated Grids) (1/2)

Since the velocity field must satisfy the continuity equation, the corrected velocity values (u_w, u_e, v_s, and v_n) must satisfy the discretized continuity equation for a general node P :

$$\left[(\rho u A)_e - (\rho u A)_w\right] + \\ + \left[(\rho v A)_n - (\rho v A)_s\right] = 0$$



2. SIMPLE Algorithm

Pressure-Correction Equation (Collocated Grids) (2/2)

Substituting in the last equation the expressions for the corrected velocities, the following discretized equation for pressure correction, p', (pressure-correction equation) is obtained

$$\boxed{a_{\mathrm{P}} p'_{\mathrm{P}} = a_{\mathrm{W}} p'_{\mathrm{W}} + a_{\mathrm{E}} p'_{\mathrm{E}} + a_{\mathrm{S}} p'_{\mathrm{S}} + a_{\mathrm{N}} p'_{\mathrm{N}} + b}$$

where,

$$a_{\mathrm{W}} = (\rho d A)_{\mathrm{w}} \qquad a_{\mathrm{S}} = (\rho d A)_{\mathrm{s}}$$

$$a_{\mathrm{E}} = (\rho d A)_{\mathrm{e}} \qquad a_{\mathrm{N}} = (\rho d A)_{\mathrm{n}}$$

$$a_{\mathrm{P}} = a_{\mathrm{W}} + a_{\mathrm{E}} + a_{\mathrm{S}} + a_{\mathrm{N}}$$

$$b = -\left\{ \left[(\rho u^* A)_{\mathrm{e}} - (\rho u^* A)_{\mathrm{w}}\right] + \left[(\rho v^* A)_{\mathrm{n}} - (\rho v^* A)_{\mathrm{s}}\right] \right\}$$

Remarks provided in Slide 20 *et seq.* are still applicable for collocated grids.

3. Boundary Layer Flows

Introduction

- A moving viscous fluid with a finite thermal conductivity in close contact with a stationary surface at a different temperature will give rise to the development of hydrodynamic (momentum) and thermal boundary layers – fluid regions along which non-negligible velocity gradients (shear stresses) and thermal gradients (heat transfer) are registered.
- The boundary layer velocity and temperature distributions are governed by the boundary layer equations – equations derived from the set of equations presented in Slide 4 after the applying the boundary layer approximations:
  - velocity component in the main flow direction much larger than the other velocity components;
  - negligible diffusion of momentum and heat along the main flow direction in comparison to the corresponding diffusive transport rates along other directions; and
  - pressure gradient across the flow much smaller than the pressure gradient along the flow main direction.

3. Boundary Layer Flows

Governing Equations

The governing equations for two-dimensional, steady-state, laminar, and non-isothermal boundary layer flows considering negligible body forces and thermal energy generation read as follows:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

$$\frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho v u) = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) - \frac{dp}{dx}$$

$$\frac{\partial}{\partial x}(\rho u u_{\mathrm{t}}) + \frac{\partial}{\partial y}(\rho v u_{\mathrm{t}}) = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu \left(\frac{\partial u}{\partial y}\right)^2$$

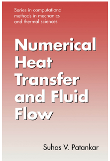
Pressure as a function of the main flow direction is provided considering free stream (inviscid) flow conditions.

3. Boundary Layer Flows

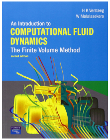
Discretization Practices

- Since dp/dx is obtained taking into account external considerations (dp/dx is assumed as a known quantity – source term for the $x-$momentum equation), the solution for the boundary layer equations can be obtained through the discretization procedures introduced for convection-diffusion problems – there is no need to apply a pressure-velocity coupling algorithm such as the SIMPLE algorithm.
- Due to the parabolic nature of the boundary layer governing equations, the solution at a particular x location only depends on the solution upstream that location – the $x-$coordinate in boundary layer equations exhibits a similar behavior to the time coordinate (in transient problems the solution at a particular time instant is only influenced by past events).
- Different spatial (along $x-$direction) interpolation methods can be considered for u and T giving rise to different discretization schemes for $x-$direction derivatives – explicit and implicit schemes.

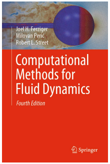
Further Reading



- Chapter 6: Calculation of the Flow Field



- Chapter 6: Solution Algorithms for Pressures-Velocity Coupling in Steady Flows



- Chapter 7: Sol. of Navier-Stokes Eqs.: Part I
- Chapter 8: Sol. of Navier-Stokes Eqs.: Part II