Casimir forces at the threshold of the Cherenkov effect

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We study the Casimir-Lifshitz forces in a strongly nonreciprocal system: a waveguide filled with a medium moving at a relativistic velocity. In such a waveguide the waves propagate dominantly along a single direction that coincides with the direction of the velocity. Our theory shows that the Casimir forces acting on a piston in such a quasi-one-way waveguide vanish when the velocity approaches the Cherenkov threshold.

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I. INTRODUCTION

Recently, the Casimir-Lifshitz interactions in systems involving moving matter have raised a growing interest. One reason for this is that at the moment there are several phenomenological and semiclassical theories of such forces—resulting from quantum fluctuations of the electromagnetic field in moving media—that sometimes give opposite predictions. For example, some semiclassical theories [1,2] predict the existence of the so-called quantum friction between two moving dielectric slabs separated by a vacuum, even at the absolute zero of temperature. Similar results were obtained within first-order perturbation quantum theories for moving plasmons [3] or particles near dielectric surfaces [4]. Nevertheless, some authors defend that there is no quantum friction at zero temperature between uniformly moving dielectric plates [5] or, in certain cases, particles [6], which shows that there are many questions in this field that remain largely unanswered.

In part, the complications arise from the fact that moving media are nonreciprocal as they are not invariant under time reversal, which would reverse the direction of the flow of matter. Such nonreciprocity may lead to interesting effects. For instance, it has been recently shown [7] that in a nonuniformly moving fluid the quantum fluctuations of the electromagnetic field may result in both attractive and repulsive interactions between the moving layers, depending on the relative velocities and directions of movement of the layers. Another effect—that we exploit in this paper—is the drag of radiation by moving matter. This effect becomes especially important when the velocity of the flow approaches the phase velocity of the electromagnetic waves in the medium, that is, when it approaches the threshold of Cherenkov radiation. When a medium moves at such a high speed, the electromagnetic radiation (as seen by a stationary observer) becomes trapped in the moving matter, so that the wave propagation in the direction opposite to the medium flow becomes forbidden.

Imagine now that one has realized the above conditions in a waveguide environment, that is, the moving fluid is confined by some waveguide. Then, in such a structure the waves propagate mainly in a single direction and, essentially, one obtains a *one-way waveguide*. Of course, filling the waveguide with real moving matter is not absolutely necessary to obtain this property, at least, if one is only interested in operation in a narrow frequency band. For instance, waveguides made of ferrites or magnetized plasma are quite well known. In these structures an applied static magnetic field plays the role of the time-odd parameter, due to which the nonreciprocity occurs. However, as we show in this work, a waveguide filled with weakly dispersive (ideally, nondispersive) moving medium may have the one-way propagation property fulfilled in a very wide range of frequencies, thus allowing for both qualitative and quantitative studies of the effect of this property on the Casimir-Lifshitz interactions in such nonreciprocal structures without resorting to particular models of dispersion in magnetized ferrites, for example.

Puzzling questions immediately come to mind when thinking about possible Casimir-Lifshitz interactions between objects in a one-way waveguide. Can this interaction become "nonreciprocal" in the sense that the Casimir pressure on two identical interacting objects embedded in the moving fluid is different? Will there be a drift force of quantum origin that may pull polarizable particles, for example, along the direction of the flow? The answer to these questions is intrinsically related to the manner in which the fields in such a waveguide are quantized. Within the usual framework of quantum electrodynamics, the Hamiltonian of the electromagnetic field is expressed as a summation over quantum harmonic oscillators that correspond to the classical electromagnetic modes of the system. This framework may be generalized to moving media, provided that the essential nonreciprocity of the electromagnetic processes in such media is properly taken into consideration. In a recently published work [7] such generalization has been performed, and, in the rest of the paper, we use this theory to answer the intriguing questions posed above.

II. THE WAVES IN A WAVEGUIDE FILLED WITH A MOVING MEDIUM

Let us consider a waveguide formed, for example, by two ideally conducting walls (PEC) at $y = \pm b/2$ and two ideally permeable walls (PMC) at $x = \pm a/2$ (Fig. 1) completely filled with a nondispersive moving dielectric fluid. As discussed in what follows, PMC plates do not play any special role here: One may as well consider a standard PEC waveguide, or a coaxial line, etc. Later in the paper we consider in detail one of such configurations. The filling fluid moves with the velocity v along the z axis, which is the axis of the waveguide. The Maxwell equations for the fields in the waveguide as seen in

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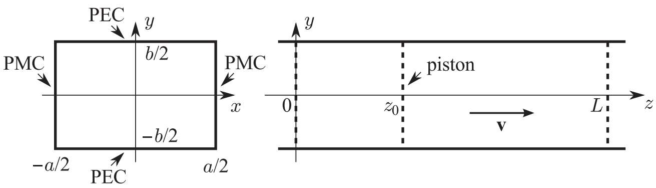

FIG. 1. Geometry of a waveguide filled with a moving dielectric fluid. (Left) The cross section of the waveguide. The waveguide is formed by two PEC plates at $y = \pm b/2$ and two PMC plates at $x = \pm a/2$. (Right) The section of the waveguide in the yz plane. The waveguide is closed with two stationary walls at $z = 0$ and $z = L$. A wall that can slide along Oz (a piston) is placed at $z = z_0$. The walls are assumed penetrable by the fluid, but impenetrable by the electromagnetic field (see further explanations in Sec. III).

the stationary frame can be written as (the time dependence is of the form $e^{-i\omega t}$) [8]

$$i\omega\overline{\overline{\mu}} \cdot \mathbf{H} = \nabla_{\rm t} \times \mathbf{E} + (i\omega\nu + \partial_z)\mathbf{z}_0 \times \mathbf{E}, \tag{1}$$

$$-i\omega\overline{\overline{\varepsilon}} \cdot \mathbf{E} = \nabla_{\rm t} \times \mathbf{H} + (i\omega\nu + \partial_z)\mathbf{z}_0 \times \mathbf{H}, \tag{2}$$

where $\nabla_{\rm t} \equiv \overline{\overline{I}}_{\rm t} \cdot \nabla$, $\partial_z \equiv \partial/\partial z$, $\overline{\overline{\varepsilon}} = \varepsilon_{\rm t}\overline{\overline{I}}_{\rm t} + \varepsilon\mathbf{z}_0\mathbf{z}_0$, $\overline{\overline{\mu}} = \mu_{\rm t}\overline{\overline{I}}_{\rm t} + \mu\mathbf{z}_0\mathbf{z}_0$, where $\overline{\overline{I}}_{\rm t}$ is the unity dyadic in the xy plane and

$$\varepsilon_{\rm t} = \varepsilon\frac{1-\beta^2}{1-n^2\beta^2}, \quad \mu_{\rm t} = \mu\frac{1-\beta^2}{1-n^2\beta^2}, \quad \nu = \frac{\beta}{c}\frac{n^2-1}{1-n^2\beta^2}, \tag{3}$$

where ε and μ are the permittivity and the permeability of the fluid filling the waveguide (at rest), $c = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light in vacuum, $\beta = v/c$, and $n^2 = \varepsilon\mu/(\varepsilon_0\mu_0)$ (see Refs. [7,8]).

Looking for plane wave solutions of Eqs. (1) and (2), it is possible to reduce Eqs. (1) and (2) to

$$\left[\omega^2\varepsilon_{\rm t}\mu_{\rm t} - (k_{z,\rm TM} + \omega\nu)^2 - \frac{\varepsilon_{\rm t}}{\varepsilon}k_{\rm t,TM}^2\right]E_z = 0, \quad ({\rm TM}_z), \tag{4}$$

$$\left[\omega^2\varepsilon_{\rm t}\mu_{\rm t} - (k_{z,\rm TE} + \omega\nu)^2 - \frac{\mu_{\rm t}}{\mu}k_{\rm t,TE}^2\right]H_z = 0, \quad ({\rm TE}_z), \tag{5}$$

where $\mathbf{k} = \mathbf{k}_{\rm t} + k_z\mathbf{z}_0\mathbf{z}_0$, $\mathbf{k}_{\rm t} \equiv \overline{\overline{I}}_{\rm t} \cdot \mathbf{k}$, is the wave vector of a partial wave and Eqs. (4) and (5) are associated with the transverse magnetic polarization with respect to the z axis (TM$_z$), for which $H_z \equiv 0$, and with the transverse electric polarization (TE$_z$), for which $E_z \equiv 0$, respectively. The waveguide supports also a transverse electromagnetic wave (TEM$_z$), with dispersion $\omega^2\varepsilon_{\rm t}\mu_{\rm t} - (k_{z,\rm TEM} + \omega\nu)^2 = 0$, which may be understood as a degenerated TE$_z$ or TM$_z$ wave with $k_{\rm t} = 0$.

The allowed transverse wave numbers of the eigenwaves in the considered waveguide are found by applying the boundary conditions at the walls of the waveguide. In doing so, one obtains

$$(k_{\rm t,TM,TE})_{lm} = \pi\sqrt{\left(\frac{l}{a}\right)^2 + \left(\frac{m}{b}\right)^2}, \tag{6}$$

where l and m are integers such that $l^2 + m^2 > 0$.

The propagation constants $\kappa_\pm$ of the eigenwaves with the z dependence of the form $e^{ik_{z,1}z} = e^{i\kappa_+ z}$ and $e^{ik_{z,2}z} = e^{-i\kappa_- z}$ propagating along the waveguide in the two opposite directions can be obtained from the dispersion Eqs. (4) and (5) (this result holds for both TM$_z$ and TE$_z$ modes):

$$\kappa_\pm = \frac{\sqrt{(1-\beta^2)\left[n^2(1-\beta^2)k_0^2 - (1-n^2\beta^2)k_{\rm t}^2\right]} \mp \beta(n^2-1)k_0}{1-n^2\beta^2}, \tag{7}$$

where $k_0 = \omega/c$. One can see from here that when $\beta \neq 0$ the propagation constants of the oppositely propagating waves differ, which is a manifestation of nonreciprocity of the system under study.

In the scope of the present study, the most interesting situation for us is the limiting case for which the velocity approaches the threshold of the Cherenkov effect: $\beta^2 \to 1/n^2$. However, we do not consider velocities *above* the Cherenkov threshold to avoid any potential instabilities related to the appearance of negative quanta [7] in media moving with such high velocities.

Without any loss of generality we may assume $\beta \geqslant 0$ and $k_0 \geqslant 0$. Then, in the mentioned limit, the propagation constant κ_- tends to infinity, as the numerator of (7) is finite in this limit. However, for the propagation constant κ_+, Eq. (7) results in an underdetermined form $\frac{0}{0}$ which can be resolved with l'Hôpital's rule. This yields a quite simple result:

$$\kappa_+ = \frac{k_0(n^2+1)}{2n} - \frac{k_{\rm t}^2}{2nk_0}. \tag{8}$$

The property $\kappa_- \to \infty$ implies that the waves propagating in our waveguide in the direction opposite to the velocity vector $\mathbf{v}$ have infinitely small phase (and also group) velocity. On the other hand, the phase and group velocities of the wave with propagation constant κ_+ are finite and may be expressed as

$$\frac{v_{\rm ph}}{c} = \frac{k_0}{\kappa_+} = \frac{2n}{1+n^2-k_{\rm t}^2/k_0^2}, \tag{9}$$

$$\frac{v_{\rm gr}}{c} = \frac{dk_0}{d\kappa_+} = \frac{2n}{1+n^2+k_{\rm t}^2/k_0^2}, \tag{10}$$

from which one may notice that, of course, $v_{\rm gr} \leqslant c$, and that $v_{\rm gr} > 0$. Quite differently, $v_{\rm ph}$ can be greater than c in absolute value and may also change sign if $k_{\rm t}$ is sufficiently large. Another interesting feature of the dispersion relation (8) is that there is no cutoff: All such modes are propagating waves starting from zero frequency and independent of the transverse wave number. In other words, at the threshold of the Cherenkov effect the electromagnetic energy of all such waves is dragged by the moving fluid.

To illustrate the dependence of the wave velocities in a waveguide filled with a moving dielectric on $\beta = v/c$, we calculated the phase velocities of the TEM waves ($k_{\rm t} = 0$) propagating in such a waveguide in two opposite directions, using Eq. (7) with different refractive indices of the fluid $n = \sqrt{\varepsilon_{\rm r}}$. In this work, the fluid permittivity is assumed independent of frequency, and, thus, the group velocity of the TEM wave is coincident with the phase velocity. The results are shown in Fig. 2 for three values of the relative permittivity of the fluid: $\varepsilon_{\rm r} = 1.0548$ (this is the permittivity of air under the pressure of 100 atm), $\varepsilon_{\rm r} = 2$, and $\varepsilon_{\rm r} = 4$. As seen, when the velocity approaches the Cherenkov threshold (which is different for

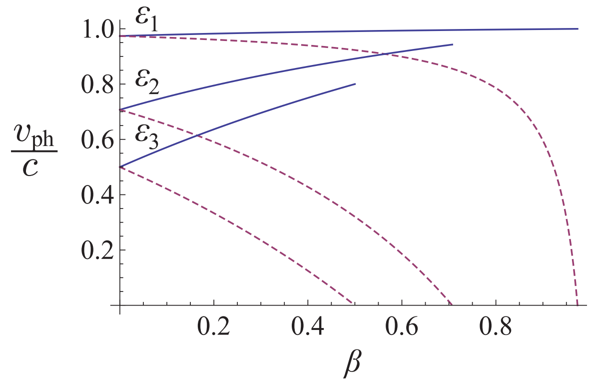

FIG. 2. (Color online) Normalized phase velocity v_{ph}/c of the TEM waves propagating in the waveguide filled with the moving fluid as a function of $\beta = v/c$ (the range of the plots is restricted to velocities below the Cherenkov's threshold). Solid curves, phase velocity of the wave copropagating with the fluid flow; dashed curves, phase velocity of the counterpropagating wave. The velocities are calculated for the following values of the dielectric constant of the fluid: $\varepsilon_1 = 1.0548$, $\varepsilon_2 = 2$, and $\varepsilon_3 = 4$.

fluids with different refractive index) the phase velocity of the counterpropagating wave tends to zero.

From the above analysis we may conclude that a waveguide filled with a medium moving with a velocity *at the threshold* of the Cherenkov effect may be a good physical model of a one-way waveguide, as it provides the required feature of such a waveguide: The waves may propagate only in a single direction along $\mathbf{v}$. Nevertheless, at a velocity *slightly below* the Cherenkov threshold, this is a quite peculiar one-way waveguide, differing in a fundamental aspect from other waveguides based, for example, on magnetized media. Indeed, strictly speaking, propagation of waves in the direction opposite to the medium flow is not really forbidden in a moving medium for velocities below the Cherenkov limit, and, moreover, it can be shown that the density of modes (per unit of frequency and volume) propagating in the direction opposite to $\mathbf{v}$ diverges to infinity as the velocity approaches c/n. However, despite the large number of counterpropagating modes, in practice they cannot be excited because they are associated with extremely large wave vectors (and extremely low group velocities), and hence it is difficult to transfer energy to these modes and ensure the conservation of momentum. In other words, these modes are highly mismatched from any realistic source, and thus in practice only modes copropagating with the medium can be excited near the Cherenkov threshold.

III. ZERO-POINT ENERGY AND CASIMIR'S FORCE AT ZERO TEMPERATURE

We can now move to the study of Casimir forces in waveguides filled with moving media, which, as discussed previously, may behave as one-way waveguides at the Cherenkov threshold. Let us close such a waveguide with two "walls" at $z = 0$ and $z = L$, where $L \gg \max(a,b)$ (Fig. 1). In addition to these two walls, we also place a sliding wall of the same type (a piston) somewhere inside the waveguide at $0 < z_0 < L$. Our purpose is to find the Casimir force acting on this piston. We assume that the walls at $z = 0$, $z = z_0$ (this one is the piston), and $z = L$ are such that they do not block the flow of the fluid, but they block the propagation of the electromagnetic waves. One may imagine such walls as dense metallic grids with holes which allow the fluid to go through. As is known, a dense metallic grid may be nearly as a good reflector for the electromagnetic waves as a sheet of PEC, particularly if the wavelength of radiation is much larger than the characteristic period of the grid [9–11].

It is well known that the Casimir interaction is mostly determined by the normal modes associated with the lowest eigenfrequency values. The upper limit of the range of relevant eigenfrequencies depends on the distance d between the interacting bodies and may be estimated as $\omega_{\max} \sim c/(nd)$. This indicates that provided the distance between the piston and the walls at $z = 0$ and $z = L$ is much larger than the characteristic period of the respective metallic grids then our assumption that these grids can be accurately modeled as PEC surfaces is justified in the context of the Casimir interaction. For instance, using the theory of Ref. [10], which applies to both dense and sparse wire grids, one may verify that in a wire grid with the wire radius of about $1/50$ of the grid period and the period on the order of $1/10$ of the characteristic wavelength, the absolute value of the reflection coefficient is greater than 0.9 and the reflection phase is close to π.

Moreover, because we are interested only in the force due to quantum-electromagnetic fluctuations in the waveguide, we neglect any mechanical friction that may appear between the moving fluid and the walls when the fluid penetrates the holes in the walls. At least from a purely theoretical perspective this does not offer any difficulties: One can easily picture the desired scenario by imagining a fluid formed by point moving dipoles whose trajectories are not blocked by the metallic grid. It is also interesting to mention that a moving medium may be to some extent mimicked in an optical fiber environment by exploiting nonlinear optical effects [12].

We would like to mention another important thing here: While in the laboratory frame (wherein the waveguide is at rest; it is the frame of Fig. 1) the reflection from the waveguide walls does not involve frequency conversion (this is valid provided the boundary conditions at the walls relate fields calculated at the *same* time instant), the reflected wave in the frame comoving with the medium has a different frequency because of the Doppler effect. The Doppler shift grows with the velocity and the frequency; therefore, if the medium is dispersive the conditions of one-way propagation may not be easily achievable, as a reflected wave will propagate at a frequency at which the refractive index may not fulfill the Cherenkov threshold condition.

From previous studies [7,13–15], it is known that the electromagnetic field in moving media is subject to the canonical quantization; therefore, we may find the zero-point energy of the quantum-electromagnetic fluctuations in the waveguide shown in Fig. 1 (at zero temperature) with the standard summation over the eigenfrequencies of all possible electromagnetic modes in this structure. Then, the Casimir force acting on the piston may be found with the principle of virtual work, that is, by differentiating the (regularized) zero-point energy with respect to the position of the piston.

Let us consider the space between one of the ends of the waveguide and the piston: $0 < z < z_0$ (the other half of the waveguide can be considered in the same way). The characteristic equation for the electromagnetic modes in this space reads

$$\kappa_+ z_0 + \kappa_- z_0 + 2\Delta\varphi = 2\pi N, \tag{11}$$

where $\Delta\varphi$ is an additional phase shift that a wave gets when reflected from the wall at $z = 0$ and the piston at $z = z_0$, and N is an arbitrary integer. For the ideally reflecting PEC piston and the wall we may let $2\Delta\varphi = 0$ (or $2\Delta\varphi = 2\pi$ which is equivalent). Thus, the characteristic equation satisfies

$$(\kappa_+ + \kappa_-)z_0 = 2\pi N. \tag{12}$$

When the formulas (7) for $\kappa_\pm$ are substituted in this equation, the β-proportional term in the numerator of (7) cancels out and we are left with

$$K z_0 = \pi N, \ \text{ where } \ K = \frac{\sqrt{1-\beta^2}}{\sqrt{1-n^2\beta^2}}\sqrt{\frac{n^2(1-\beta^2)}{1-n^2\beta^2}k_0^2 - k_t^2}. \tag{13}$$

In the limit $\beta^2 \to 1/n^2$, K behaves as $K \sim \frac{n(1-\beta^2)k_0}{1-n^2\beta^2}$; therefore, it grows without limit when velocity approaches the Cherenkov threshold.

To obtain the modal eigenfrequencies, we solve Eq. (13) for the free space wave number k_0. This gives

$$\frac{\omega_N}{c} = \frac{\sqrt{1-n^2\beta^2}}{n\sqrt{1-\beta^2}}\sqrt{\frac{1-n^2\beta^2}{1-\beta^2}\left(\frac{\pi N}{z_0}\right)^2 + k_t^2}. \tag{14}$$

We may introduce a notation,

$$z_0^{\text{eff}} = z_0 \frac{\sqrt{1-\beta^2}}{\sqrt{1-n^2\beta^2}}, \tag{15}$$

so that the eigenfrequency (14) is expressed as

$$\frac{\omega_N}{c} = \frac{\sqrt{1-n^2\beta^2}}{n\sqrt{1-\beta^2}}\sqrt{\left(\frac{\pi N}{z_0^{\text{eff}}}\right)^2 + k_t^2}. \tag{16}$$

Therefore, the zero-point energy associated with the modes in the space $0 < z < z_0$ reads

$$\mathcal{E}(z_0) = \frac{\hbar c}{2}\frac{\sqrt{1-n^2\beta^2}}{n\sqrt{1-\beta^2}}\sum_{\mathbf{k}_t}\sum_N \sqrt{\left(\frac{\pi N}{z_0^{\text{eff}}}\right)^2 + k_t^2}. \tag{17}$$

In the trivial case when $v = 0$ and the fluid is at rest, the expression for the zero-point energy may be written as

$$\mathcal{E}_0(z_0) = \frac{\hbar c}{2n}\sum_{\mathbf{k}_t}\sum_N \sqrt{\left(\frac{\pi N}{z_0}\right)^2 + k_t^2}. \tag{18}$$

We understand the infinite sums over eigenstates (which are diverging in a strict mathematical sense) as regularized ones. When regularized, (17) and (18) represent the interaction part of the zero-point energy which vanishes when $z_0 \to \infty$. We would like to stress that our derivation is not limited to a particular regularization procedure (for the existing methods, see, for instance, [16,17]).

Comparing (17) with (18) we realize that

$$\mathcal{E}(z_0) = \frac{\sqrt{1-n^2\beta^2}}{\sqrt{1-\beta^2}}\mathcal{E}_0\left(z_0\frac{\sqrt{1-\beta^2}}{\sqrt{1-n^2\beta^2}}\right); \tag{19}$$

that is, the zero-point energy in a waveguide filled with a moving medium can be expressed through the formula for the zero-point energy in a system at rest.

The Casimir force (to be precise, the part of it related to the quantum fluctuations in the space $0 < z < z_0$), which is the derivative of the energy with respect to z_0, can, therefore, be found as

$$F_{\text{C}}(z_0) = F_{\text{C}0}\left(z_0\frac{\sqrt{1-\beta^2}}{\sqrt{1-n^2\beta^2}}\right), \tag{20}$$

where $F_{\text{C}0}(z_0)$ is the Casimir force in a system with $v = 0$. Because from physical reasons $F_{\text{C}0}(z_0)$ must vanish in the limit $z_0 \to \infty$ (the same holds for the contribution to the force from the fluctuations in the space $z_0 < z < L$ when $L - z_0 \to \infty$), we obtain the following result: In the limit of one-way propagation that happens in the considered waveguide when the velocity tends to the Cherenkov threshold, $\beta^2 \to 1/n^2$, the Casimir forces acting on the piston in the one-way waveguide vanish.

To illustrate the above general results—which are applicable to *arbitrary waveguides* (with PEC or PMC walls, or with any combination of them)—with a particular example, let us consider a metallic PEC waveguide with a square cross section ($b = a$), similar to the geometry considered in Ref. [18]. The Casimir force $F_{\text{C}0}$ acting on the piston (per unity of area) in this waveguide is given by Eq. (6) of Ref. [18], which, in our notation, reads

$$\begin{aligned} F_{\text{C}0}(z_0) = \frac{\hbar c}{n}\Bigg[& -\frac{3\zeta(4)}{8\pi^2 z_0^4} + \frac{\zeta(2)}{8\pi a^2 z_0^2} - \frac{2Z_2(1,1;4)}{32\pi^2 a^4} \\ & + \frac{\pi^2}{16 z_0^4}\sum_{l,m\neq 0,0}\frac{\coth\left(\frac{\pi a}{z_0}\sqrt{l^2+m^2}\right)}{\frac{\pi a}{z_0}\sqrt{l^2+m^2}\,\sinh^2\left(\frac{\pi a}{z_0}\sqrt{l^2+m^2}\right)}\Bigg], \end{aligned} \tag{21}$$

where $\zeta(x)$ is Riemann's ζ function, $2Z_2(1,1;4) \approx 12.053$, and we expressed the force in SI units (in Ref. [18] $\hbar = 1$ and $c = 1$) and took into account that the waveguide is filled with a nondispersive stationary medium with the refractive index n.

As one can check by a direct calculation using Eq. (21), the force quickly and monotonically decays with z_0, and vanishes practically completely at $z_0 \gtrsim 2a$. Mathematically, this happens because at large z_0 the first three addends of (21) are compensated by the contribution of the series (for more details, see Ref. [18]). Physically, the force vanishes at the distance on the order of the transverse dimension of the waveguide due to the fact that the propagating modes of this waveguide are cut off at low frequencies, and, as discussed previously, the force for a distance d is mainly determined by normal modes with eigenfrequencies smaller than $\omega \sim c/(nd)$, and there are no modes with such properties when $d > a$.

Equation (21) was obtained assuming that the walls of the waveguide and the piston are ideally conducting, nevertheless, as explained in the beginning of this section, we may apply it also to the case when some of the walls and the piston are

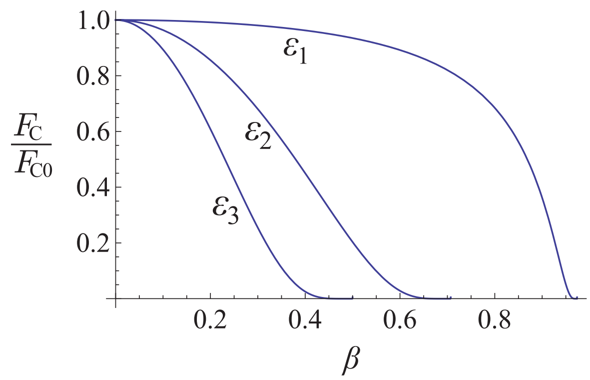

FIG. 3. (Color online) Casimir force acting on the piston [Eq. (20)] in a PEC waveguide with $a \times a$ cross section filled with the moving fluid as a function of $\beta = v/c$. The force is calculated at $z_0 = a$ and normalized to the same force at $\beta = 0$. The three curves correspond to the three different values of the dielectric constant of the fluid (same as in Fig. 2).

made penetrable by a moving fluid. Therefore, in the case of the same waveguide filled with moving medium ($\beta \neq 0$) we may apply the relation (20) and get to the same conclusion that at the limit $\beta^2 \to 1/n^2$ the force acting on the piston vanishes. This property is illustrated in Fig. 3 for different values of the index of refraction at rest, n, of the moving fluid.

IV. CASIMIR INTERACTION AT FINITE TEMPERATURE

In the previous section, we considered the Casimir force acting on a piston in the waveguide filled with a moving medium at zero temperature and found that the force vanishes when the velocity of the medium approaches the Cherenkov limit. Naturally, it is interesting to study if this phenomenon persists also at finite temperatures.

At finite temperatures, the ensemble of quantum oscillators formed by all available electromagnetic modes in a system may be conveniently described by its free energy which is defined as

$$\mathcal{F} = \mathcal{E} - TS, \tag{22}$$

where $\mathcal{E}$ is the internal energy of the system, T is the absolute temperature, and S is the entropy. At a constant temperature, the generalized macroscopic force F_q associated with a generalized coordinate q (a parameter of the system) is found from the free energy as $F_q = -\partial_q \mathcal{F}$.

As is known, when dealing with finite temperatures one may use Matsubara's formalism, so that the free energy of a system in thermal equilibrium at temperature T is expressed as a summation over imaginary Matsubara's frequencies [17,19]:

$$\mathcal{F}(z_0,T) = \frac{k_B T}{2} \sum_{\mathbf{k}_t} \sum_{\omega_m} \ln \mathcal{D}(\omega_m, z_0), \tag{23}$$

where $\omega_m = i(2\pi k_B T/\hbar)m$, $m \in \mathcal{Z}$, and k_B is Boltzmann's constant. In our waveguide, the free energy depends on the position of the piston z_0 and the temperature. As before, we understand the infinite sums in (24) as renormalized ones, that is, representing only the interaction part of the free energy. The result (23) may be derived by considering an ensemble of quantum oscillators representing the eigenmodes of the quasi-one-way waveguide and taking into account that the occupation numbers of the states satisfy the Bose-Einstein statistics [20]. An experimental verification of the theory of thermal corrections of the Casimir effect has been recently reported [21], but received certain criticism [22,23].

The function $\mathcal{D}(\omega,z_0)$ that occurs in (23) is a suitable characteristic function of the system such that the eigenfrequencies of the electromagnetic modes are the roots of the equation $\mathcal{D}(\omega,z_0) = 0$. Similarly to the previous section, we may consider the two halves of the waveguide separately. Then, for the waveguide segment at $0 < z < z_0$ the characteristic function may be chosen in the following form:

$$\mathcal{D}(\omega,z_0) = 1 - e^{i2Kz_0}, \tag{24}$$

where K is defined by (13). Besides evident analytical reasons, Eq. (24) can be also inferred from physical considerations. Indeed, the characteristic equation for the modes in a segment of a waveguide bounded by two reflectors with the reflection coefficients r_1 and r_2 may be written as $r_1 r_2 \exp(i\kappa_+ z_0)\exp(i\kappa_- z_0) = 1$, from which one obtains (24) when the reflectors are PEC. Since only the distance-dependent terms should appear in (23), the term with both $k_t = 0$ and $m = 0$ at which $K = 0$ must be excluded from the summation. Notice that in Eq. (23) the summation is over both positive and negative imaginary frequencies and that Eq. (23) with $\mathcal{D}(\omega,z_0)$ of the form (24) reduces to the well-known Lifshitz integral for the Casimir energy when $T \to 0$.

Comparing the expression (13) for K at an arbitrary velocity with the same expression at $\beta = 0$, $K_0(\omega) = \sqrt{n^2(\omega/c)^2 - k_t^2}$, one may notice that

$$K(\omega) = \frac{\sqrt{1-\beta^2}}{\sqrt{1-n^2\beta^2}} K_0\left(\frac{\omega\sqrt{1-\beta^2}}{\sqrt{1-n^2\beta^2}}\right). \tag{25}$$

Therefore, the characteristic function at $\beta \neq 0$ can be expressed through the corresponding function for a system without movement:

$$\mathcal{D}(\omega,z_0) = \mathcal{D}_0\left(\frac{\omega\sqrt{1-\beta^2}}{\sqrt{1-n^2\beta^2}}, \frac{z_0\sqrt{1-\beta^2}}{\sqrt{1-n^2\beta^2}}\right). \tag{26}$$

When this expression is substituted into (23), it becomes evident that because Matsubara's frequencies are proportional to the temperature one may introduce the effective temperature,

$$T^{\text{eff}} = T\frac{\sqrt{1-\beta^2}}{\sqrt{1-n^2\beta^2}}, \tag{27}$$

and the effective distance (15), so that the free energy in a waveguide filled with a moving fluid can be expressed through the formula for the free energy in the system where the fluid is at rest:

$$\mathcal{F}(z_0,T) = \frac{\sqrt{1-n^2\beta^2}}{\sqrt{1-\beta^2}} \mathcal{F}_0\left(z_0^{\text{eff}}, T^{\text{eff}}\right). \tag{28}$$

This result is the generalization of (19) to the case of finite temperatures. From both physical and analytical reasons, $\mathcal{F}_0(z_0,T) \to 0$ when $z_0 \to \infty$.

When the velocity of the medium approaches the Cherenkov threshold ($\beta^2 \to 1/n^2$) both the effective distance z_0^{eff} and the effective temperature T^{eff} tend to infinity. The growth of $T^{\rm eff}$ [which is the multiplier in front of the double sum (23) for $\mathcal{F}_0$] is compensated by the factor $\sqrt{1-n^2\beta^2}/\sqrt{1-\beta^2}$ in front of (28), while the infinite growth in $z_0^{\rm eff}$ results in the same phenomenon as we observed at zero temperature: The Casimir interaction of the piston with the walls closing the waveguide vanishes in the Cherenkov limit.

V. DISCUSSION AND CONCLUSIONS

In this work we obtained an interesting result: The Casimir force acting on a piston in a waveguide filled with a medium moving at the velocity approaching the Cherenkov threshold vanishes (at both zero and finite temperatures). This result may be physically understood as follows.

One hint is given by the characteristic Eq. (13). Because $K \sim \frac{n(1-\beta^2)k_0}{1-n^2\beta^2}$ when $\beta^2 \to 1/n^2$, the solution of (13) behaves in this limit as $\frac{\omega_N}{c} \sim \frac{\pi N}{z_0}\frac{1-n^2\beta^2}{n(1-\beta^2)} \to 0$. Therefore, the eigenfrequencies of the electromagnetic modes with indices from within a finite range $N \in [1, N_{\rm max}]$ with an arbitrary $N_{\rm max} \gg 1$ *all* tend to zero when the velocity approaches the Cherenkov limit. In essence, one may say that these modes simply disappear, as they do not contribute to the zero-point energy anymore. As the *regularized* infinite sums for the Casimir energy converge and can be truncated at a certain large-enough $N_{\rm max}$, the above observation means that the Casimir energy itself vanishes at the threshold of the Cherenkov velocity. Additionally, at finite temperatures, when the interaction part of the free energy is given by (28) and (23) with an effective temperature expressed by (27) one may notice that the interval between the Matsubara frequencies becomes larger and the respective summation over them in (23) becomes sparser when $T^{\rm eff}$ increases with $\beta^2 \to 1/n^2$, which indicates that the Casimir interaction will decrease even faster at finite temperatures than at zero temperature.

One may also picture that when the medium moves with a velocity equal to the velocity of the waves in the same medium, the electromagnetic field becomes trapped within the moving matter so that it becomes impossible for the waves to travel back and forward in between the stationary reflectors and form a standing wave (i.e., a mode). Such a physical picture is very intuitive and also helps to understand why the Casimir force in a system of two parallel plates separated by a moving fluid that moves *tangentially* to the plates *does not* vanish in the same limit. In such a scenario the partial waves can still reach the plates, be reflected, and form a standing wave (notice also that in this scenario there are no reflectors oriented perpendicularly to the flow, and thus, evidently, the characteristic equation for the eigenmodes is different). The theory for this configuration was developed in Ref. [7], which, in particular, predicts that the Casimir force per unit area (i.e., the pressure) in such a scenario is invariant in all inertial frames moving in a direction parallel to the plates. Specifically, for a pair of PEC reflectors separated by a tangentially moving nondispersive fluid, this force is independent of the velocity of the fluid and is the same as in the frame comoving with the fluid: $|F_{\rm C}/A_0| = \pi^2\hbar c/(240nd^4)$ (at zero temperature).

It is interesting to point out that at the threshold of the Cherenkov effect some of the effective parameters of the moving medium [Eq. (3)] approach infinity. Typically, the Casimir interaction between two bodies is reduced if these bodies are embedded in some fluid as compared to the case where the bodies stand in vacuum, because of the larger value of n in the fluid. Thus, it could be thought that the reason for the suppression of the Casimir interaction at the threshold of the Cherenkov effect could also be explained by the increased value of the effective index of refraction of the moving fluid, as seen from the stationary frame. However, such picture is not really accurate because it fails to describe the fact that the Casimir interaction does not vanish when the fluid flows tangentially to the plates, as discussed in the previous paragraph.

Thus, we may conclude that the simple theory developed in this paper answers the questions posed in the Introduction: In the limit of one-way propagation in the considered *electromagnetically closed* system the quantum fluctuation-originated forces acting on the piston simply vanish. There is no "mysterious" drift force or any other force that might "self-accelerate" the piston and violate the conservation laws for such a closed system (recall that we consider only the electromagnetic part of the force and neglect any possible friction due to the contact of the piston with the moving fluid). The obtained result, however, does not pose any restrictions on the behavior of *open systems* wherein there may exist intermediate agents (for example, real photons that may leave the system) that may contribute to the energy and to the momentum of the system.

One may also wonder if the developed theory can be directly applied to one-way waveguides realized with magnetized ferrites or a plasma. The general answer is negative, because of the strong dispersion in these materials. In these structures the one-way operation may happen only in a narrow band of frequencies. In particular, there are no eigenmodes in the frequency band associated with one-way waveguiding, which therefore does not contribute to the zero-point energy. However, the modes outside this band can contribute to the zero-point interaction energy. Nevertheless, one may still expect a *reduction* in the value of the interaction energy in this case and a respective reduction in the value of the Casimir force.

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