On sublimits, limit superior, and limit inferior

Contents

1 The closed number line

In this document, $\mathbb{R}$ stands for the real line $]-\infty, \infty[$ and $\overline{\mathbb{R}}$ stands for the closed real line $[-\infty, \infty]$.

We remind ourselves of the axiom of the least upper bound property in $\mathbb{R}$:

Least Upper Bound Property: Any nonempty set $X \subseteq \mathbb{R}$ with a real upper bound has a least upper bound, denoted $\sup X$.

This property can easily be extended to the closed real line in a rather nicer way:

Least Upper Bound Property (in $\overline{\mathbb{R}}$): Any set $X \subseteq \overline{\mathbb{R}}$ has a least upper bound, denoted $\sup X$.

We now conclude the latter principle from the former:

Proof. Let $X \subseteq \overline{\mathbb{R}}$.

If $\infty \in X$, the least (and in fact, the only) upper bound is precisely ∞.

If $\infty \notin X$, let $Y = X \cap \mathbb{R}$. If Y is empty, then X is either empty or $\{-\infty\}$; in both cases any element of $\overline{\mathbb{R}}$ is an upper bound, and the smallest of these is precisely $-\infty$.

If Y is not empty, it either has a real upper bound or it does not. If not, since ∞ is an upper bound and there are no real upper bounds, ∞ is the least upper bound.

If, on the other hand, it does have a real upper bound, we are in the conditions of the least upper bound property for $\mathbb{R}$, and thus, Y has a least (real) upper bound M in $\mathbb{R}$. Notice this is still least in the context of $\overline{\mathbb{R}}$, since $-\infty$ is not an upper bound (because Y is nonempty) and, while ∞ is an upper bound, it is larger than M, hence we conclude M is still, in this context, the least upper bound of Y, and ergo of X.

Thus, we showed that, in every case, there exists a least upper bound for X. $\square$

Notice that, as shown in the proof, if we are in the conditions of the real LUBP, then the suprema obtained in $\mathbb{R}$ and in $\overline{\mathbb{R}}$ are the same, and thus there is no ambiguity in saying $\sup X$.

We use $\inf X$ to denote the greatest lower bound, which, as one can easily show, always exists, equaling $-\sup(-X)$, where $-X$ denotes $\{\, -x \mid x \in X \,\}$.

2 Limits

2.1 Neighbourhoods

We remind ourselves of the notion of *neighbourhood of a real number*:

Given a real number a and a positive real ε, we define

$$V_\varepsilon(a) := \{\, x \mid |x - a| < \varepsilon \,\}$$

This definition does not make sense if we try to put $a = \pm\infty$, so in an effort to define neighbourhoods in the real closed number line we must define them for the infinities separately. As such, we define:

$$V_\varepsilon(\infty) :=]1/\varepsilon, \infty]$$
$$V_\varepsilon(-\infty) := [-\infty, -1/\varepsilon[$$

The reason behind using $1/\varepsilon$ rather than ε is simply so that we can assert that if we decrease the radius of the neighbourhood we also decrease the set, a property that is sometimes handy, albeit never strictly necessary.

2.2 Convergence

Given a sequence a_n of numbers in $\overline{\mathbb{R}}$, we say it *converges towards* $L \in \overline{\mathbb{R}}$, denoted $a_n \to L$ if:

$$\forall_\varepsilon \exists_N \forall_{n \geq N} a_n \in V_\varepsilon(L)$$

Where it is implied that $\varepsilon \in \mathbb{R}^+$ and $N \in \mathbb{N}$.

Prop 1. *If a_n converges towards L and L' then $L = L'$. That is: the limit, when it exists, is unique.*

Proof. To do this, it is enough to show that if $L \neq L'$ then there exist $\varepsilon, \varepsilon'$ such that $V_\varepsilon(L)$ and $V_{\varepsilon'}(L')$ are disjoint. (We say that $\overline{\mathbb{R}}$ is a *Hausdorff space*)

This is enough because we simply notice that, for such ε, by definition there exists N such that for all $n \geq N$ we have that $a_n \in V_\varepsilon(L)$, and also there exists N' such that for $n \geq N'$ we have $a_n \in V_{\varepsilon'}(L')$. Taking p as the maximum of N and N', we have a_p is in both $V_\varepsilon(L)$ and $V_{\varepsilon'}(L')$; an impossibility if these two are disjoint.

So all we need to show now is that $\overline{\mathbb{R}}$ is Hausdorff. We do this on a case-by-case basis.

If L and L' are both finite (i.e. in $\mathbb{R}$) let $\varepsilon = \varepsilon' = \frac{|L-L'|}{2} > 0$. It is a simple application of the triangular inequality to show the respective neighbourhoods are disjoint.

If L is finite and L' is ∞, pick a positive number s greater than L. Put $\varepsilon = s - L$ and $\varepsilon' = 1/s$. The case where $L' = -\infty$ is completely analogous, and if L is infinite and L' is finite one simply swaps the letters to return to this case.

Finally, if $L = -L' = \pm\infty$, $\varepsilon = \varepsilon' = 1$ works. □

We can then write, without ambiguity, $\lim a_n$ to denote the limit of the sequence a_n when it exists.

2.3 Operations

One of the main disadvantages of working with $\overline{\mathbb{R}}$ instead of $\mathbb{R}$ is that the former is not a field: while we are used to adding, subtracting, and multiplying elements of $\mathbb{R}$ in very nice ways, infinity does not take kindly to being messed with in this manner. Indeed, there is no way to extend the operations $+$ and $\times$ on $\mathbb{R}$ to $\overline{\mathbb{R}}$ while staying in an ordered field.

An important thing to notice is that the symbol $-\infty$ should not be taken in the same sense something like $-x$ should be: the latter stands for the additive inverse of a real number; the number y such that $x + y = 0$. The former, on the other hand, is simply a symbol distinct from ∞.

However, this distinction can be easily blurred once we try to define multiplication on $\overline{\mathbb{R}}$.

To make life easier on ourselves, we will temporarily adopt the following formalism: instead of working with $\pm\infty$, we will work with ordered pairs $(\pm 1, \infty)$, where $(1, \infty)$ should be understood as ∞ and $(-1, \infty)$ as $-\infty$.

For elements different from zero, we define the *sign function*, sgn, as

$$\operatorname{sgn} x := 1 \text{ if } x > 0, \ -1 \text{ if } x < 0$$

We define multiplication of numbers in $\overline{\mathbb{R}}$ as follows: let $x, y \in \overline{\mathbb{R}}$.

If both are real, their product is defined as their product as real numbers.

We define, *in this context*, $0 \times \infty$ to be 0.

And finally, if one of them is an infinity and the other is different from zero, we define xy to be $(\operatorname{sgn} x \operatorname{sgn} y, \infty)$.

One can easily check the following properties of this new multiplication on $\overline{\mathbb{R}}$:

Prop 2. *The product as we just defined it on $\overline{\mathbb{R}}$ is such that, for all $x, y, z \in \overline{\mathbb{R}}$:*

i) $xy = yx$

ii) $x(yz) = (xy)z$

iii) If $x \geq y$ *and* $z \geq 0$ *then* $xz \geq yz$

iv) However, it is possible that $x > y$ *and* $z > 0$*, yet* $xz \not> yz$

v) If $x \geq y$ *and* $z \leq 0$ *then* $xz \leq yz$

Proof. This is left as an exercise. □

Under this notion of multiplication, one can settle the situation with minuses: if we take $-x$ to be an abbreviation for $(-1)x$, then the symbol $-\infty$ is the same as $-x$ for $x = \infty$.

One can also check it (mostly) plays nice with limits.

Prop 3. *If* $a_n \to L$ *and* $c \in \mathbb{R}$, $ca_n \to cL$

Proof. If $c = 0$ this is trivial, so we assume $c \neq 0$.

Pick some ε. We wish to find N such that for $n \geq N$ we have $ca_n \in V_\varepsilon(cL)$.

If $c > 0$, as the reader may easily check, $ca_n \in V_\varepsilon(cL)$ iff $a_n \in V_{\varepsilon/c}(L)$, which happens for all large enough n.

If, on the other hand, $c < 0$, we have $ca_n \in V_\varepsilon(cL)$ iff $a_n \in V_{-\varepsilon/c}(L)$, which also happens for all large enough n. □

Notice that we excluded the case $c = \pm\infty$. It is instructive to find an example of a sequence a_n converging to some number L, and some c such that ca_n does not converge to cL.

A particular case which will be useful in the sequence:

Prop 4. *If* $a_n \to L$ *then* $-a_n \to -L$

2.4 Sublimits

Unfortunately, not every sequence converges (shocker). However, as we will soon show, given any sequence, we can find a subsequence of it that does converge.

First, an easy consequence of the supremum axiom.

Prop 5. *Any monotone sequence converges in $\overline{\mathbb{R}}$.*

In particular, if a_n is increasing it converges to $\sup a_n$ *and if it is decreasing it converges to* $\inf a_n$.

Proof. We will do this proof only for an increasing sequence a_n; if a_n is decreasing, simply consider the sequence $-a_n$, which shall converge to some limit L, implying a_n will converge to $-L$.

Let $L = \sup a_n$. If $L = -\infty$ then all a_n must be $-\infty$, so in this case the proposition is trivially true.

Suppose, then, $L \neq -\infty$. Fix any $\varepsilon > 0$. Let s be an element of $V_\varepsilon(L)$ that is lesser than L. By definition of sup, there exists some $a_N > s$. But then, since the sequence is increasing, and L is greater than or equal to all a_n, we have that, for $n \geq N$, $a_n \in]s, L] \subseteq V_\varepsilon(L)$, as we wished to show. □

Because of this, to show that any sequence has a converging subsequence it is enough to show it has a monotone one.

Prop 6. *Any sequence has a monotone subsequence.*

Proof. Let a_n be a sequence, and let S be the set of all n such that a_n is greater than or equal to all elements after it. In symbols:

$$S = \{\, n \mid a_n \geq a_m \text{ for all } m > n \,\}$$

Either S is finite or infinite. If it is infinite, let $i_0 < i_1 < i_2 < \cdots$ be an infinitude of elements of S. Then, the sequence a_{i_n} is decreasing, by definition of 'element of S'.

If, on the other hand, S is finite, pick i_0 greater than all elements of S. Since i_0 is not in S, there exists $i_1 > i_0$ such that $a_{i_0} < a_{i_1}$. Likewise, there exists $i_2 > i_1$ such that $a_{i_1} < a_{i_2}$, and so on and so forth. Collecting an infinitude of such i_n, the sequence a_{i_n} is increasing. □

This allows us to conclude

Prop 7. *Any sequence has a converging subsequence.*

3 Sublimits

We proceed to investigate the concept of sublimit in more detail.

Given a sequence a_n, denote the set of its sublimits by S_a. We know by the previous proposition that $S_a \neq \varnothing$.

A more or less natural question is the following: does this set have a maximum? Is there a single greatest sublimit?

The answer, somewhat surprisingly, is yes. But to answer this question, we will delve a little bit into topology.

3.1 Topology

We will say a set $X \subseteq \overline{\mathbb{R}}$ is *open* if for all $x \in X$ there exists an ε-neighbourhood of x contained in X. In symbols:

$$\forall_{x \in X} \exists_\varepsilon V_\varepsilon(x) \subseteq X$$

We say a set X is *closed* if its complement, X^c, is open.

The reader may easily check that, for all $\varepsilon \in \mathbb{R}^+$, $x \in \overline{\mathbb{R}}$, we have $V_\varepsilon(x)$ is open.

Closed sets have the nice property of always (if not empty) having extrema, as we proceed to show:

Prop 8. *If $X \neq \varnothing$ is closed, it has a maximum and a minimum.*

Proof. We show only the existence of a maximum; the existence of a minimum is done much similarly, with inf instead of sup and a few signal swaps.

Let $s = \sup X$. We wish to show s is an element of X. If we do, we will have shown s is an element of X greater than all others, i.e. the maximum of X.

If $s = -\infty$, then $s \in X$, as X contains at least one element, which must be $\leq -\infty$, and thus must be $-\infty$ itself, showing $s \in X$.

Suppose, then, $s > -\infty$. Since X is closed, if s were not in X, there would be an ε-neighbourhood of s disjoint from X. But then, we could pick an element of said neighbourhood lesser than s, which would also be an upper bound of X. This contradicts the definition of s as the least upper bound of X, which means s must be in X, as we wished to show. □

As such, to show there is a 'biggest sublimit' it is enough to show the set of sublimits is closed.

Prop 9. *Given a sequence a_n, S_a is closed.*

Proof. To do this, we will fix a point $L \in \overline{\mathbb{R}}$ and suppose that there exists no $\varepsilon > 0$ such that $V_\varepsilon(L)$ is disjoint from S_a. In other words, for all ε we have $V_\varepsilon(L)$ is not disjoint from S_a. We will conclude $L \in S_a$.

Ergo, by counterreciprocal, for any $L \notin S_a$ there exists ε such that $V_\varepsilon(L)$ is disjoint from S_a; i.e. S_a is closed.

So, fix $L \in \overline{\mathbb{R}}$ in these conditions. We will construct explicitly a subsequence of a_n that converges to L.

First, fix any $i_1 \in \mathbb{N}$. Then, fixed i_{n-1}, construct i_n as follows:

Consider the $\frac{1}{n}$-neighbourhood of L. By hypothesis, there exists an element of S_a in this neighbourhood. Let ℓ be such element, and let a_{j_n} be a subsequence of a_n that converges to ℓ.

Since $V_{\frac{1}{n}}(L)$ is open, there exists ε such that $V_\varepsilon(\ell) \subseteq V_{\frac{1}{n}}(L)$. Since $a_{j_n} \to \ell$, we have that, for all big enough N, $a_{j_N} \in V_\varepsilon(\ell) \subseteq V_{\frac{1}{n}}(L)$. Define i_n to be j_N, for some N large enough that $j_N > i_{n-1}$.

In this manner, we have constructed a sequence, a_{i_n}, that converges to L, as, fixed any $\varepsilon > 0$, we can pick $N \in \mathbb{N}$ such that $1/N < \varepsilon$ and therefore we will have that, for $n \geq N$, $a_{i_n} \in V_{\frac{1}{n}}(L) \subseteq V_{\frac{1}{N}}(L) \subseteq V_\varepsilon(L)$. Therefore, $L \in S_a$. □

Prop 10. *Fixed a sequence a_n, there exists a largest and a least sublimit.*

We will denote the largest sublimit of a_n by $\overline{\lim}\, a_n$, and the least sublimit by $\underline{\lim}\, a_n$.

3.2 Limit superior and limit inferior

Suppose we are interested in finding out the largest and the least sublimit of a sequence a_n. One possible way to try and guess at it is as follows:

Suppose all elements of a_n are less than or equal to b_n, where b_n is some converging sequence. As such, if a_{i_n} is a converging subsequence of a_n, we have $a_{i_n} \leq b_{i_n}$, and as the reader may easily check:

Prop 11. *Any subsequence of a converging sequence converges to the same limit.*

Prop 12. *Suppose a_n and b_n converge to A and B respectively. If $a_n \leq b_n$ for all n, then $A \leq B$.*

And so, we conclude $\lim a_{i_n} \leq \lim b_n$. In particular, $\overline{\lim}\, a_n \leq \lim b_n$.

As such, if we have some b_n in these conditions, and find a subsequence of a_n that converges towards $\lim b_n$, we must necessarily have this number be the largest sublimit of a_n.

That raises an idea. For a basic estimate of $\overline{\lim}\, a_n$, we could try finding some sequence b_n, greater than a_n, that does converge. To assert convergence, perhaps requiring it to be monotone would make our life easier.

After some mental gymnastics, one could potentially come up with the sequence $b_n = \sup_{k \geq n} a_k$. We certainly have $b_n \geq a_n$, and since the sequence is the supremum of a sequence of sets that are getting smaller and smaller, it's certainly decreasing, and therefore converges. We now have our first estimate:

$$\overline{\lim}\, a_n \leq \lim \sup_{k \geq n} a_k$$

Analogourly, for the least sublimit:

$$\underline{\lim}\, a_n \geq \lim \inf_{k \geq n} a_n$$

These estimates are so important they deserve their own names, creatively denoted *limit superior* ($\limsup a_n$) and *limit inferior* ($\liminf a_n$).

One last thing to notice is that, because of the way we showed convergence of monotone sequences, we have

$$\limsup a_n = \inf_n \sup_{k \geq n} a_k \text{ and } \liminf a_n = \sup_n \inf_{k \geq n} a_k$$

That is, we can write these estimates in terms of merely suprema and infima.

3.3 Oops

As it turns out, these estimates we just made are so good that we can replace the inequalities by equalities. That is:

Prop 13.

$$\overline{\lim}\, a_n = \limsup a_n \textit{ and } \underline{\lim}\, a_n = \liminf a_n$$

Proof. We will do only the proof for the limit superior, as the other one is perfectly analogous.

We already know $\overline{\lim}\, a_n \leq \limsup a_n$, so we need only find a subsequence of a_n that converges towards $\limsup a_n$.

Let $\limsup_{k \geq n} a_k = L$. For all m, there exists N such that, for $n \geq N$, $\sup_{k \geq n} a_k \in V_{1/m}(L)$.

Define i_m as follows: fix any i_0. Defined i_{m-1}, let n be some number greater than i_{m-1} such that $\sup_{k \geq n} a_k \in V_{1/m}(L)$. By openness, there is a neighbourhood of $\sup_{k \geq n} a_k$ contained in $V_{1/m}(L)$; pick $i_m \geq n$ such that a_{i_m} is in this neighbourhood. (It must exist because for any nonempty set there are elements arbitrarily close to the supremum.)

For i_n defined this way, we have that, for all $m \in N$, $a_{i_m} \in V_{1/m}(L)$, and hence a_{i_m} converges to L, as we wished to show. □

4 Applications

In this section, we show the versatility of having both definitions at our disposal. We begin with a simple proposition:

Prop 14. *$a_n \to L$ iff $\overline{\lim}\, a_n = \underline{\lim}\, a_n = L$.*

Proof. We do this proof in two parts, each part being done two ways to show how the different definitions compare.

($\to$)

(a) Suppose $a_n \to L$. Then, any subsequence of a_n also converges to L. Therefore, the smallest and greatest possible sublimits are both L, hence $\overline{\lim}\, a_n = \underline{\lim}\, a_n = L$.

(b) Suppose $a_n \to L$. Then, for any $\varepsilon > 0$ there exists N such that $n \geq N$ implies $a_n \in V_{\varepsilon/2}(L)$. Therefore, since the supremum and infimum of S are contained in $\overline{S}$, we have that if $S \subseteq A$ then the supremum and infimum of S are contained in $\overline{A}$. In particular, the sup and inf of $\{a_n\}_{n \geq N}$ are contained in $V_\varepsilon(L)$ Since ε is arbitrary, we get that the sequences $\sup_{k \geq n} a_k$ and $\inf_{k \geq n} a_k$ converge to L, hence $\limsup a_n = \liminf a_n = L$.

($\leftarrow$)

(a) Supose a_n does not converge to L. Then, there exists a $\varepsilon > 0$ such that there are infinite i such that a_i is not in $V_\varepsilon(L)$. Pick an increasing sequence of i_n satisfying this. Then a_{i_n} is a subsequence of a_n. Suppose without loss of generality it converges to some limit L'. (We can suppose this for, if it did not converge, we could pass to a converging subsequence of it.)

Notice L' cannot equal L, since, for the ε we fixed earlier, the terms of a_{i_n} are never within ε of L. Therefore, we have a subsequence that converges to something different from L, which shows L is either not the greatest sublimit (if $L' > L$) or it is not the least (if $L' < L$).

(b) Fix some arbitrary $\varepsilon > 0$. There exists, by hypothesis, N such that $n \geq N$ implies $\sup_{k \geq n} a_k, \inf_{k \geq n} a_k \in V_\varepsilon(L)$. In particular, $\sup_{k \geq N} a_k, \inf_{k \geq N} a_k \in V_\varepsilon(L)$, hence, since for all $n \geq N$ we have a_n is between these two numbers, a_n will be in $V_\varepsilon(L)$, which shows a_n converges to L as desired.

□

The reader should notice that for the right-implication, the proof with the sublimit definition was the most direct, while for the left-implication the proof with suprema and infima was slightly easier. This shows that sometimes one definitions might be more convenient than another, and we should allow ourselves to jump freely between them.

A trivial corollary from this proposition is the following:

Prop 15. *a_n converges iff* $\overline{\lim}\, a_n = \underline{\lim}\, a_n$