Optimal Investment Decision Under Switching regimes of Subsidy Support

Carlos Oliveira[1]

joint work with: Nicolas Perkowski[2]

[1]GFMUL - Grupo de Física Matemática da Universidade de Lisboa
[2]Department of Mathematics, Humboldt University of Berlin

Probability and Statistics Seminar
Instituto Superior Técnico, Lisbon, 8 November 2018

Carlos Oliveira is partially supported by the Fundação para a Ciência e a Tecnologia (FCT)

Motivation: subsidies

- A firm intends to undertake a new project in a subsidized market;
- However, the program of subsidies will be removed sometime in the future;
- This makes the decision of investment more uncertain.

When?

The stochastic process

1) There are k different levels of subsidy:

$$\begin{cases} \theta_s \in \Theta \equiv \{1, \dots, k\}, & \text{for each } s \geq 0, \\ \theta \text{ is a càdlàg process.} \end{cases}$$

We introduce the process $\{\nu_n \,:\, n \in \mathbb{N}_0\}$, where

$$\nu_1 = \inf\{s > 0 \,:\, \theta_{s^-} \neq \theta_s\} \quad \text{and} \quad \nu_n = \inf\{s > \nu_{n-1} \,:\, \theta_{s^-} \neq \theta_s\}.$$

(ν_n is the time until the $n^{\text{th}}-$transition of Markov chain θ.)

The stochastic process

- We assume that for every $j, m \in \Theta$

$$P(\nu_n - \nu_{n-1} \leq s \,|\, \theta_{\nu_{n-1}} = j) = 1 - e^{-\int_0^s \lambda_j(u)du}, \quad \text{for all } s \geq 0,$$

  where $\lambda_j : [0, \infty) \to [0, \infty)$ is continuous;

- the random variables $(\nu_{n_1} - \nu_{n_1-1})$ and $(\nu_{n_2} - \nu_{n_2-1})$, when $n_1 \neq n_2$, are independent;

- and, finally, $P(\theta_{\nu_n} = m \,|\, \theta_{\nu_{n-1}} = j) = p_{j,m}(\nu_n - \nu_{n-1})$, with $p_{j,k} : [0, \infty) \to [0, 1]$ is a continuous function such that $\sum_{m=1}^{k} p_{j,m}(s) = 1$ and $p_{j,j}(s) = 0$ for all $s > 0$.

We will use the following notation: $\lambda_{i,j}(t) = p_{i,j}(t) \times \lambda_i(t)$.

The stochastic process

- $X = \{X_s : s \geq 0\}$ is a random economic indicator and solves the switching stochastic differential equation

$$dX_s = \alpha(X_s, \theta_s)ds + \sigma(X_s, \theta_s)dW_s,$$

taking values in the open set $D \subseteq \mathbb{R}^n$;

- $W = \{W_s, s \geq 0\}$ is a $m-$dimensional Brownian motion;

- W and θ are independent processes;

- the Borel measurable functions α and σ are such that

$$|\alpha(x,i) - \alpha(y,i)| + \|\sigma(x,i) - \sigma(y,i)\| \leq L|x - y|,$$

and the SDE, for each initial condition, has a unique strong solution (W, X) on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, P)$ that remains in D for all times.

Markovian representation of the process (X, θ)

- The process (X, θ) is not, in general, a Markov process, because it is never known how much time was spent since the last transition in the Markov chain;

- we introduce the process $\zeta = \{\zeta_s,\, s \geq t\}$, which represents the time spent from the last change in the level of subsidy until the moment s, defined by

$$\zeta_s = s - \nu^s,\ s > 0,$$

  where $\nu^s \equiv \sup\{\nu_n : \nu_n \leq s, n \in \mathbb{N}\}$ is an $\mathcal{F}_s-$stopping time.

- we will work with the process (X, ζ, θ), which is the Markovian representation of the process (X, θ).

Optimal stopping problem

- We consider the functions;
  - $\Pi : D \times \Theta \to \mathbb{R}$ is the running payoff;
  - $h : D \times \Theta \to \mathbb{R}$ is the cost of abandonment;
  - $r : D \times \Theta \to \mathbb{R}$ the instantaneous interest rate.
- The functions $\Pi, h, r : D \times \Theta \to \mathbb{R}$ are such that

$$\Pi(\cdot, i), h(\cdot, i), r(\cdot, i) \in C(D), \quad \text{for all } i \in \Theta$$
$$\exists \epsilon_i > 0 \text{ such that } r(\cdot, i) > \epsilon_i, \quad \text{for all } i \in \Theta.$$

- If the investment project is permanently abandoned at the moment τ, its revenue is given by

$$\int_0^\tau e^{-\rho_s} \Pi(X_s, \theta_s) ds - e^{-\rho_\tau} h(X_\tau, \theta_\tau) 1_{\{\tau < \infty\}},$$
$$\rho_s = \int_0^s r(X_u, \theta_u) du, \quad s \geq 0.$$

Optimal stopping problem

- Our problem is to seek τ^* such that

$$V^*(x,t,i) = \sup_{\tau \in \mathcal{S}} J(x,t,i,\tau) = J(x,t,i,\tau^*)$$

where

$$J(x,t,i,\tau) = E_{x,t,i}\left[\int_0^{\tau \wedge T^I} e^{-\rho_s}\Pi(X_s,\theta_s)ds - e^{-\rho_\tau} h(X_\tau,\theta_\tau)1_{\{\tau<\infty\}}\right]$$

  - $E_{x,t,i}[\cdot]$ is the expected value conditional on $X_0 = x$, $\zeta_0 = t$ and $\theta_0 = i$;
  - $T^I = \inf\{s \geq 0 : X_t \notin I\}$ with $I \subset D$

Optimal stopping problem

The problem's well-posedness is guaranteed by introducing the following integrability conditions:

- The functions $\Pi, h, r : D \times \Theta \to \mathbb{R}$ are such that

$$E_{x,t,i}\left[\int_0^{T^l} e^{-\rho_s}\Pi^+(X_t,\theta_t)ds\right] < \infty \quad and$$

$$\{h(X_\tau,\theta_\tau)\}_{\tau\in\mathcal{S}} \quad \text{is a uniformly integrable family of random variables.}$$

where, for every real function f, we set that

$$f^+ = \max(0, f), \quad \text{and} \quad f^- = \max(0, -f).$$

Uniformly integrability

- **Definition:** Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family $\{f_j\}_{\{j \in J\}}$ of real, measurable function f_j on Ω is called uniformly integrable if

$$\lim_{M \to \infty} \left(\sup_{j \in J} E \left[|f_j| 1_{\{|f_j| > M\}} \right] \right) = 0$$

Uniformly integrability

- **Definition:** Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family $\{f_j\}_{\{j \in J\}}$ of real, measurable function f_j on Ω is called uniformly integrable if

$$\lim_{M \to \infty} \left(\sup_{j \in J} E\left[|f_j| 1_{\{|f_j| > M\}} \right] \right) = 0$$

- **Proposition:** Let V^* be the value function defined before. Then, $\{V^*(X_\tau, \zeta_\tau, \theta_\tau)\}_{\tau \in \mathcal{S}}$ is a uniformly integrable family of random variables, for every initial condition $(X_0, \zeta_0, \theta_0) = (x, t, i)$.

Infinitesimal Generator

- Infinitesimal generation:

$$(\mathcal{L}\phi)(x,t,i) = \lim_{u\downarrow 0} \frac{1}{u} E_{x,t,i} \left[\phi(X_u, \zeta_u, \theta_u) - \phi(x,t,i) \right],$$

where ϕ is a sufficiently smooth function.

- The operator takes the form

$$\begin{aligned}(\mathcal{L}\phi)(x,t,i) =& \frac{\partial \phi}{\partial t}(x,i,t) + \alpha(x,i) D\phi(x,t,i) \\ &+ \frac{1}{2} Tr \left[\sigma\sigma^T(x,i) D^2\phi(x,t,i) \right] + (Q\phi)(x,t,i)\end{aligned}$$

$$(Q\phi)(x,t,i) = \sum_{j\neq i} \lambda_{i,j}(t) \left(\phi(x,0,j) - \phi(x,t,i) \right), \quad \text{for a fixed } i \in \Theta.$$

- It can be useful to define the operator

$$\begin{aligned}(\tilde{\mathcal{L}}\phi)(x,t,i) =& \lim_{u\downarrow 0} \frac{1}{u} E_{x,t,i} \left[e^{-\rho_u} \phi(X_u, \zeta_u, \theta_u) - \phi(x,t,i) \right], \\ =& - r(x,i)\phi(x,t,i) + (\mathcal{L}\phi)(x,t,i)\end{aligned}$$

Dynamic Programming Principle

The basic Idea:

- The principle of optimality is the basic principle of dynamic programming, which was developed by Richard Bellman in 1957;
- an optimal path has the property that whatever the initial conditions and control variables (choices) over some initial period, the control (or decision variables) chosen over the remaining period must be optimal for the remaining problem, with the state resulting from the early decisions taken to be the initial condition.

Dynamic Programming Principle

Informal argument:

- We fix (x, t, i) and a "small" time increment u, where later on $u \to 0$. We will compare three different strategies:
  - We use the optimal strategy τ^*;
  - We stop immediately;
  - We wait until time u, and from time u we behave optimally, i.e. we use the stopping time τ^*.

This implies that:

$$V^*(x, t, i) \geq -h(x, i);$$

$$V^*(x, t, i) \geq E_{x,t,i}\left[\int_0^u e^{-\rho_s}\Pi(X_s, \theta_s)ds + e^{-\rho_u}J(X_u, \zeta_u, \theta_u, \tau^*)1_{\{\tau<\infty\}}\right].$$

Dynamic Programming Principle

Making the argument more formal:

- If V^* is sufficiently smooth, then:

$$V^*(x,t,i) = \sup_{\tau \in \mathcal{S}} E_{x,t,i}\Bigg[\int_0^{\tau\wedge\delta} e^{-\rho_s}\Pi(X_s,\theta_s)ds$$

$$+ e^{-\rho_{\tau\wedge\delta}}\left(-h(X_\tau,\theta_\tau)1_{\{\tau<\delta\}} + V^*(X_\delta,\zeta_\delta,\theta_\delta)1_{\{\tau\geq\delta\}}\right)\Bigg],$$

where $\delta < \infty$ is an arbitrary but fixed stopping time.

Dynamic Programming Principle

Making the argument more formal:

- Since we do not have any information about the regularity of V^* we proved a **weak** dynamic programming principle:

$$V^*(x,t,i) \leq \sup_{\tau \in \mathcal{S}} E_{x,t,i}\Bigg[\int_0^{\tau\wedge\delta} e^{-\rho_s}\Pi(X_s,\theta_s)ds$$
$$+e^{-\rho_{\tau\wedge\delta}}\left(-h(X_\tau,\theta_\tau)1_{\{\tau<\delta\}} + \overline{V}^*(X_\delta,\zeta_\delta,\theta_\delta)1_{\{\tau\geq\delta\}}\right)\Bigg],$$

and

$$V^*(x,t,i) \geq \sup_{\tau \in \mathcal{S}} E_{x,t,i}\Bigg[\int_0^{\tau\wedge\delta} e^{-\rho_s}\Pi(X_s,\theta_s)ds$$
$$+e^{-\rho_{\tau\wedge\delta}}\left(-h(X_\tau,\theta_\tau)1_{\{\tau<\delta\}} + \underline{V}^*(X_\delta,\zeta_\delta,\theta_\delta)1_{\{\tau\geq\delta\}}\right)\Bigg].$$

Dynamic Programming Principle

Some references about weak dynamic programming principles:

- B Bouchard and N Touzi. *Weak dynamic programming principle for viscosity solutions.* (2011)
- N Touzi. *Optimal stochastic control, stochastic target problems, and backward SDE.* (2012)
- M Ehrhardt, M Gunther and E Maten. *Novel methods in computational finance.* (2017)
- R Dumitrescu, MC Quenez, A Sulem. *A Weak Dynamic Programming Principle for Combined Optimal Stopping/Stochastic Control with $\mathcal{E}^f$-expectations.* (2016)

HJB equations

Assuming that V^* is sufficiently regular, it is not difficult to show that V^*, in the classical sense, satisfies the system of HJB equations

$$\begin{cases} F_i(x,t,\{v(x,t,j):\ j\in\Theta\},\partial_t v(x,t,i),Dv(x,t,i),D^2v(x,t,i))=0 \\ (x,t,i)\in I\times\Theta\times(0,\infty) \end{cases},$$

where

$$F_i(x,t,\{v(x,t,j):\ j\in\Theta\},\partial_t v(x,t,i),Dv(x,t,i),D^2v(x,t,i))\equiv$$
$$\equiv \min\Big\{-(\tilde{\mathcal{L}}v)(x,t,i)-\Pi(x,i),\ v(x,t,i)+h(x,i)\Big\},$$

What happens when V^* "is not sufficiently regular"?

Viscosity solutions: Motivation

- Let $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^d \mapsto \mathbb{R}$, with $\mathcal{O} \subset \mathbb{R}^d$ be a function such that

$$F(x, r, p, A) \leq F(x, r, p, B), \quad \text{se} \quad A \geq B.$$

- Now, consider the PDE

$$F(x, u(x), Du(x), D^2 u(x)) = 0.$$

- If u is a subsolution to the previous equation and $\hat{x}$ is a local maximizer to the function $u - \phi$ with $\phi \in C^2$, then

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \leq F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2 u(\hat{x})) \leq 0$$

- If u is a supersolution to the previous equation and $\hat{x}$ is a local minimizer to the function $u - \psi$ with $\psi \in C^2$, then

$$F(\hat{x}, u(\hat{x}), D\psi(\hat{x}), D^2\psi(\hat{x})) \geq F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2 u(\hat{x})) \geq 0$$

Viscosity solutions: Motivation

u is a viscosity solution to the equation

$$F(x, u(x), Du(x), D^2u(x)) = 0$$

if

- u is a viscosity subsolution, i.e, if for any $\phi \in C^2$ and $\hat{x}$, such that $\hat{x}$ is a local maximizer to the function $u - \phi$ we have

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \leq 0,$$

- u is a viscosity supersolution, i.e, if for any $\phi \in C^2$ and $\hat{x}$ such that $\hat{x}$ is a local minimizer to the function $u - \phi$ we have

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \geq 0.$$

Viscosity solutions

Consider a locally bounded function $v : I \times (0, \infty) \times \Theta \to \mathbb{R}$. Then, v is a

(a) viscosity subsolution if whenever $\psi \in C^2(I \times [0, \infty))$, $i \in \Theta$ and $\overline{v}(\cdot, \cdot, i) - \psi(\cdot, \cdot)$ has a local maximum at $(x, t) \in I \times [0, \infty)$, such that $\overline{v}(x, t, i) = \psi(x, t)$, then

$$F_i(x, t, \{\overline{v}(x, t, j) : j \in \Theta\}, D\psi(x, t), D^2\psi(x, t)) \leq 0.$$

(b) viscosity supersolution if whenever $\psi \in C^2(I \times [0, \infty))$, $i \in \Theta$ and $\underline{v}(\cdot, \cdot, i) - \psi(\cdot, \cdot)$ has a local minimum at $(x, t) \in I \times [0, \infty)$, such that $\underline{v}(x, t, i) = \psi(x, t)$, then

$$F_i(x, t, \{\underline{v}(x, t, j) : j \in \Theta\}, D\psi(x, t), D^2\psi(x, t)) \geq 0.$$

(c) viscosity solution if it is simultaneously a viscosity subsolution and a viscosity supersolution.

Viscosity solutions: Existence

Proposition: Let V^* be the value function. Then V^* is a viscosity solution to the system of HJB equations and the boundary condition

$$v(x,t,i) = -h(x,i), \quad \text{for all } x \in \partial I$$

is satisfied.

Remark: The set ∂I is obtained by considering the topology on D, which is the trace of the usual topology on $\mathbb{R}^n$.

Viscosity solutions: Example

Consider the following system of HJB equations:

$$\min\left\{(2+\lambda(t))v(x,t,2)-\partial_t v(x,t,2)-\tilde{\sigma}^2(x)\frac{\partial^2}{\partial x^2}v(x,t,2)-\lambda(t)v(x,t,1),\right.$$
$$\left.v(x,t,2)+h(x,2)\right\}=0,$$

$$\min\left\{2v(x,t,1)-\partial_t v(x,t,1)-x^2\frac{\partial^2}{\partial x^2}v(x,t,1),v(x,t,1)+\tilde{h}(x,1)\right\}=0,$$

$$\lambda(t)\geq 0,\;\; -\tilde{h}(x,1)=(x-1)\times 1_{\{1\leq x\leq 2\}}+1_{\{x>2\}},\;\; \textit{and}\;\; -\tilde{h}(x,2)=\frac{x^2}{1+x^2}.$$

It is straightforward to see that any function $v(x,t,i)=bxe^{2t}$, with $b\geq 1$ and $i=1,2$, is a classical solution (and, consequently, a viscosity solution) to the system of HJB equations.

Viscosity solutions: Uniqueness

Theorem: Suppose that v is a viscosity solution to the system of HJB equations and satisfies the boundary condition

$$v(x,t,i) = -h(x,i), \quad \text{for all } x \in \partial I. \tag{1}$$

Additionally, assume that

$\{v(X_\tau, \zeta_\tau, \theta_\tau)\}_{\tau \in \mathcal{S}}$ is a uniformly integrable family of random variables.

Then, v is the unique solution to the system of HJB equations and boundary conditions that satisfies the previous integrability condition and verifies $v = V^*$. Furthermore,

$$\tau^* = \inf\{s \geq 0 : v(X_s, \zeta_s, \theta_s) \leq -h(X_s, \theta_s)\}.$$

Viscosity solutions: Example

Optimal stopping problem:

$$V^*(x,t,i) = \sup_{\tau} E_{x,t,i}\left[-e^{-2\tau} h(X_\tau, \theta_\tau)\right]$$
$$dX_s = \alpha(X_s, \theta_s)ds + \sigma(X_s, \theta_s)dW_s$$

State Space: $\Theta = \{1, 2\}$

Parameters of the diffusion:

$\theta = 1 \Rightarrow \sigma(x,1) = \sqrt{2}x,\ \alpha(x,1) = 0;$
$\theta = 2 \Rightarrow \sigma(x,2) = 0,\ \alpha(x,2) = 0.$

Parameters associated with the Markov chain:

$\lambda_{1,2}(s) = 0,\ \lambda_{2,1}(s) = 2s$, for all $s \geq 0$.

Cost function:

$$-h(x,i) = \begin{cases} (x-1) \times 1_{\{1 \leq x \leq 2\}} + 1_{\{x>2\}}, & i = 1 \\ \frac{x^2}{1+x^2}, & i = 2 \end{cases}, \quad \text{for all } x > 0.$$

Example

Some easy consequences of our choice of parameters:

- the HJB equations characterizing the correspondent value function, V^*, are given in the last example;
- when $\theta_0 = 1$, there is no possibility to have $\theta_s = 2$, for any $s > 0$, which means that, in this case, the process (θ, ζ) does not explicitly influence the resolution of the optimization problem;
- we can guess that the value function V^* is such that $V^*(x, t, 1) \equiv V^*(x, 1)$;
- if $\theta_0 = 2$, then X_t is frozen at the level $X_0 = x$ for every $t \leq \nu_1$;

Example

- **Firstly, we try to find** $V^*(\cdot, \cdot, 1)$.
- We guess that, when $(x, t, i) \in [x_0, \infty) \times [0, \infty) \times \{1\}$, for some $x_0 \in (1, 2]$, it is optimal to stop immediately;
- we should find a function $v(x, t, 1) \equiv v(x, 1)$, such that $2v(x, 1) - x^2 \frac{\partial^2}{\partial x^2} v(x, 1) = 0$, for all $x \in (0, x_0)$ and $v(x, 1) = -h(x, 1)$, for all $x \in [x_0, \infty)$ i.e.

$$v(x,1) = \begin{cases} Ax^{-1} + Bx^2, & x \in (0, x_0) \\ -h(x,1), & x \in [x_0, \infty) \end{cases}$$

- Fixing the parameter A:
  - From the supersolution property, we get that $v(x, 1) + h(x, 1) \geq 0$, which implies that $A \geq 0$;
  - Additionally, we want to find a function $v(\cdot, 1)$ satisfying the property $\{v(X_\tau, 1)\}_{\tau \in \mathcal{S}}$ is a uniformly integrable family of random variables, which is only possible when $A = 0$;

Example

- Fixing the parameter B and the threshold x_0:
  - we postulate that $x_0 = 2$ and $B = 1/4$, which guarantees that the function $v(\cdot, 1)$ is continuous at x_0;
  - without those conditions, we would obtain a function $v(\cdot, 1)$ with a "downward kink"or with a discontinuity.
- Therefore, we propose the following function as suitable candidate

$$v(x,1) = \begin{cases} \frac{x^2}{4}, & x \in (x,2) \\ -h(x,1), & x \in [2,\infty) \end{cases}.$$

- **Now, we can deduce** $V^*(\cdot,\cdot,2)$:
- if $\theta_0 = 2$, then X_t is frozen at the level $X_0 = x$ for every $t \leq \nu_1$;
- since $-h(x,1) \leq -h(x,2)$, when $x \leq \sqrt{3}$ and $t \in [0,\infty)$, we guess that, in this case, stopping immediately is optimal;
- if $x > \sqrt{3}$, then $-h(x,2) < -h(x,1)$, and, consequently, as $\lambda(s) \to \infty$ when $s \to \infty$, it might be optimal to wait until stopping the process, at least for $\zeta_0 = t > \tilde{t}(x)$, where $\tilde{t} : (\sqrt{3},\infty) \to (0,\infty)$ is a function to be determined.

Example

- we postulate that the candidate function $v(x,t,2)$ is given by

$$v(x,t,2) = \begin{cases} \tilde{v}(x,t), & (x,t) \in (\sqrt{3},\infty) \times (\tilde{t}(x),\infty) \\ -h(x,2), & (x,t) \in (0,\sqrt{3}] \times [0,\infty) \cup (\sqrt{3},\infty) \times [0,\tilde{t}(x)] \end{cases},$$

where

$$\tilde{v}(x,t) = e^{t^2+2t-\tilde{t}(x)^2-\tilde{t}(x)} \left(\frac{x^2}{1+x^2} - v(x,1) \int_{\tilde{t}(x)}^{t} 2se^{\tilde{t}(x)^2+2\tilde{t}(x)-s^2-2s} ds \right),$$

- the function $\tilde{t} : (\sqrt{3},\infty) \to (0,\infty)$ should be such that $\{v(X_\tau,\zeta_\tau,\theta_\tau)\}_{\tau\in\mathcal{S}}$ is a uniformly integrable family of random variables
- which holds true if, and only if, for each $x > \sqrt{3}$, the function $\tilde{t}$ is implicitly defined by the equation

$$\int_{\tilde{t}(x)}^{\infty} 2se^{\tilde{t}(x)^2+2\tilde{t}(x)-s^2-2s} ds = \frac{x^2}{(1+x^2)v(x,1)}.$$

Example

From the arguments above, the value function V^* is such that $V^* = v$, if v is a viscosity solution to the system of HJB equations, which holds trivially true if the following statements are satisfied:

1) $v(x,t,2) \geq -h(x,2)$, for every $(x,t) \in (0,\infty) \times [0,\infty)$;
2) v satisfies the viscosity property at the points $(2,t,1)$ and $(x,\tilde{t}(x),2)$, with $t \geq 0$ and $x > \sqrt{3}$.

Conclusion

$$V^*(x,1) = \begin{cases} \frac{x^2}{4}, & x \in (x,2) \\ -h(x,1), & x \in [2,\infty) \end{cases}.$$

$$V^*(x,t,2) = \begin{cases} \tilde{v}(x,t), & (x,t) \in (\sqrt{3},\infty) \times (\tilde{t}(x),\infty) \\ -h(x,2), & (x,t) \in (0,\sqrt{3}] \times [0,\infty) \cup (\sqrt{3},\infty) \times [0,\tilde{t}(x)] \end{cases},$$

Thank you for your attention!