# Radiation from elementary sources in a uniaxial wire medium

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We investigate the radiation properties of two types of elementary sources embedded in a uniaxial wire medium: a short dipole parallel to the wires and a lumped voltage source connected across a gap in a generic metallic wire. It is demonstrated that the radiation pattern of these elementary sources may have quite anomalous and unusual properties. Specifically, the radiation pattern of a short vertical dipole resembles that of an isotropic radiator close to the effective plasma frequency of the wire medium, whereas the radiation from the lumped voltage generator is characterized by an infinite directivity and a nondiffractive far-field distribution.

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## I. INTRODUCTION

Wire media, generically defined as structured materials formed by arrays of long metallic wires,[1–3] are perhaps the class of metamaterials whose effective response is better understood. Particularly, during the last decade a vast number of theoretical methods and analytical tools have been developed which enable characterizing the effective electromagnetic response of wire-based materials in different scenarios with great accuracy.[4–18] However, a bit surprisingly, the problem of radiation by localized external sources embedded within wire media has only been cursorily discussed in the literature.[19–23]

In part, this gap can be explained by the peculiar electromagnetic response of wire media, which are typically characterized by strong spatial dispersion in the long-wavelength limit,[6] and this property complicates analytical modeling. In simple terms, a medium is spatially dispersive if the polarization vector at some generic point in space depends not only on the macroscopic electric field, but also on the gradient of the field and, possibly, higher order derivatives.[24]

The objective of this work is to characterize the radiation properties of elementary localized sources placed within a wire medium using an effective medium approach. Specifically, we are interested in the following two scenarios: (i) a short vertical dipole is embedded in the wire medium [Fig. 1(a)], and (ii) an external lumped voltage source is connected across a gap in a generic metallic wire [Fig. 1(b)]. As detailed in Sec. II, these sources are modeled in terms of the Dirac $\delta$ function. It is, however, important to make clear at the outset that, strictly speaking, an effective medium description of the radiation problem is only possible if the source is localized on a larger scale than the characteristic dimension of the metamaterial (e.g., the lattice constant $a$; see Fig. 1). Hence, the short vertical dipole considered here should be understood as some external current distributed over a region of space whose characteristic diameter in the $xoy$ plane is larger than or equal to $a$ but much smaller than the wavelength. The simplest way to visualize our macroscopic dipolar source inside the wire medium is to consider a dense array of microscopic (i.e., with the dimensions much smaller than the lattice period) dipoles (all with the dipole moment oriented along the wires and oscillating in phase), spread over several unit cells of the structure. The radiation from such a cloud of dipolar particles is equivalent to averaging the radiation field of a single particle placed at random locations within a certain volume of the wire medium. At microwaves, the macroscopic dipolar source may be realized in practice with a single "structured" dipole-type antenna that is flexible enough so that it can let the inclusions of the wire medium go through it without much disturbance.

Similarly, even though, for the purposes of illustration and discussion, we say that in case (ii) the voltage source is connected across the gap of a single wire, it is more accurate to imagine such a source as an array of voltage generators, distributed over a region of space whose characteristic diameter is larger than $a$, with each voltage generator being connected across a gap in a metallic wire lying within the mentioned region. With the exception of the immediate vicinity of these sources, the solution determined with our theory (based on the Dirac-$\delta$ distribution) should describe the radiated fields accurately.

One of the challenges in the characterization of the radiation by a localized source within a wire medium is related to the calculation of quantities such as the Poynting vector or the radiation intensity (i.e., the power radiated per unit of solid angle). Indeed, in general the usual form of the Poynting vector, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, does not hold in the case of spatially dispersive materials.[24,25] Moreover, there is no known theory to determine the Poynting vector in a general spatially dispersive material, and the only case that is actually understood, and for which closed analytical formulas are available, is when the electromagnetic fields have a spatial variation of the form $e^{-j\mathbf{k}\cdot\mathbf{r}}$ (plane waves).[24] In this work, we derive closed analytical formulas that enable calculating explicitly the Poynting vector and the electromagnetic energy density in uniaxial wire media for *arbitrary* electromagnetic field distributions. This is one of the key results of the paper.

To do this, we rely on the theory of our earlier works,[17,26] where we have shown that the effective medium response of the wire medium can be modeled using a quasistatic model, based on the introduction of two additional variables, $I$ and $\varphi_w$. What is remarkable about such a model is that the material response (e.g., the macroscopic polarization vector) can be expressed through the macroscopic electromagnetic fields, and the additional variables $I$ and $\varphi_w$ through *local* relations in space. Therefore, such a formalism enables describing the unconventional nonlocal electrodynamics of the wire medium using local material relations, without requiring the definition of an effective spatially dispersive dielectric function, which

 

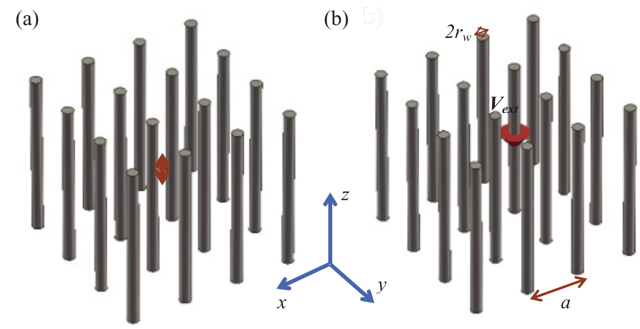


FIG. 1. (Color online) Uniaxial wire medium formed by a square lattice of metallic wires oriented along the $z$ direction. (a) Excitation based on a short vertical dipole embedded in the wire medium. (b) Excitation based on a discrete voltage source connected directly at the center of one of the wires.

would lead to nonlocal relations between the polarization vector and the macroscopic electric field.[6] It is important to mention that the introduction of the additional variables is not just a trick that simplifies the modeling of the wire medium: it is actually full of physical significance and clarifies the microscopic mechanisms that determine the macroscopic response of the metamaterial. Indeed, the variable $I$ can be understood as the electric current that flows along the metallic wires (interpolated in such a manner that it becomes a continuous function defined in all space), whereas the variable $\varphi_w$ can be understood as the average potential drop from a given wire to the boundary of the respective unit cell (the potential is interpolated in the same manner as the current). For more details, the reader is referred to Refs. 17 and 26.

This paper is organized as follows. In Sec. II, we briefly review the quasistatic model of the wire medium and formulate the radiation problem for the two excitations of interest. In Sec. III we solve the pertinent radiation problem in the spectral domain. First, we discuss the general case of a stratified (along $z$) structure, and after this we analyze in detail the particular case of an unbounded uniform structure. In Sec. IV, we show that, for an unbounded uniform structure, the fields radiated by the elementary external sources can also be directly determined from the nonlocal dielectric function of the metamaterial. After this, in Sec. V we derive a general Poynting theorem that expresses the conservation of energy in wire media, and in Sec. VI we use these results to obtain the asymptotic form of the Poynting vector in the far field, as well as the directive gain, directivity, and power radiated by a short vertical dipole. Conclusions are drawn in Sec. VII. In this work, we assume that in the case of a time harmonic regime, the time variation is of the form $e^{j\omega t}$.

## II. FRAMEWORK BASED ON THE INTRODUCTION OF ADDITIONAL VARIABLES

In Refs. 17 and 26 it was shown that the internal physical processes that determine the macroscopic response of a wide class of wire media are intrinsically related to the dynamics of the electric current $I$ along the wires and the additional potential $\varphi_w$, whose physical meaning is discussed in Sec. I. In particular, it was proven that for the case of straight wires oriented along the $z$ direction, the macroscopic electromagnetic fields satisfy

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H}, \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{ext}} + \frac{I}{A_c}\hat{\mathbf{z}} + j\omega\varepsilon_h\mathbf{E}, \tag{2}$$

where $\varepsilon_h$ is the permittivity of the host material, $A_c = a^2$ is the area of the unit cell, and $a$ is the period of the wire medium (Fig. 1). Note that, unlike in previous work,[17,26] here we admit the possibility of an external distributed current source $\mathbf{J}_{\text{ext}}$. The electromagnetic fields are coupled to the current $I$ and additional potential $\varphi_w$ via a set of transmission line-type equations:

$$\frac{\partial\varphi_w}{\partial z} = -(Z_w + j\omega L)I + E_z + V_{\text{ext}}A_c\delta(x,y,z), \tag{3}$$

$$\frac{\partial I}{\partial z} = -j\omega C\varphi_w. \tag{4}$$

In the above, $C$, $L$, and $Z_w$ represent the capacitance, inductance, and self-impedance of a wire per unit of length, respectively, and explicit formulas for these parameters can be found in our previous papers. It should be noted that $C$ and $L$ depend exclusively on the properties of the host medium and on the geometry of the wire medium. The effects of the metal are described by the self-impedance $Z_w$. The real part of $Z_w$ is determined by the ohmic loss in the metallic wires, whereas its imaginary part is related to the kinetic inductance of the electrons in the metal. Compared to Refs. 17 and 26, now we allow for an external lumped voltage source (with amplitude $V_{\text{ext}}$) to be placed across a gap in the wire in the central unit cell. It is simple to check, based on the theory of Ref. 17, that this lumped voltage source is modeled by the term $V_{\text{ext}}A_c\delta(x,y,z)$. Note that, similarly to the current and additional potential, the lumped generator is interpolated so that it becomes a function defined over all space.

In the next section, we determine the solution of the radiation problems sketched in Fig. 1, based on the system of Eqs. (1)–(4). It should be mentioned that the present framework provides a *local* description of the uniaxial wire medium because the material response represented by the expressions on the right-hand side of Eqs. (1)–(4) (without the source terms) is independent of the gradient and higher order derivatives of the electromagnetic fields, $I$, and $\varphi_w$. More specifically, our system of equations may be rewritten in the form $\hat{\mathrm{L}} \cdot \mathbf{F} = -j\omega\hat{\mathrm{M}} \cdot \mathbf{F}$, where $\hat{\mathrm{L}}$ is a linear differential operator (fully independent of the medium response), $\mathbf{F} = (\mathbf{E},\mathbf{H},\varphi_w,I)^T$ is our eight-component state vector, and $\hat{\mathrm{M}}$ is a linear matrix operator that is determined by the wire medium response and is written in terms of the constitutive parameters $\varepsilon_h$, $\mu_0$, $C$, $L$, and $Z_w$. In the present framework the medium response is local because $\hat{\mathrm{M}}$ does not include any integrodifferential operators. This contrasts with the usual formulation based on the effective dielectric function, which does not introduce any additional variables but in which the dielectric function depends explicitly on the spatial gradient.[6,12] This is discussed further in Sec. IV.

For future reference, we note that from Eqs. (3) and (4) it follows that

$$\frac{\partial}{\partial z}\frac{1}{C}\frac{\partial I}{\partial z} + (\omega^2 L - j\omega Z_w)I = -j\omega[E_z + V_{\text{ext}}A_c\delta(x,y,z)]. \tag{5}$$

In the above, it was supposed that $L$, $C$, and $Z_w$ may depend on $z$ (but not on $x$ and $y$), which can happen in the case of a stratified wire medium (with the direction of stratification along $z$), such that either the permittivity of the host medium or the radii of the wires vary with $z$.

## III. THE RADIATION PROBLEM

Next, we derive the solution of the radiation problem in terms of two potential functions. We admit that the external current density describes a short vertical dipole, so that $\mathbf{J}_{\text{ext}} = j\omega p_e \delta(x,y,z)\,\hat{\mathbf{z}}$, where $p_e$ represents the electric dipole moment. Since the Maxwell equations are linear it is possible to solve the two radiation problems sketched in Fig. 1 simultaneously. This is done in what follows.

### A. Solution in terms of two potential functions for the general case of a stratified structure

For generality, in this subsection, we admit that $L$, $C$, $Z_w$, and $\varepsilon_h$ may depend on $z$, which, as discussed previously, may be useful to study problems of radiation in stratified media. We look for a solution of Eqs. (1)–(4) such that the macroscopic electromagnetic fields are written in terms of a potential function $\Phi$ so that

$$\mathbf{H} = \nabla \times \{j\omega\Phi\hat{\mathbf{z}}\}, \tag{6}$$

$$\mathbf{E} = \omega^2\mu_0\Phi\,\hat{\mathbf{z}} + \nabla\left(\frac{1}{\varepsilon_h}\frac{\partial\Phi}{\partial z}\right). \tag{7}$$

The usual Hertz potential is written in terms of $\Phi$ as $\mathbf{\Pi}_e = \hat{z}\Phi/\varepsilon_h$. The above representation of the fields in terms of a potential function is applicable for the excitations considered in Fig. 1 but would not be valid if the short dipole was perpendicular to the wires. It can be easily verified that Eqs. (6) and (7) satisfy the Maxwell equations, (1) and (2), provided that

$$\varepsilon_h\frac{\partial}{\partial z}\frac{1}{\varepsilon_h}\frac{\partial\Phi}{\partial z} + \nabla_t^2\Phi + \omega^2\mu_0\varepsilon_h\Phi + \frac{I}{j\omega A_c} = -p_e\delta(x,y,z), \tag{8}$$

where $\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Hence, substituting Eq. (7) into Eq. (5) and using the above result, it follows that

$$C\frac{\partial}{\partial z}\frac{1}{C}\frac{\partial}{\partial z}\left(\frac{I}{j\omega A_c}\right) + (\omega^2 LC - j\omega Z_w C)\left(\frac{I}{j\omega A_c}\right) = -\frac{C}{A_c\varepsilon_h}[\varepsilon_h E_z + \varepsilon_h V_{\text{ext}} A_c \delta(x,y,z)] = -\frac{C}{A_c\varepsilon_h}\left[-\nabla_t^2\Phi - p_{\text{ef}}\delta(x,y,z) - \frac{I}{j\omega A_c}\right], \tag{9}$$

where we defined the effective dipole moment for the combined excitations:

$$p_{\text{ef}} = p_e - \varepsilon_h A_c V_{\text{ext}}. \tag{10}$$

Note that $p_{\text{ef}}$ depends on both $p_e$ and $V_{\text{ext}}$, because we allow for the simultaneous excitation of the wire medium with the two pertinent types of elementary sources.

For convenience, let us introduce the auxiliary potential function

$$\psi = \frac{1}{k_p^2}\frac{I}{j\omega A_c}, \tag{11}$$

where $k_p = \sqrt{\mu_0/(LA_c)}$ is the so-called plasma wave number of the wire medium,[6,12,17] which may be calculated using, for example, the approximate formula applicable to both square and hexagonal wire lattices $k_p \approx (1/a)\sqrt{2\pi/\log[a^2/4r_w(a-r_w)]}$, where $r_w$ is the radius of the metallic wires. Using the fact that for straight unloaded wires $LC = \mu_0\varepsilon_h$, it follows that Eqs. (8) and (9) are equivalent to

$$\varepsilon_h\frac{\partial}{\partial z}\frac{1}{\varepsilon_h}\frac{\partial\Phi}{\partial z} + \nabla_t^2\Phi + k_h^2\Phi + k_p^2\psi = -p_e\delta(x,y,z), \tag{12}$$

$$C\frac{\partial}{\partial z}\frac{1}{C}\frac{\partial}{\partial z}\psi + \left(k_h^2 + \beta_c^2 - k_p^2\right)\psi - \nabla_t^2\Phi = p_{\text{ef}}\delta(x,y,z), \tag{13}$$

where we put $k_h^2 = \omega^2\mu_0\varepsilon_h$ and $\beta_c^2 = -j\omega Z_w C$. Hence, to determine the solution of our problem, we need to solve this coupled system of partial differential equations with unknowns $\Phi$ and $\psi$.

To do this, it is most convenient to work in the Fourier domain. Defining $\tilde{\Phi}$ and $\tilde{\psi}$ as the Fourier transform of $\Phi$ and $\psi$ in the $xy$ plane, respectively, so that

$$\tilde{\Phi} = \iint \Phi e^{j(k_x x + k_y y)} dx dy, \tag{14}$$

and $\tilde{\psi}$ is defined similarly, it follows that

$$\varepsilon_h\frac{\partial}{\partial z}\frac{1}{\varepsilon_h}\frac{\partial\tilde{\Phi}}{\partial z} + \left(k_h^2 - k_t^2\right)\tilde{\Phi} + k_p^2\tilde{\psi} = -p_e\delta(z), \tag{15}$$

$$C\frac{\partial}{\partial z}\frac{1}{C}\frac{\partial}{\partial z}\tilde{\psi} + \left(k_h^2 + \beta_c^2 - k_p^2\right)\tilde{\psi} + k_t^2\tilde{\Phi} = p_{\text{ef}}\delta(z), \tag{16}$$

where $k_t^2 = k_x^2 + k_y^2$. Thus, we have reduced the radiation problem to the solution of a system of linear ordinary differential equations.

### B. The case of a homogeneous medium

Hereafter, we restrict our attention to the particular case of a homogeneous and uniform medium, for which the structural parameters $\varepsilon_h$, $C$, $L$, and $Z_w$ can be assumed to be independent of $z$. In such a case, the system of Eqs. (15) and (16) can be rewritten in a compact matrix notation as follows:

$$\frac{\partial^2}{\partial z^2}\begin{pmatrix}\tilde{\Phi}\\ \tilde{\psi}\end{pmatrix} = \begin{pmatrix}k_t^2 - k_h^2 & -k_p^2\\ -k_t^2 & k_p^2 - k_h^2 - \beta_c^2\end{pmatrix}\cdot\begin{pmatrix}\tilde{\Phi}\\ \tilde{\psi}\end{pmatrix} + \delta(z)\begin{pmatrix}-p_e\\ p_{\text{ef}}\end{pmatrix}. \tag{17}$$

The general solution of the homogeneous problem, when $p_e = p_{\text{ef}} = 0$, can be easily found using standard methods and is given by

$$\begin{pmatrix}\tilde{\Phi}\\ \tilde{\psi}\end{pmatrix} = (C_1^+ e^{-\gamma_{\text{TM}} z} + C_1^- e^{+\gamma_{\text{TM}} z})\begin{pmatrix}k_p^2\\ \gamma_h^2 - \gamma_{\text{TM}}^2\end{pmatrix} + (C_2^+ e^{-\gamma_{\text{qT}} z} + C_2^- e^{+\gamma_{\text{qT}} z})\begin{pmatrix}k_p^2\\ \gamma_h^2 - \gamma_{\text{qT}}^2\end{pmatrix}, \tag{18}$$

where $\gamma_h^2 = k_t^2 - k_h^2$, $C_i^{\pm}$ with $i = 1,2$ are integration constants, and $\gamma_{\mathrm{qT}}$ and $\gamma_{\mathrm{TM}}$ are the propagation constants along the $z$ direction of the so-called quasitransverse electromagnetic (qT) and transverse magnetic (TM) modes supported by the bulk wire medium. These parameters are defined consistently with Refs. 12 and 27 and satisfy

$$\gamma_{\mathrm{TM}} = j\left[k_h^2 - \tfrac{1}{2}\left(k_p^2 + k_t^2 - \beta_c^2 + \sqrt{\left(k_p^2 + k_t^2 - \beta_c^2\right)^2 + 4k_t^2\beta_c^2}\right)\right]^{\frac{1}{2}}, \quad (19)$$

$$\gamma_{\mathrm{qT}} = j\left[k_h^2 - \tfrac{1}{2}\left(k_p^2 + k_t^2 - \beta_c^2 - \sqrt{\left(k_p^2 + k_t^2 - \beta_c^2\right)^2 + 4k_t^2\beta_c^2}\right)\right]^{\frac{1}{2}}. \quad (20)$$

In the case of perfectly conducting wires, we have $Z_w = 0$, and thus $\beta_c = 0$. In such a case the propagation constants of the qT and TM modes reduce to the well-known forms, $\gamma_{\mathrm{qT}} = jk_h$ and $\gamma_{\mathrm{TM}} = \sqrt{k_p^2 + k_t^2 - k_h^2}$, respectively.[6]

Since the solution of Eq. (17) is obviously an even function of $z$, we may try a solution of the form

$$\begin{pmatrix} \tilde{\Phi} \\ \tilde{\psi} \end{pmatrix} = C_{\mathrm{qT}} e^{-\gamma_{\mathrm{qT}}|z|} \begin{pmatrix} k_p^2 \\ \gamma_h^2 - \gamma_{\mathrm{qT}}^2 \end{pmatrix} + C_{\mathrm{TM}} e^{-\gamma_{\mathrm{TM}}|z|} \begin{pmatrix} k_p^2 \\ \gamma_h^2 - \gamma_{\mathrm{TM}}^2 \end{pmatrix}. \quad (21)$$

By direct substitution into Eq. (17), it is readily found that the unknown constants $C_{\mathrm{qT}}$ and $C_{\mathrm{TM}}$ are required to satisfy

$$\gamma_{\mathrm{qT}} C_{\mathrm{qT}} \begin{pmatrix} k_p^2 \\ \gamma_h^2 - \gamma_{\mathrm{qT}}^2 \end{pmatrix} + \gamma_{\mathrm{TM}} C_{\mathrm{TM}} \begin{pmatrix} k_p^2 \\ \gamma_h^2 - \gamma_{\mathrm{TM}}^2 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -p_e \\ p_{\mathrm{ef}} \end{pmatrix}. \quad (22)$$

This yields

$$C_{\mathrm{qT}} = \frac{1}{2\gamma_{\mathrm{qT}}} \frac{\left(\gamma_h^2 - \gamma_{\mathrm{TM}}^2\right) p_e + k_p^2 p_{\mathrm{ef}}}{\gamma_{\mathrm{qT}}^2 - \gamma_{\mathrm{TM}}^2} \frac{1}{k_p^2}, \quad (23)$$

$$C_{\mathrm{TM}} = \frac{1}{2\gamma_{\mathrm{TM}}} \frac{\left(\gamma_h^2 - \gamma_{\mathrm{qT}}^2\right) p_e + k_p^2 p_{\mathrm{ef}}}{\gamma_{\mathrm{TM}}^2 - \gamma_{\mathrm{qT}}^2} \frac{1}{k_p^2}. \quad (24)$$

Substituting this result into Eq. (21), we finally obtain the desired solution:

$$\begin{pmatrix} \tilde{\Phi} \\ \tilde{\psi} \end{pmatrix} = \frac{1}{2\gamma_{\mathrm{qT}}} \frac{\left(\gamma_h^2 - \gamma_{\mathrm{TM}}^2\right) p_e + k_p^2 p_{\mathrm{ef}}}{\gamma_{\mathrm{qT}}^2 - \gamma_{\mathrm{TM}}^2} e^{-\gamma_{\mathrm{qT}}|z|} \begin{pmatrix} 1 \\ \frac{\gamma_h^2 - \gamma_{\mathrm{qT}}^2}{k_p^2} \end{pmatrix} + \frac{1}{2\gamma_{\mathrm{TM}}} \frac{\left(\gamma_h^2 - \gamma_{\mathrm{qT}}^2\right) p_e + k_p^2 p_{\mathrm{ef}}}{\gamma_{\mathrm{TM}}^2 - \gamma_{\mathrm{qT}}^2} e^{-\gamma_{\mathrm{TM}}|z|} \begin{pmatrix} 1 \\ \frac{\gamma_h^2 - \gamma_{\mathrm{TM}}^2}{k_p^2} \end{pmatrix}. \quad (25)$$

The inverse Fourier transform of $\tilde{\Phi}$ is given by

$$\Phi = \frac{1}{(2\pi)^2} \int\int \tilde{\Phi} e^{-j(k_x x + k_y y)} dk_x dk_y = \frac{1}{2\pi} \int_0^{+\infty} \tilde{\Phi}\, J_0(k_t \rho) k_t dk_t, \quad (26)$$

where $J_0$ is the zero-order Bessel function of the first kind, $\rho = \sqrt{x^2 + y^2}$, and in the second identity we used the fact that $\tilde{\Phi}$ is a function of $k_t$. In general, this Sommerfeld-type integral can only be evaluated using numerical methods. Obviously, it is possible to write a similar formula for $\psi$.

### C. Perfectly electric conducting wires

Let us now study what happens when, to a first approximation, the metal can be modeled as a perfect electric conductor (PEC), so that $Z_w \approx 0$. In this situation, Eq. (25) simplifies to

$$\begin{pmatrix} \tilde{\Phi} \\ \tilde{\psi} \end{pmatrix} = \frac{1}{2\gamma_{\mathrm{qT}}} k_p^2 \frac{p_e - p_{\mathrm{ef}}}{k_p^2 + k_t^2} e^{-\gamma_{\mathrm{qT}}|z|} \begin{pmatrix} 1 \\ \frac{k_t^2}{k_p^2} \end{pmatrix} + \frac{1}{2\gamma_{\mathrm{TM}}} \frac{k_t^2 p_e + k_p^2 p_{\mathrm{ef}}}{k_p^2 + k_t^2} e^{-\gamma_{\mathrm{TM}}|z|} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (27)$$

We discuss the two scenarios of interest separately. Let us consider first that $V_{\mathrm{ext}} = 0$, so that the metamaterial is excited solely with the short vertical dipole. In this case, $p_{\mathrm{ef}} = p_e$, and thus we obtain simply

$$\begin{pmatrix} \tilde{\Phi} \\ \tilde{\psi} \end{pmatrix} = p_e \frac{1}{2\gamma_{\mathrm{TM}}} e^{-\gamma_{\mathrm{TM}}|z|} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (28)$$

The corresponding inverse Fourier transforms can be evaluated analytically in a trivial manner. This yields

$$\begin{pmatrix} \Phi \\ \psi \end{pmatrix} = p_e \frac{1}{4\pi r} e^{-jk_{\mathrm{ef}} r} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (29)$$

with $k_{\mathrm{ef}} = \sqrt{k_h^2 - k_p^2}$ and $r = \sqrt{x^2 + y^2 + z^2}$.

The result, Eq. (29), implies two unexpected things. First, despite the anisotropy of the wire medium, the wave fronts are spherical surfaces! Second, the emission of radiation is possible only above the effective plasma frequency of the metamaterial, $\omega_p = k_p/\sqrt{\varepsilon_h \mu_0}$. These two properties result from the surprising fact that the qT mode does not contribute to the radiation field of the short vertical dipole. To explain this, we start by noting that in the PEC case the qT mode is exactly the transverse electromagnetic (TEM) mode with respect to the $z$ direction.[6] On the other hand, the electric Green dyadic in a periodic structure (e.g., a photonic crystal or a metamaterial) can be written as a weighted summation of terms such as $\mathbf{E}_n \otimes \mathbf{E}_n$, where $\mathbf{E}_n$ stands for a generic natural mode of the system and $\otimes$ represents the tensor product.[28] In particular, this implies that a TEM mode (with respect to the $z$ direction) cannot contribute to the field radiated by a short vertical dipole, because its contribution would be proportional to $\mathbf{E}_{\mathrm{TEM}} (\mathbf{E}_{\mathrm{TEM}} \cdot \hat{\mathbf{z}})$, whereas for a TEM mode $\mathbf{E}_{\mathrm{TEM}} \cdot \hat{\mathbf{z}} = 0$. Even though this discussion applies to the microscopic electromagnetic fields (before homogenization on the scale of the lattice constant), it clearly indicates that the TEM mode cannot contribute as well to the radiated field in the framework of a macroscopic theory, in agreement with Eq. (29). It is interesting to note that in the presence of loss the contribution of the qT mode to the radiation field does not vanish [the first addend in Eq. (25) does not vanish when $p_e = p_{\mathrm{ef}}$], which is fully consistent with the microscopic theory, because in the case of loss the electric field associated

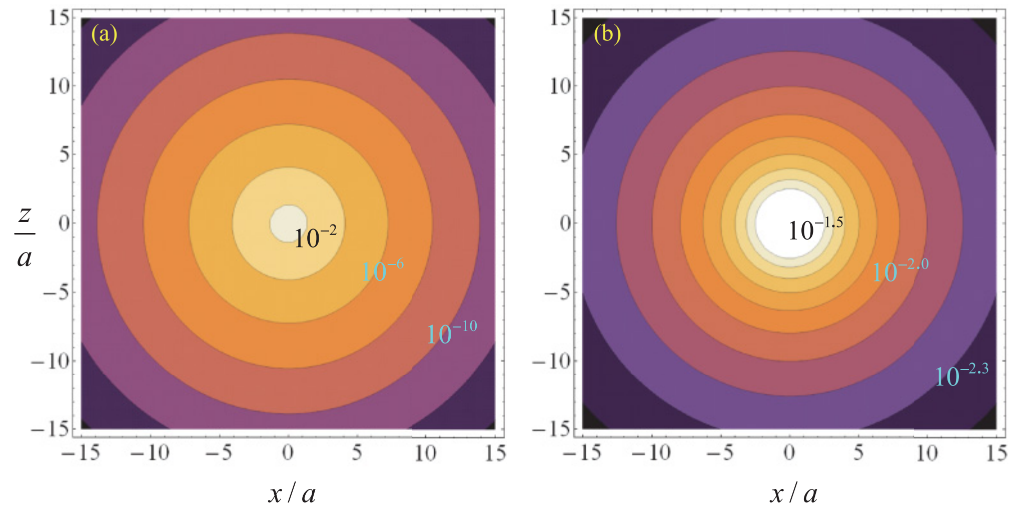


FIG. 2. (Color online) Contour plots of the amplitude of the potential $\Phi$ (arbitrary logarithmic unities). (a) $\omega a/c = 0.5$. (b) $\omega a/c = 1.5$. The wire medium is formed by PEC wires with $r_w = 0.01a$ standing in a vacuum and is excited by a short vertical dipole.

with the qT mode has a small longitudinal component along the $z$ direction.

To illustrate variation in space of the potential $\Phi$, we plot in Fig. 2 the contour plots of $\Phi$ for two different frequencies of operation. The wire medium is formed by PEC wires with $r_w = 0.01a$ standing in a vacuum. The plasma wave number of the effective medium is $k_p = 1.38/a$. Thus, the example in Fig. 2(a) corresponds to a frequency below $\omega_p$, whereas the example in Fig. 2(b) corresponds to a frequency above $\omega_p$. This explains that in the former case the potential is strongly localized in the vicinity of the dipole, whereas in the latter case the potential decays much more slowly as $1/r$. At first glance, the result, Eq. (29), could suggest that the electric-field radiation pattern should be similar to that of a Hertz dipole standing in a homogeneous isotropic plasma with a Drude dispersion. As shown in Sec. VI, this is not true.

Next, we consider the case $p_e = 0$, so that the wire medium is excited by a lumped voltage source [Fig. 1(b)]. In this scenario, the contribution of the TEM mode to the radiation field does not vanish. Indeed, the inverse Fourier transform of the first term on the right-hand side of Eq. (27) can be readily calculated and is equal to

$$\Phi_{\mathrm{qT}} = -p_{\mathrm{ef}} \frac{1}{2\gamma_{\mathrm{qT}}} e^{-\gamma_{\mathrm{qT}}|z|} \frac{k_p^2}{2\pi} K_0(k_p \rho), \qquad (30)$$

where $K_0$ is the modified Bessel function of the second kind. On the other hand, the auxiliary potential, Eq. (11), associated with the wire current satisfies $\psi_{\mathrm{qT}} = -\frac{1}{k_p^2}\nabla_t^2 \Phi_{\mathrm{qT}}$.

The result, Eq. (30), is quite remarkable, because it predicts that the Hertz potential, and hence the electromagnetic fields, varies with $z$ simply as $e^{-\gamma_{\mathrm{qT}}|z|} = e^{-jk_h|z|}$, and hence the radiated field is simply guided along $z$, without any form of decay or diffraction. Moreover, $\Phi_{\mathrm{qT}}$ is strongly localized in the vicinity of the $z$ axis, within a spatial region whose characteristic diameter is determined by $\lambda_p = 2\pi/k_p$. It should be mentioned that $\Phi_{\mathrm{qT}}$ is actually singular over the $z$ axis (it has a logarithmic singularity). Such a singularity occurs because of the adopted $\delta$-function model for the lumped voltage generator. The singularity disappears if one considers a less localized model for the discrete source, e.g., if $\delta(x,y,z)$ is replaced by $g(\rho)\delta(z)$, where $g$ is some function of $\rho$ concentrated near the origin. Even for such a source, the electromagnetic fields are characterized by a diffraction-free pattern. This is related to the "canalization" properties of the wire medium, which enable the transport of the near field with no diffraction.[27,29] It should be mentioned that local indefinite (hyperbolic) materials[30] may also enable a "canalization" effect, however, unlike PEC wire media, they do not support diffraction-free beam propagation. As far as we could check, the inverse Fourier transform of the second addend in Eq. (27), i.e., the contribution of the TM mode when $p_e = 0$, cannot be written in terms of the standard special functions, and hence it needs to be calculated numerically using Eq. (26). In Fig. 3, we plot the contour plots of $\Phi$, for the same example as in Fig. 2. It is shown that when $\omega a/c = 0.5$ [Fig. 3(a)], i.e., below the effective plasma frequency, the fields are strongly concentrated close to the $z$ axis and are guided away from the source with no diffraction. The contribution from the TM mode appears to be residual. On the other hand, above the plasma frequency [Fig. 3(b)], there are clearly two distinct emission channels, one associated with the TEM mode and the other with the TM mode.

The results of this section reveal that the spatially dispersive properties of wire media imply a dual behavior of the electromagnetic response. For some excitations, the wire medium has a response that is more close to that of a plasma with negative permittivity. For example, if the source is a short vertical dipole, the emission of radiation is only possible above an effective plasma frequency. On the other hand, for other excitations (e.g., lumped voltage source) the wire medium behaves as a material with extreme anisotropy, with $\varepsilon_{zz} = -\infty$, and enables diffraction-free wave propagation. Thus, depending on the excitation, the metamaterial reveals different electromagnetic characteristics. This dual behavior makes the metamaterial response quite unique and totally different from that of any local material.

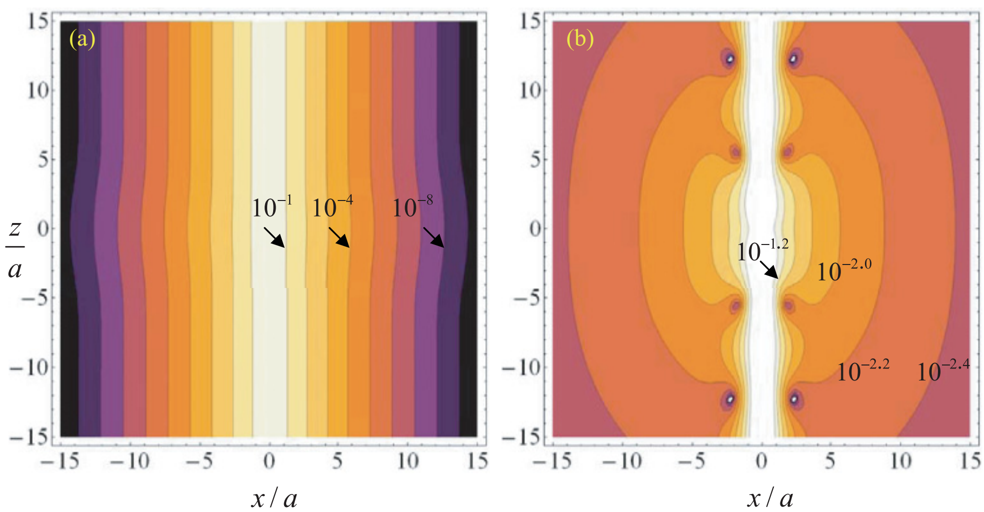


FIG. 3. (Color online) Contour plots of the amplitude of the potential $\Phi$ (arbitrary logarithmic unities). (a) $\omega a/c = 0.5$. (b) $\omega a/c = 1.5$. The wire medium is formed by PEC wires with $r_w = 0.01a$ standing in a vacuum and is excited by a lumped voltage generator. In (b) an interference pattern of the TEM and TM modes is observed.

## IV. NONLOCAL DIELECTRIC FUNCTION APPROACH

The objective of this section is to prove that the radiation problem in an unbounded uniform structure can be as well solved using the standard nonlocal dielectric function formalism.[6,12,17] The case of a stratified structure, which can be easily handled with the theory of Sec. III, is out of reach of the nonlocal framework (it could, however, be handled with a combination of mode matching and additional boundary conditions,[11,26] but a detailed discussion of it is out of the scope of this paper).

In a spatially dispersive medium, the Maxwell equations may be written in a compact form in the space domain as follows:

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H}, \tag{31}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_{\text{ext}} + j\omega\overline{\overline{\varepsilon}}(\omega, j\nabla)\cdot\mathbf{E}, \tag{32}$$

where the dyadic operator $\overline{\overline{\varepsilon}}(\omega, j\nabla)$ represents the effective dielectric function of the material. Note that in the space domain the effective dielectric function should be regarded as a function of the gradient $\nabla$. This contrasts with the formulation in Sec. II, where all the constitutive parameters are independent of $\nabla$. It is also possible to write the term $\overline{\overline{\varepsilon}}(\omega, j\nabla)\cdot\mathbf{E}$ as a spatial convolution.[24] In the spectral (Fourier) domain, in which $j\nabla \leftrightarrow \mathbf{k}$, in the particular case of a uniaxial wire medium formed by straight wires, the effective dielectric function is[6,12,17]

$$\frac{\overline{\overline{\varepsilon}}(\omega,\mathbf{k})}{\varepsilon_h} = \overline{\overline{I}} - \frac{k_p^2\hat{\mathbf{z}}\hat{\mathbf{z}}}{k_h^2 - j\xi k_h - k_z^2}, \tag{33}$$

where $\xi = (Z_w/L)\sqrt{\varepsilon_h\mu_0}$, and the rest of the symbols are defined as in Sec. II. Note that the effective dielectric function depends explicitly on $k_z \leftrightarrow j\frac{\partial}{\partial z}$.

Despite the apparently complicated form of Eqs. (31) and (32), the radiation problem can be readily solved in the spectral domain in the case of an unbounded uniform structure. Indeed, by calculating the Fourier transform of both sides of Eqs. (31) and (32) with respect to all the space coordinates, so that $j\nabla \leftrightarrow \mathbf{k}$, it is readily found that

$$\mathbf{k}\times\mathbf{E}(\omega,\mathbf{k}) = \omega\mu_0\mathbf{H}(\omega,\mathbf{k}), \tag{34}$$

$$\mathbf{k}\times\mathbf{H}(\omega,\mathbf{k}) = -\omega\overline{\overline{\varepsilon}}(\omega,\mathbf{k})\cdot\mathbf{E}(\omega,\mathbf{k}) - \omega\mathbf{P}_{\text{ext}}(\omega,\mathbf{k}), \tag{35}$$

where $j\omega\mathbf{P}_{\text{ext}}(\omega,\mathbf{k}) = \mathbf{J}_{\text{ext}}(\omega,\mathbf{k})$ is the Fourier-transformed source term. After some straightforward manipulations, we find that the Fourier transform of the electric field is

$$\mathbf{E} = j\omega\mu_0[\omega^2\mu_0\overline{\overline{\varepsilon}}(\omega,\mathbf{k}) + \mathbf{kk} - k^2\overline{\overline{\mathbf{I}}}]^{-1}\cdot\mathbf{J}_{\text{ext}}, \tag{36}$$

and hence the electric field in the space domain can be formally written as

$$\mathbf{E}(\mathbf{r}) = \frac{j\omega\mu_0}{(2\pi)^3}\int[\omega^2\mu_0\overline{\overline{\varepsilon}}(\omega,\mathbf{k}) + \mathbf{kk} - k^2\overline{\overline{\mathbf{I}}}]^{-1} \cdot\mathbf{J}_{\text{ext}}(\mathbf{k})e^{-j\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{k}. \tag{37}$$

Note that, at least *a priori*, in the nonlocal dielectric function framework we can only consider excitations based on an external density of current [Fig. 1(a)]. The characterization of the excitation based on a lumped voltage source requires the knowledge of internal degrees of freedom of the wire medium (e.g., the current along the wires and the additional potential), which are not described by the effective medium model. Nevertheless, in the following we show that a lumped source $V_{\text{ext}}$ can also be modeled by a suitable equivalent $\mathbf{J}_{\text{ext}}$.

Next, we obtain the solution of the radiation problem when $\mathbf{J}_{\text{ext}}(\mathbf{r}) = j\omega p_e\hat{\mathbf{z}}\delta(\mathbf{r})$ or, equivalently, when $\mathbf{P}_{\text{ext}}(\omega,\mathbf{k}) = p_e\hat{\mathbf{z}}$. Instead of attempting to calculate integral (37) directly, we instead solve Eqs. (34) and (35) by introducing the Hertz potential $\mathbf{\Pi}_e$. In this manner, we write the Fourier-transformed fields as follows:

$$\mathbf{E} = \omega^2\varepsilon_h\mu_0\mathbf{\Pi}_e - \mathbf{k}(\mathbf{k}\cdot\mathbf{\Pi}_e), \tag{38}$$

$$\mathbf{H} = \omega\varepsilon_h\mathbf{k}\times\mathbf{\Pi}_e. \tag{39}$$

This form immediately satisfies Eq. (34). From Eq. (35) we find

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{\Pi}_e) = -\frac{\overline{\overline{\varepsilon}}}{\varepsilon_h}\cdot\left[k_h^2\mathbf{\Pi}_e - \mathbf{k}(\mathbf{k}\cdot\mathbf{\Pi}_e)\right] - \frac{p_e\hat{\mathbf{z}}}{\varepsilon_h}. \quad (40)$$

Using Eq. (33), after some trivial vector algebra, we obtain

$$\left(k_h^2-k^2\right)\mathbf{\Pi}_e = -\chi_{zz}\hat{\mathbf{z}}\hat{\mathbf{z}}\cdot\left[k_h^2\mathbf{\Pi}_e - \mathbf{k}(\mathbf{k}\cdot\mathbf{\Pi}_e)\right] - \frac{p_e\hat{\mathbf{z}}}{\varepsilon_h}, \quad (41)$$

where

$$\chi_{zz} = -\frac{k_p^2}{k_h^2 - j\xi k_h - k_z^2}. \quad (42)$$

Calculating the vector product of Eq. (41) by $\hat{\mathbf{z}}$ we find that

$$\hat{\mathbf{z}}\times\mathbf{\Pi}_e = 0, \quad (43)$$

and therefore, $\mathbf{\Pi}_e = \Pi_z\hat{\mathbf{z}} \equiv \hat{\mathbf{z}}\Phi/\varepsilon_h$. From this and Eq. (41),

$$\left[k_h^2 - k^2 + \chi_{zz}\left(k_h^2-k_z^2\right)\right]\Phi = -p_e. \quad (44)$$

For PEC wires $\xi = 0$, and thus from Eq. (42) we have

$$\left(k_h^2 - k_p^2 - k^2\right)\Phi = -p_e, \quad (45)$$

from which

$$\Phi(\omega,\mathbf{k}) = -\frac{p_e}{\left(k_h^2-k_p^2-k^2\right)}, \quad (46)$$

and thus,

$$\Phi(\omega,\mathbf{r}) = p_e\frac{e^{-j\sqrt{k_h^2-k_p^2}\,r}}{4\pi r}, \quad (47)$$

which is the same as Eq. (29).

In the general case in which the metal has a plasmonic-type response, $\xi \neq 0$. Introducing the notation $\beta_c^2 = -j\xi k_h$, we obtain from Eq. (44)

$$\begin{aligned}\Phi(\omega,\mathbf{k}) &= -p_e\frac{k_h^2+\beta_c^2-k_z^2}{\left(k_h^2-k^2\right)\left(k_h^2+\beta_c^2-k_z^2\right) - k_p^2\left(k_h^2-k_z^2\right)}\\ &= -p_e\frac{k_h^2+\beta_c^2-k_z^2}{\left(k_z^2+\gamma_{\mathrm{TM}}^2\right)\left(k_z^2+\gamma_{\mathrm{qT}}^2\right)},\end{aligned} \quad (48)$$

where $\gamma_{\mathrm{TM}}$ and $\gamma_{\mathrm{qT}}$ are given by Eqs. (19) and (20).

Calculating the inverse Fourier transform with respect to $k_z$, we find

$$\Phi(\omega,\mathbf{k}_t,z) = \frac{p_e}{2}\left(\frac{k_h^2+\beta_c^2+\gamma_{\mathrm{qT}}^2}{\gamma_{\mathrm{qT}}\left(\gamma_{\mathrm{qT}}^2-\gamma_{\mathrm{TM}}^2\right)}e^{-\gamma_{\mathrm{qT}}|z|} + \frac{k_h^2+\beta_c^2+\gamma_{\mathrm{TM}}^2}{\gamma_{\mathrm{TM}}\left(\gamma_{\mathrm{TM}}^2-\gamma_{\mathrm{qT}}^2\right)}e^{-\gamma_{\mathrm{TM}}|z|}\right). \quad (49)$$

At first glance, this result looks different from Eq. (25), but one may verify that $k_h^2+\beta_c^2+\gamma_{\mathrm{qT}}^2 = (\gamma_h^2-\gamma_{\mathrm{TM}}^2)+k_p^2$, and, similarly, $k_h^2+\beta_c^2+\gamma_{\mathrm{TM}}^2 = (\gamma_h^2-\gamma_{\mathrm{qT}}^2)+k_p^2$. Thus, we recover Eq. (25) with $p_e = p_{\mathrm{ef}}$, which corresponds to the case $V_{\mathrm{ext}} = 0$, consistent with our assumptions at the beginning of this section.

Surprisingly, the lumped voltage source $V_{\mathrm{ext}}$ in Eqs. (3) and (4) can be equivalently represented *within the nonlocal dielectric function model* with some distributed current density $\mathbf{J}_{\mathrm{ext},V}$ in the unbounded wire medium. To show this, we consider the Fourier-transformed equations, (3) and (4) (for simplicity, we let $Z_w = 0$), from which the Fourier-transformed current $I(\omega,\mathbf{k})$ can be expressed as

$$I(\omega,\mathbf{k}) = -j\omega\varepsilon_h A_c\frac{k_p^2}{k_h^2-k_z^2}[E_z(\omega,\mathbf{k}) + V_{\mathrm{ext}}A_c]. \quad (50)$$

When this expression is substituted into the Fourier-transformed equations (1) and (2), the $E_z$-proportional term in Eq. (50) is combined with the term $j\omega\varepsilon_h\mathbf{E}$, which results in the spatially dispersive permittivity (33), and the $V_{\mathrm{ext}}$-proportional term occurs as an additional external current density,

$$\mathbf{J}_{\mathrm{ext},V}(\omega,\mathbf{k}) = -j\omega\varepsilon_h V_{\mathrm{ext}}\frac{k_p^2A_c}{k_h^2-k_z^2}\hat{\mathbf{z}}. \quad (51)$$

Therefore, applying the inverse Fourier transform, we find that

$$\mathbf{J}_{\mathrm{ext},V}(\omega,\mathbf{r}) = \frac{k_p^2A_cV_{\mathrm{ext}}e^{-jk_h|z|}}{2\eta_h}\delta(x,y)\hat{\mathbf{z}}, \quad (52)$$

where $\eta_h = \sqrt{\mu_0/\varepsilon_h}$. Thus, a lumped voltage source inserted into a wire of the unbounded uniaxial wire medium (with PEC wires, $Z_w = 0$) may be equivalently represented with a line of $z$-directed wave-like current, Eq. (52). It is curious to note that while the lumped voltage source excitation is localized at the origin, the equivalent current density is distributed over the entire $z$ axis. At first sight, this may look inconsistent with causality. However, it is simple to verify that such a current is just a wave emerging from the discontinuity point at $z = 0$. Indeed, if one calculates the inverse Fourier transform of Eq. (52) with respect to time, it is found that

$$\mathbf{J}_{\mathrm{ext},V}(t,\mathbf{r}) = k_p^2\frac{A_c}{2\eta_h}\tilde{V}_{\mathrm{ext}}\left(t-\frac{|z|}{v_h}\right)\hat{\mathbf{z}}\delta(x,y), \quad (53)$$

with $v_h = 1/\sqrt{\varepsilon_h\mu_0}$ the velocity of propagation in the host material and $\tilde{V}_{\mathrm{ext}}(t)$ the inverse Fourier transform of $V_{\mathrm{ext}}(\omega)$. The above formula is manifestly consistent with causality, because the excitation at a given point $z$ only depends on the excitation at the origin with a delay $|z|/v_h$.

## V. ENERGY CONSERVATION IN THE UNIAXIAL WIRE MEDIUM AND POYNTING THEOREM

In what follows, we prove that the framework in Sec. II based on the introduction of additional variables enables formulating an energy conservation theorem and defining a Poynting vector in the uniaxial wire medium. We start with Eqs. (1)–(4) written in the time domain. The host permittivity $\varepsilon_h$ is assumed to be dispersionless and lossless, and the wires are modeled by a self-impedance of the form $Z_w(\omega) = j\omega L_{\mathrm{kin}} + R$, where the parameters $L_{\mathrm{kin}}$ and $R$ are independent of frequency. For metallic wires with radius $r_w$ standing in air and described by the Drude model with plasma frequency $\omega_m$ and collision frequency $\Gamma$, these parameters may be estimated as $L_{\mathrm{kin}} = 1/(\varepsilon_0\pi r_w^2\omega_m^2)$ and $R = \Gamma/(\varepsilon_0\pi r_w^2\omega_m^2)$ when $\varepsilon_h = \varepsilon_0$.

Thus, in the time domain Eqs. (1)–(4) may be written as

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \tag{54}$$

$$\nabla \times \mathbf{H} = \varepsilon_h \frac{\partial \mathbf{E}}{\partial t} + \frac{I}{A_c}\hat{\mathbf{z}} + \mathbf{J}_{\text{ext}}, \tag{55}$$

and

$$\frac{\partial \varphi_w}{\partial z} = -(L + L_{\text{kin}})\frac{\partial I}{\partial t} - RI + E_z + \mathcal{E}_{\text{ext}}, \tag{56}$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial \varphi_w}{\partial t}, \tag{57}$$

where $\mathcal{E}_{\text{ext}}$ is the effective EMF of the voltage sources inserted into the wires, per unit length of the wires [e.g., for a lumped source $V_{\text{ext}}$ inserted into a wire at $\mathbf{r} = 0$, $\mathcal{E}_{\text{ext}} = V_{\text{ext}} A_c \delta(\mathbf{r})$].

Following a standard procedure, we obtain from Eqs. (54) and (55)

$$\nabla \cdot [\mathbf{E} \times \mathbf{H}] = -\frac{\partial}{\partial t}\left[\frac{\varepsilon_h \mathbf{E}^2}{2} + \frac{\mu_0 \mathbf{H}^2}{2}\right] - \mathbf{E}\cdot\mathbf{J}_{\text{ext}} - \frac{E_z I}{A_c}. \tag{58}$$

On the other hand, from Eqs. (56) and (57) we have

$$\frac{\partial(\varphi_w I)}{\partial z} = -\frac{\partial}{\partial t}\left[\frac{C\varphi_w^2}{2} + \frac{L_{\text{tot}} I^2}{2}\right] - RI^2 + E_z I + \mathcal{E}_{\text{ext}} I, \tag{59}$$

where $L_{\text{tot}} = L + L_{\text{kin}}$.

Diving the last relation by $A_c$ and adding it to Eq. (58), we obtain the conservation law

$$\nabla \cdot \mathbf{S} = -\frac{\partial W}{\partial t} - P_{\text{loss}} + P_{\text{ext}}, \tag{60}$$

where

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} + \frac{\varphi_w I}{A_c}\hat{\mathbf{z}}, \tag{61}$$

$$W = \frac{\varepsilon_h \mathbf{E}^2}{2} + \frac{\mu_0 \mathbf{H}^2}{2} + \frac{C\varphi_w^2}{2A_c} + \frac{L_{\text{tot}} I^2}{2A_c}, \tag{62}$$

$$P_{\text{loss}} = \frac{RI^2}{A_c}, \tag{63}$$

$$P_{\text{ext}} = \frac{\mathcal{E}_{\text{ext}} I}{A_c} - \mathbf{E}\cdot\mathbf{J}_{\text{ext}}. \tag{64}$$

The vectorial quantity $\mathbf{S}$ in Eqs. (60) and (61) may be understood as the Poynting vector in the uniaxial wire medium, and $P_{\text{ext}}$ as the volume density of the power transferred by the external sources to the medium. In the absence of loss, i.e., when $R = 0$, the term $W$ is univocally identified with the density of stored energy. In contrast, if loss is present, then it is generally impossible to separate the energy storage rate from the energy loss rate when a metamaterial is considered *macroscopically*.

However, if *the microstructure* of a metamaterial is known, the stored energy can be found from a consistent physical model that fully describes the processes within a unit volume of the metamaterial. Thus, if we assume that the Drude model is such a consistent model for the dynamics of the free electron plasma in metals, then Eq. (62) preserves the meaning of the stored energy density even when $R > 0$. In this case, the quantity $P_{\text{loss}}$ has the physical meaning of an instantaneous power loss density.

Evidently, in a time-harmonic regime the time-averaged Poynting vector is given by

$$\mathbf{S}_{\text{av}} = \frac{1}{2}\text{Re}\left\{\mathbf{E} \times \mathbf{H}^* + \frac{\varphi_w I^*}{A_c}\hat{\mathbf{z}}\right\}. \tag{65}$$

It can be checked that in the lossless case (Re$\{Z_w\} = 0$), and for the case of fields with a spatial dependence of the form $e^{-j\mathbf{k}\cdot\mathbf{r}}$ with $\mathbf{k}$ real-valued, this reduces to the formula

$$\mathbf{S}_{\text{av},l} = \frac{1}{2}\text{Re}\{(\mathbf{E} \times \mathbf{H}^*)_l\} - \frac{\omega}{4}\mathbf{E}^* \cdot \frac{\partial \overline{\overline{\varepsilon}}}{\partial k_l}(\omega,\mathbf{k}) \cdot \mathbf{E}, \tag{66}$$

with $l = x,y,z$ and $\overline{\overline{\varepsilon}}$ defined as in Eq. (33), which is applicable to plane waves in general lossless spatially dispersive media.[24,25,31] The application of the above formula to wire media has been considered in several works.[10,32]

## VI. RADIATION PATTERN IN THE PEC CASE

Next, we obtain the radiation pattern, directive gain, directivity, and radiation resistance for the case of a short vertical dipole radiating in a wire medium formed by PEC wires [Fig. 1(a)]. We do not discuss in detail the case wherein the metamaterial is excited by a lumped voltage source because, as discussed in Sec. III C, in such a scenario the radiated field is guided along the $z$ axis with no decay. In particular, this implies immediately that the directivity in such a configuration is infinite.

### A. Asymptotic form of the radiated fields

To begin with, we obtain the asymptotic form of the field radiated by a short vertical dipole ($V_{\text{ext}} = 0$) embedded in a wire medium formed by PEC wires when $r \to \infty$. Evidently, from the results in Sec. III C, unless the frequency of operation is higher than the plasma frequency of the effective medium, the radiated fields will decay exponentially away from the source. Hence, in what follows we assume that $\omega > \omega_p = k_p/\sqrt{\mu_0 \varepsilon_h}$, so that $k_{\text{ef}} > 0$ in Eq. (29). Substituting Eq. (29) into Eqs. (6) and (7), it can be easily checked that

$$\mathbf{H} \doteq -\omega k_{\text{ef}} \Phi \sin\theta\, \hat{\varphi}, \tag{67}$$

$$\mathbf{E} \doteq k_h^2 \frac{\Phi}{\varepsilon_h}\left[\left(1 - \frac{k_{\text{ef}}^2}{k_h^2}\right)\cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\theta}\right], \tag{68}$$

where the symbol $\doteq$ indicates that the identities are asymptotic ($r \to \infty$), $\Phi = p_e \frac{1}{4\pi r} e^{-jk_{\text{ef}} r}$, and $(\hat{\mathbf{r}},\hat{\theta},\hat{\varphi})$ define an orthogonal reference system associated with the usual spherical coordinate system $(r,\theta,\varphi)$. As shown, unlike what happens in an isotropic medium, the electric far field has a radial component. The amplitude of the electromagnetic fields varies asymptotically as $1/r$, and $E_\theta = \eta_{\text{ef}} H_\varphi$ with $\eta_{\text{ef}} = \omega\mu_0/k_{\text{ef}}$.

Similarly, substituting Eq. (29) into Eqs. (3), (4), and (11), it is found that the asymptotic forms of the current and additional potential are

$$I \doteq -j\omega A_c k_p^2 \Phi, \tag{69}$$

$$\varphi_w \doteq I\frac{k_{\text{ef}}}{\omega C}\cos\theta \doteq -\frac{j\,k_{\text{ef}}}{\varepsilon_h}\cos\theta\,\Phi. \tag{70}$$

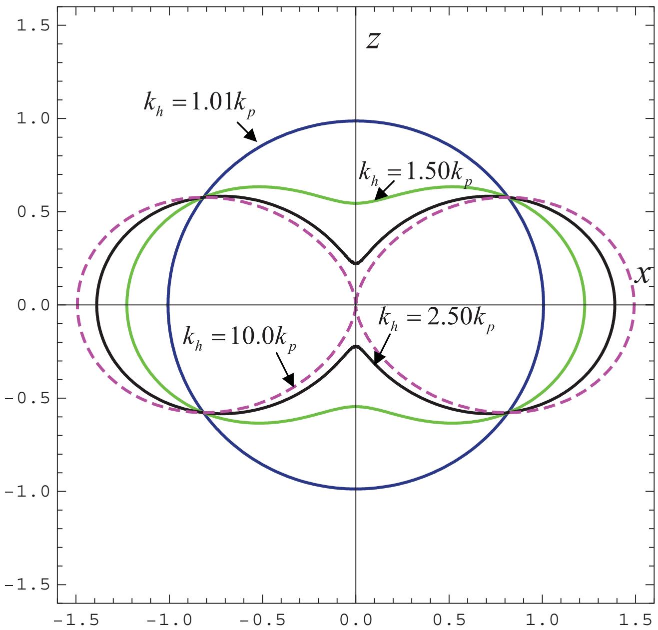


FIG. 4. (Color online) Polar plot of the directive gain of a short vertical dipole embedded in the uniaxial wire medium for different frequencies of operation ($\omega = k_h/\sqrt{\varepsilon_h \mu_0}$).

Thus, from Eqs. (68) and (70), we see that the time-averaged Poynting vector, Eq. (65), in the far field is

$$\mathbf{S}_{\rm av} \doteq \frac{1}{2}\omega^3\mu_0|\Phi|^2 k_{\rm ef}\sin\theta\left(\frac{k_p^2}{k_h^2}\cos\theta\,\hat{\theta} + \sin\theta\,\hat{\mathbf{r}}\right) + \frac{1}{2}\frac{k_{\rm ef}}{\varepsilon_h}k_p^2\omega\cos\theta\,|\Phi|^2\,\hat{\mathbf{z}}. \quad (71)$$

Straightforward calculations show that the Poynting vector only has a radial component:

$$\mathbf{S}_{\rm av} \doteq \frac{1}{2}k_{\rm ef}\omega^3\mu_0|\Phi|^2\left(\sin^2\theta + \frac{k_p^2}{k_h^2}\cos^2\theta\right)\hat{\mathbf{r}}. \quad (72)$$

Hence, in part surprisingly, it follows that a short vertical dipole embedded in a wire medium *can* radiate energy along the direction of vibration, i.e., along the $z$ axis! Moreover, in the limit $\omega \to \omega_p$ the radiation pattern becomes isotropic: $\mathbf{S}_{\rm av} \approx \frac{1}{2}k_{\rm ef}\omega^3\mu_0\,|\Phi|^2\,\hat{\mathbf{r}}$. Note, however, that for $\omega = \omega_p$, we have $k_{\rm ef} = 0$ and thus the Poynting vector vanishes in the far field. However, slightly above $\omega_p$ the emission of radiation is certainly possible. Note also that in the limit where $k_p \to 0$, we recover the far field of a short vertical dipole embedded in a dielectric with permittivity $\varepsilon_h$.

### B. Radiation intensity, directive gain, directivity, and radiation resistance

The radiation intensity of the short vertical dipole, $U = \lim_{r\to\infty} r^2\mathbf{S}_{\rm av}$, is given by

$$U = \frac{|p_e|^2}{32\pi^2}k_{\rm ef}\omega^3\mu_0\left(\sin^2\theta + \frac{k_p^2}{k_h^2}\cos^2\theta\right). \quad (73)$$

Hence, the power radiated by the dipole, $P_{\rm rad} = \int U d\Omega = 2\pi\int U\sin\theta d\theta$, is such that

$$P_{\rm rad} = \frac{|p_e|^2}{12\pi}k_{\rm ef}\omega^3\mu_0\left(1 + \frac{k_p^2}{2k_h^2}\right). \quad (74)$$

The directive gain, $g = 4\pi U/P_{\rm rad}$, is

$$g(\theta,\varphi) = \frac{3}{2 + k_p^2/k_h^2}\left(\sin^2\theta + \frac{k_p^2}{k_h^2}\cos^2\theta\right). \quad (75)$$

Since $k_h \geqslant k_p$, it can be checked that the direction of maximal radiation is $\theta = \pi/2$. The directivity of the short vertical dipole is, thus,

$$D = \frac{3}{2 + k_p^2/k_h^2}, \quad (76)$$

which therefore increases from unity (for $k_h \approx k_p$) up to 3/2 in the limit $k_h \gg k_p$.

In Fig. 4 we show a polar plot of the directive gain of the short vertical dipole for different frequencies of operation, normalized to the effective plasma frequency. In agreement with the previous discussion, it can be seen that the radiation pattern becomes more directive for increasing values of the frequency and that, for $\omega \approx \omega_p$, the radiator resembles an isotropic radiator.

To conclude, we note that if the dipole is fed by a current $I_0$ and has infinitesimal height $dl$, then the corresponding dipole moment is such that $\omega\,|p_e| = |I_0|\,dl$. Thus, it follows that the radiation resistance ($R_{\rm rad} = 2P_{\rm rad}/\,|I_0|^2$) of such an elementary source is given by

$$R_{\rm rad} = \frac{(dl)^2}{6\pi}k_{\rm ef}\omega\mu_0\left(1 + \frac{k_p^2}{2k_h^2}\right) = \eta_h\frac{(dl)^2}{6\pi}k_{\rm ef}k_h\left(1 + \frac{k_p^2}{2k_h^2}\right), \quad (77)$$

where $\eta_h = \sqrt{\mu_0/\varepsilon_h}$ is the impedance of the host material.

## VII. CONCLUSION

In this work we have studied the radiation of two types of elementary sources embedded in a uniaxial wire medium and derived a general energy conservation theorem. The main challenge of the radiation problem is related to the metamaterial being spatially dispersive. We have shown that the radiation problem can be solved by considering either a nonlocal dielectric function framework or, alternatively, a framework based on the introduction of additional variables where the medium response may be regarded as local. However, only the latter approach enables considering stratified media and calculating quantities such as the Poynting vector or the directive gain. It was shown that the emission of radiation by a short dipole in a wire medium has several anomalous features, such as a uniform directive gain near the effective plasma frequency. On the other hand, the radiation by a lumped voltage generator results in a nondiffractive beam that is localized in the vicinity of the $z$ axis and corresponds to an infinite directivity.

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[1] J. Brown, Proceedings of the IEE—Part IV: Institution Monographs **100**(5), 51 (1953).
[2] W. Rotman, IRE Trans. Antennas Propag. **10**, 82 (1962).
[3] J. B. Pendry, A. J. Holden, W. J. Stewart, and I. Youngs, Phys. Rev. Lett. **76**, 4773 (1996).
[4] S. I. Maslovski, S. A. Tretyakov, and P. A. Belov, Microw. Opt. Techn. Lett. **35**, 47 (2002).
[5] A. L. Pokrovsky and A. L. Efros, Phys. Rev. B **65**, 045110 (2002).
[6] P. A. Belov, R. Marques, S. I. Maslovski, I. S. Nefedov, M. Silveirinha, C. R. Simovski, and S. A. Tretyakov, Phys. Rev. B **67**, 113103 (2003).
[7] M. G. Silveirinha and C. A. Fernandes, IEEE Trans. Microwave Theory Tech. **53**, 1418 (2005).
[8] C. R. Simovski and P. A. Belov, Phys. Rev. E **70**, 046616 (2004).
[9] I. S. Nefedov, A. J. Viitanen, and S. A. Tretyakov, Phys. Rev. E **71**, 046612 (2005).
[10] I. S. Nefedov, A. J. Viitanen, and S. A. Tretyakov, Phys. Rev. B **72**, 245113 (2005).
[11] M. G. Silveirinha, IEEE Trans. Antennas Propag. **54**, 1766 (2006).
[12] M. G. Silveirinha, Phys. Rev. E **73**, 046612 (2006).
[13] M. G. Silveirinha, in *Theory and Phenomena of Artificial Materials*,, edited by F. Capolino, Vol. 1 (CRC Press, Boca Raton, FL, 2009).
[14] M. G. Silveirinha and C. A. Fernandes, Phys. Rev. B **78**, 033108 (2008).
[15] O. Luukkonen, M. G. Silveirinha, A. B. Yakovlev, C. R. Simovski, I. S. Nefedov, and S. A. Tretyakov, IEEE Trans. Microwave Theory Tech. **57**, 2692 (2009).
[16] A. B. Yakovlev, M. G. Silveirinha, O. Luukkonen, C. R. Simovski, I. S. Nefedov, and S. A. Tretyakov, IEEE Trans. Microwave Theory Tech. **57**, 2700 (2009).
[17] S. I. Maslovski and M. G. Silveirinha, Phys. Rev. B **80**, 245101 (2009).
[18] I. S. Nefedov, Phys. Rev. B **82**, 155423 (2010).
[19] I. Nefedov and A. Viitanen, Electronic Lett. **43**, 1206 (2007).
[20] P. Ikonen, M. Karkkainen, C. Simovski, P. Belov, and S. Tretyakov, IEE Proc. Microw. Antennas Propag. **153**, 163 (2006).
[21] G. Lovat, P. Burghignoli, F. Capolino, D. R. Jackson, and D. R. Wilton, IEEE Trans. Antennas Propag. **54**, 1017 (2006).
[22] P. Burghignoli, G. Lovat, F. Capolino, D. R. Jackson, and D. R. Wilton, IEEE Trans. Antennas Propag. **56**, 1329 (2008).
[23] P. Burghignoli, G. Lovat, F. Capolino, D. R. Jackson, and D. R. Wilton, IEEE Trans. Microwave Theory Tech. **58**, 1112 (2008).
[24] V. M. Agranovich and V. Ginzburg, *Spatial Dispersion in Crystal Optics and the Theory of Excitons* (Wiley-Interscience, New York, 1966).
[25] J. T. Costa, M. G. Silveirinha, and A. Alu, Phys. Rev. B **83**, 165120 (2011).
[26] S. I. Maslovski, T. Morgado, M. G. Silveirinha, C. S. R. Kaipa, and A. B. Yakovlev, New J. Phys. **12**, 113047 (2010).
[27] M. G. Silveirinha, P. A. Belov, and C. R. Simovski, Phys. Rev. B **75**, 035108 (2007).
[28] K. Sakoda, *Optical Properties of Photonic Crystals* (Springer-Verlag, Berlin, 2001).
[29] P. A. Belov, Y. Hao, and S. Sudhakaran, Phys. Rev. B **73**, 033108 (2006).
[30] L. P. Felsen and N. Marcuvitz, *Radiation and Scattering of Waves*, IEEE Series on Electromagnetic Theory (IEEE Press, Piscataway, NJ, 1994).
[31] M. G. Silveirinha, Phys. Rev. B **80**, 235120 (2009).
[32] M. G. Silveirinha, New J. Phys. **11**, 113016 (2009).