Physical restrictions on the Casimir interaction of metal-dielectric metamaterials: An effective-medium approach

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Using an effective-medium approach, we demonstrate that the Casimir interaction of structured metal-dielectric metamaterial slabs which effectively behave as either uniform nongyrotropic materials or bi-isotropic materials is attractive at all distances, independent of the emergence of artificial magnetism or strong magnetoelectric coupling, when the slabs stand in a vacuum. In particular, it is shown that the magnetic response of a metal-dielectric metamaterial is always diamagnetic at imaginary frequencies, and this explains in simple physical terms the impossibility of Casimir repulsion.

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I. INTRODUCTION

The Casimir effect is a manifestation of the quantum fluctuations of the electromagnetic field [1]. These fluctuations are modified by the presence of material boundaries, and typically result in an attractive Casimir interaction. However, the Casimir force can in principle be repulsive, even when the materials are separated by a vacuum. For example, in 1974 Boyer showed theoretically that a paramagnetic body may repel a conducting body [2]. Recently, it was shown that by microstructuring conventional metals and dielectrics it may be possible to tailor their electromagnetic properties and induce an effective magnetic response, at least in some frequency range (artificial magnetism). Thus, it is only natural to ask if such "metamaterials" may provide a route for Casimir repulsion [3–6]. Driven by this exciting possibility, several ideas for "quantum levitation" based on either magnetic or chiral metamaterials have been suggested [7–9].

In a recent work [10], we have theoretically demonstrated, by extending the $TGTG$ formalism of Ref. [11] to the case of periodic structures, that as long as the metamaterials are formed by either dielectric or metallic inclusions with no intrinsic magnetism, the Casimir force is necessarily attractive for $d > d_0$, where d_0 is some distance comparable to the transverse lattice constant of the metamaterial. We argued that if the structured slabs can be described using effective-medium theory, then their interaction is necessarily in the far zone relative to the length scale of the lattice constant, and thus the bodies must attract each other. In a related study, Rahi *et al.* investigated whether fluctuation-induced forces can lead to stable levitation, and found that in the case of nonmagnetic dielectric objects in a vacuum the equilibrium position is always unstable [12]. It is important to point out that in general it is possible to have a repulsive force without any stable point of equilibrium.

In this paper, we further develop the theory of Ref. [10] and explicitly demonstrate using an effective-medium approach that if the structured slabs may be regarded as uniform effective media with either a nongyrotropic or bi-isotropic response the Casimir force is attractive at all distances, independent of the emergence of optical magnetism or strong chirality. In particular, we demonstrate that the magnetic response of a metal-dielectric metamaterial at imaginary frequencies is always diamagnetic.

II. RESTRICTIONS ON THE EFFECTIVE-MEDIUM PARAMETERS AT IMAGINARY FREQUENCIES

To begin with, we consider an arbitrary metamaterial formed by metal-dielectric inclusions. Even though some metals have an extremely tiny magnetic response, it can be safely neglected since $|\varepsilon|/\varepsilon_0 \gg |\mu|/\mu_0$ (i.e., the electromagnetic response of metals is determined by the electrical properties). Thus, it is supposed in what follows that the inclusions do not have an intrinsic magnetic response ($\mu = \mu_0$). However, the metamaterial can have an effective magnetic response.

Typically, it is assumed that for long wavelengths the effective response of the metamaterial can be characterized by an effective permittivity $\overline{\overline{\varepsilon}}$, an effective permeability $\overline{\overline{\mu}}$, and possibly by some parameters, $\overline{\overline{\vartheta}}$ and $\overline{\overline{\zeta}}$, that describe magnetoelectric coupling ("bianisotropic model" [13]). However, here we find it useful to consider first a more general framework that describes the metamaterial by a dielectric function denoted in the spectral domain by $\overline{\overline{\varepsilon}}_{\rm eff}(\omega,\mathbf{k})$, $\mathbf{k}$ being the wave vector [14,15]. Such approach takes into account the spatial dispersion effects, and thus provides a more accurate description of the electrodynamics of the structured material. Next, we prove that because the inclusions are passive and have a causal response the nonlocal dielectric function evaluated for an arbitrary *real* valued $\mathbf{k}$ and imaginary frequencies $\omega = i\xi$ must be such that $\overline{\overline{\varepsilon}}_{\rm eff}(i\xi,\mathbf{k})/\varepsilon_0 - \overline{\overline{\mathbf{I}}} > 0$ (i.e., the "susceptibility function" must be positive definite). The proof is a generalization of a similar result for local isotropic materials [16]. Ahead, we will use this property to derive physical restrictions on the usual effective parameters associated with the bianisotropic model.

We consider an arbitrary uniform linear reciprocal spatially dispersive material and write $\overline{\overline{\varepsilon}}_{\rm eff} = \overline{\overline{\varepsilon}}' + i\,\overline{\overline{\varepsilon}}''$, where $\overline{\overline{\varepsilon}}' = (\overline{\overline{\varepsilon}}_{\rm eff} + \overline{\overline{\varepsilon}}_{\rm eff}^{\dagger})/2$ and $i\,\overline{\overline{\varepsilon}}'' = (\overline{\overline{\varepsilon}}_{\rm eff} - \overline{\overline{\varepsilon}}_{\rm eff}^{\dagger})/2$. From the definition, both $\overline{\overline{\varepsilon}}'$ and $\overline{\overline{\varepsilon}}''$ are Hermitian symmetric. By thermodynamical considerations, it is known that the heat liberated per unit volume must be positive [14]. For electromagnetic fields

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with a spatial variation of the form $e^{i\mathbf{k}\cdot\mathbf{r}}$ the heating rate is given by [14,17] $q = \frac{1}{2}\mathrm{Re}\{-i\omega\mathbf{E}^*\cdot\overline{\overline{\varepsilon}}_{\mathrm{eff}}(\omega,\mathbf{k})\cdot\mathbf{E}\} = \frac{1}{2}\omega\mathbf{E}^*\cdot\overline{\overline{\varepsilon}}''(\omega,\mathbf{k})\cdot\mathbf{E}$. Hence, since ω and $\mathbf{k}$ are independent parameters, the passivity of the material implies that for every $\omega > 0$ and $\mathbf{k}$ real valued, the dyadic $\overline{\overline{\varepsilon}}''$ must be positive definite [14].

On the other hand, $\overline{\overline{\varepsilon}}_{\mathrm{eff}}$ can be regarded as the response function of the material, and thus, because of causality considerations, it cannot have singularities in the upper half plane of the complex variable ω [14]. Moreover, it must satisfy $\lim_{\omega\to\infty}\overline{\overline{\varepsilon}}_{\mathrm{eff}}(\omega,\mathbf{k}) = \varepsilon_0\overline{\overline{\mathbf{I}}}$. Here, we are interested in the properties of the nonlocal dielectric function for imaginary frequencies $\omega = i\xi$, which, as shown next, are intrinsically related to the behavior of $\overline{\overline{\varepsilon}}''$ over the real axis. Indeed, consider the auxiliary function $\overline{\overline{\chi}}(\omega,\mathbf{k}) = \overline{\overline{\varepsilon}}_{\mathrm{eff}}(\omega,\mathbf{k})/\varepsilon_0 - \overline{\overline{\mathbf{I}}}$ and calculate the integral of $\overline{\overline{\chi}}(\omega,\mathbf{k})\omega/(\omega^2+\xi^2)$ over a closed contour in the complex ω plane that consists of the real axis and an arc of circumference with infinite radius in the upper half plane, as in Ref. [16]. Applying Cauchy's theorem, and noting that the considered function has a single pole (at $\omega = i\xi$) in the region enclosed by integration contour, it can be proven that

$$\overline{\overline{\chi}}(i\xi,\mathbf{k}) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\omega}{\omega^2+\xi^2}\overline{\overline{\chi}}(\omega,\mathbf{k})\,d\omega = \frac{2}{\pi}\int_0^{\infty}\frac{\omega}{\omega^2+\xi^2}\frac{\overline{\overline{\varepsilon}}''(\omega,\mathbf{k})}{\varepsilon_0}\,d\omega. \qquad (1)$$

The second identity is a consequence of the definition of $\overline{\overline{\chi}}$ and of the fact that $\overline{\overline{\varepsilon}}'$ ($\overline{\overline{\varepsilon}}''$) is an even (odd) function of ω, because for reciprocal materials and ω and $\mathbf{k}$ real valued the dielectric function satisfies $\overline{\overline{\varepsilon}}_{\mathrm{eff}}(\omega,\mathbf{k}) = \overline{\overline{\varepsilon}}_{\mathrm{eff}}^{\dagger}(-\omega,\mathbf{k})$ [14]. Thus, since $\overline{\overline{\varepsilon}}''$ is positive definite for $\omega > 0$, as discussed before, it follows that $\overline{\overline{\varepsilon}}_{\mathrm{eff}}(i\xi,\mathbf{k})/\varepsilon_0 - \overline{\overline{\mathbf{I}}}$ is also positive definite for every ξ and $\mathbf{k}$ real valued, and this concludes the proof.

In the derivation of this result we have implicitly assumed that the function $\omega\overline{\overline{\chi}}$ does not have poles in the real frequency axis, which in the presence of dissipation is surely the case for $\omega \neq 0$. At $\omega = 0$, the dielectric function $\overline{\overline{\varepsilon}}_{\mathrm{eff}}$ may have a pole, but the singularity of $\overline{\overline{\varepsilon}}_{\mathrm{eff}}$ at $\omega = 0$ cannot be more severe than that of the microscopic dielectric function ε of the inclusions. The most critical situation corresponds to the case where the inclusions are characterized by a lossy Drude dispersion model, which corresponds to a pole of order one. In any case, it is clear that for a metamaterial made of lossy dielectric or metallic inclusions (with no intrinsic magnetism) $\omega\overline{\overline{\chi}}$ does not have, indeed, poles in the real frequency axis.

Let us now suppose that the considered metamaterial can be as well characterized by a bianisotropic model with constitutive relations [13]

$$\mathbf{D} = \varepsilon_0\overline{\overline{\varepsilon}}\cdot\mathbf{E} + \frac{1}{c}\overline{\overline{\vartheta}}\cdot\mathbf{H}, \qquad \mathbf{B} = \frac{1}{c}\overline{\overline{\zeta}}\cdot\mathbf{E} + \mu_0\overline{\overline{\mu}}\cdot\mathbf{H}, \qquad (2)$$

where c is the speed of light in vacuum. Since the material is reciprocal, $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$ are symmetric tensors and $\overline{\overline{\zeta}} = -\overline{\overline{\vartheta}}^{t}$ [13]. It is well known that in this case the dielectric function associated with the nonlocal homogenization model must satisfy [15,17,18]

$$\frac{\overline{\overline{\varepsilon}}_{\mathrm{eff}}}{\varepsilon_0}(\omega,\mathbf{k}) = \overline{\overline{\varepsilon}} - \overline{\overline{\vartheta}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}} + \frac{c^2}{\omega^2}\mathbf{k}\times(\overline{\overline{\mu}}^{-1} - \overline{\overline{\mathbf{I}}})\times\mathbf{k} + \frac{c}{\omega}(\overline{\overline{\vartheta}}\cdot\overline{\overline{\mu}}^{-1}\times\mathbf{k} - \mathbf{k}\times\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}}), \qquad (3)$$

which for $\omega = i\xi$ may be rewritten in a compact form as

$$\frac{\overline{\overline{\varepsilon}}_{\mathrm{eff}}}{\varepsilon_0}(i\xi,\mathbf{k}) - \overline{\overline{\mathbf{I}}} = \tilde{U}^{\dagger}\hat{\chi}_{\mathrm{EB}}\tilde{U}, \quad \tilde{U} = \begin{pmatrix}\overline{\overline{\mathbf{I}}}\\ \frac{ic}{\xi}\mathbf{k}\times\overline{\overline{\mathbf{I}}}\end{pmatrix}, \qquad (4)$$

where (the superscript t represents the transpose matrix)

$$\hat{\chi}_{\mathrm{EB}} = \begin{pmatrix}\overline{\overline{\varepsilon}} - \overline{\overline{\mathbf{I}}} - \overline{\overline{\vartheta}}\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}} & -\overline{\overline{\vartheta}}\cdot\overline{\overline{\mu}}^{-1}\\ \overline{\overline{\mu}}^{-1}\cdot\overline{\overline{\zeta}} & \overline{\overline{\mu}}^{-1} - \overline{\overline{\mathbf{I}}}\end{pmatrix} \equiv \begin{pmatrix}\overline{\overline{A}} & \overline{\overline{B}}\\ \overline{\overline{B}}^{t} & \overline{\overline{C}}\end{pmatrix}. \qquad (5)$$

It can be easily checked that the matrix $\hat{\chi}_{\mathrm{EB}}(i\xi)$ is real valued and symmetric.[1]

Next, we use the very general property $\overline{\overline{\varepsilon}}_{\mathrm{eff}}(i\xi,\mathbf{k})/\varepsilon_0 - \overline{\overline{\mathbf{I}}} > 0$ to derive the restrictions on the parameters associated with the bianisotropic model at imaginary frequencies. Specifically, we will show that if the metamaterial is reciprocal and either bi-isotropic (i.e., ε, μ, and ϑ and ζ are scalars) or nongyrotropic ($\overline{\overline{\vartheta}} = \overline{\overline{\zeta}} = 0$) then $\hat{\chi}_{\mathrm{EB}}(i\xi) \geqslant 0$. First, we note that $\overline{\overline{\varepsilon}}_{\mathrm{eff}}(i\xi,\mathbf{k})/\varepsilon_0 - \overline{\overline{\mathbf{I}}} > 0$ implies that $\hat{\chi}_{\mathrm{EB}}(i\xi)$ is such that $\langle\tilde{U}\mathbf{E}_0|\hat{\chi}_{\mathrm{EB}}|\tilde{U}\mathbf{E}_0\rangle > 0$, for every nontrivial complex (constant) vector $\mathbf{E}_0$. However, we should not rush to the conclusion that $\hat{\chi}_{\mathrm{EB}} > 0$, because the range of mappings of the form $\tilde{U} = \tilde{U}(\mathbf{k})$ (with $\mathbf{k}$ real valued) is not necessarily the whole complex vector space of dimension 6.

To circumvent this difficulty, we note that for every real valued scalar α we have $\langle\tilde{U}(\alpha\mathbf{k})\mathbf{E}_0|\hat{\chi}_{\mathrm{EB}}|\tilde{U}(\alpha\mathbf{k})\mathbf{E}_0\rangle > 0$. Notice that $\tilde{U}$ is evaluated for the wave vector $\alpha\mathbf{k}$ and that $\tilde{U}(\mathbf{k})\mathbf{E}_0 = \begin{pmatrix}\mathbf{E}_0\\ \frac{c}{\xi}i\mathbf{k}\times\mathbf{E}_0\end{pmatrix}$. But, since the expression $\langle\tilde{U}(\alpha\mathbf{k})\mathbf{E}_0|\hat{\chi}_{\mathrm{EB}}|\tilde{U}(\alpha\mathbf{k})\mathbf{E}_0\rangle$ is a quadratic polynomial in α, ($\mathbf{E}_0$ is an arbitrary complex vector and $\mathbf{k}$ is an arbitrary real vector), a straightforward analysis shows that in order that this quadratic form is always positive it is necessary that $\overline{\overline{A}} > 0$, $\overline{\overline{C}} \geqslant 0$, and that

$$|\mathrm{Im}\{\langle\mathbf{E}_0|\overline{\overline{B}}|\mathbf{k}\times\mathbf{E}_0\rangle\}|^2 \leqslant \langle\mathbf{E}_0|\overline{\overline{A}}|\mathbf{E}_0\rangle\langle\mathbf{k}\times\mathbf{E}_0|\overline{\overline{C}}|\mathbf{k}\times\mathbf{E}_0\rangle, \qquad (6)$$

where the dyadics $\overline{\overline{A}}$, $\overline{\overline{B}}$, $\overline{\overline{C}}$ are defined as in Eq. (5). Hence, it is clear that for nongyrotropic media ($\overline{\overline{B}} = 0$) we have $\hat{\chi}_{\mathrm{EB}}(i\xi) \geqslant 0$. On the other hand, for an isotropic material all the dyadics reduce to scalars and choosing, for example, $\mathbf{E}_0 = (1,i,0)/\sqrt{2}$ and $\mathbf{k} = (0,0,1)$, it follows that $B^2 \leqslant AC$. It can be verified that this ensures that $\hat{\chi}_{\mathrm{EB}}(i\xi)$ is nonnegative in the bi-isotropic case, as we wanted to prove.

An immediate but extremely important implication of the condition $\hat{\chi}_{\mathrm{EB}} \geqslant 0$ is that an isotropic metamaterial is

[1]For $\mathbf{k}$ real valued the nonlocal dielectric function satisfies $\overline{\overline{\varepsilon}}^{*}(i\xi,\mathbf{k}) = \overline{\overline{\varepsilon}}(i\xi,-\mathbf{k})$ [14], and thus the effective parameters $\overline{\overline{\varepsilon}}$, $\overline{\overline{\mu}}$, $\overline{\overline{\vartheta}}$, and $\overline{\overline{\zeta}}$ must be real valued for imaginary frequencies, $\omega = i\xi$.

necessarily diamagnetic at imaginary frequencies: $0 < \mu(i\xi) \leqslant 1$. Thus, it follows that by structuring either dielectrics or metals it is impossible to obtain isotropic paramagnetism at imaginary frequencies, and consequently any form of "quantum levitation" based on the mechanism proposed by Boyer [2]!

It is important to note that our theory does not invalidate in any manner the existence of paramagnetic materials at imaginary frequencies. It only establishes that it is impossible to synthesize a metal-dielectric metamaterial with such effective response. In fact, one could very well argue that *any* bianisotropic material (not necessarily a metamaterial) can always be described by a nonlocal dielectric function defined as in Eq. (3), and thus that our conclusion that $\bar{\bar{\varepsilon}}_{\rm eff}(i\xi,\mathbf{k})/\varepsilon_0 - \bar{\bar{\mathbf{I}}} > 0$ would imply that any isotropic material is necessarily diamagnetic at imaginary frequencies. There is, however, a flaw in such reasoning. Indeed, as discussed before, the proof that $\bar{\bar{\varepsilon}}_{\rm eff}(i\xi,\mathbf{k})/\varepsilon_0 - \bar{\bar{\mathbf{I}}} > 0$ is valid only if $\omega\bar{\bar{\chi}}$ is regular in the real frequency axis. But, for a general bianisotropic material [such that the nonlocal dielectric function and the local parameters are linked as in Eq. (3)] the finiteness of $\omega\bar{\bar{\chi}}$ at the origin is equivalent to the requirement that the effective permeability of the material in the static limit is trivial ($\mu = 1$), because otherwise $\omega\bar{\bar{\chi}}$ has a pole at $\omega = 0$. Thus, in the case of a general bianisotropic material, the conclusion that $\bar{\bar{\varepsilon}}_{\rm eff}(i\xi,\mathbf{k})/\varepsilon_0 - \bar{\bar{\mathbf{I}}}$ is positive definite holds only if the material does not have a magnetic response in the static limit. This precludes the direct application of the theory to conventional materials with an intrinsic magnetic response in the static limit (e.g., paramagnetics).

However, as discussed before, in case of metal-dielectric metamaterials $\omega\bar{\bar{\chi}}$ is always regular at the origin. In fact, due to the dissipation effects the metamaterial cannot have a magnetic response in the limit $\omega = 0$ when the origin of the magnetism is the circulation of microscopic electric currents. For example, the current I induced in a closed loop of area S, by an external magnetic field $B_{\rm ext}$, vanishes in the static limit in the presence of ohmic loss. Indeed, supposing that the resistance of the loop is R and that its inductance is L we have $(R - i\omega L)I = i\omega B_{\rm ext} S$. Thus, in the quasistatic limit the magnetic dipole moment of the loop is such that $m = i\omega S^2 B_{\rm ext}/(R - i\omega L)$, which evidently vanishes in the presence of realistic loss ($R \neq 0$) at $\omega = 0$, and thus the static permeability is trivial. Note, however, that if the inclusions were made of perfect electric conductors the metamaterial could have a diamagnetic response at $\omega = 0$. Very differently, natural magnetics can have a nontrivial permeability in the static limit, even in the presence of dissipation, because the magnetic response is rooted in other physical mechanisms (e.g., spin magnetic moments). The restrictions on the effective parameters of (natural) materials with intrinsic magnetism in the static limit are discussed in the appendix.

It should be noted that the property $\mu(\omega = 0^+) = 1$ is compatible with Kramers-Kronig relations for the permeability as formulated in the book of Landau and Lifshitz [19]. However, such property also indicates that the effective-medium model must break down for a sufficiently large frequency, because if the permeability had meaning in all the frequency spectrum it should satisfy $\mu(\omega = 0^+) > 1$.

III. CASIMIR INTERACTION OF TWO NONUNIFORM BIANISOTROPIC METAMATERIAL SLABS

In the second part of the paper, we extend the theory of Ref. [10] and study the Casimir interaction of two generic structured bianisotropic metamaterial slabs (Fig. 1). The structuring is such that the system is invariant under translations along the primitive vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, which are assumed to lie in the $x0y$ plane. Thus, the slabs are described by a continuous material bianisotropic model, such that the effective parameters $\bar{\bar{\varepsilon}}$, $\bar{\bar{\mu}}$, $\bar{\bar{\vartheta}}$, and $\bar{\bar{\zeta}}$ may depend on $\mathbf{r}$ in an arbitrary way, apart from being periodic in the transverse (x and y) coordinates. The metamaterial slabs are formed by the repetition of the unit cell $\Omega = \Omega_T \times [-\infty,+\infty]$ where $\Omega_T = \{(x,y) = \alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 : |\alpha_i| \leqslant 1/2\}$ is the transverse unit cell. The region of space $0 < z < d$ is a vacuum. The intersection of Ω with the semispace $z < 0$ ($z > d$) is denoted by A (B).

Next, we derive a $TGTG$ formulation [11] of the Casimir interaction of the bianisotropic bodies. As in Ref. [10], first we obtain the dispersion equation $D(\omega,\mathbf{k}_{||}) = 0$ of the Bloch-Floquet electromagnetic modes associated with the transverse wave vector $\mathbf{k}_{||} = (k_x,k_y,0)$. The zero-temperature Casimir interaction energy per unit of area, $\delta\mathcal{E}/A_s$, can be written in terms of D as follows [10]:

$$\frac{\delta\mathcal{E}}{A_s} = \frac{\hbar}{(2\pi)^3}\int_{\rm BZ} d^2\mathbf{k}_{||}\int_0^{+\infty} d\xi\,\ln D(i\xi,\mathbf{k}_{||}), \qquad (7)$$

where BZ stands for Brillouin zone. Formula (7) is derived exactly in the same way as in our previous work [10] by regularizing the calculation of the zero-point energy of the system using the argument principle [20,21].

It is well known that the Bloch modes (associated with the frequency ω and with the real valued wave vector $\mathbf{k}_{||}$) of a transverse periodic structure satisfy the generalized Lippmann-Schwinger integral equation,

$$\begin{pmatrix}\mathbf{E}\\ -\eta_0\mathbf{H}\end{pmatrix} = \left(\frac{\omega}{c}\right)^2\begin{pmatrix}\hat{G}_{\rm p} & \frac{ic}{\omega}\nabla\times\hat{G}_{\rm p}\\ \frac{ic}{\omega}\nabla\times\hat{G}_{\rm p} & -\hat{G}_{\rm p}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{P}/\varepsilon_0\\ \eta_0\mathbf{M}\end{pmatrix}, \qquad (8)$$

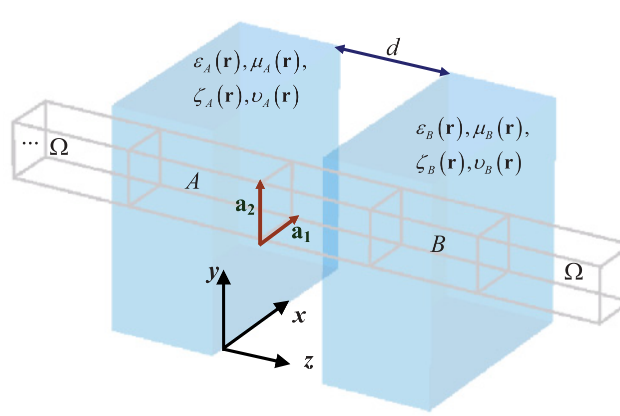

FIG. 1. (Color online) Two planar (transverse periodic) bianisotropic slabs (formed by metal-dielectric inclusions at the "microscopic" level) stand in a vacuum and are separated by a distance d. The framed region represents the basic cell Ω.

where **E**, **H**, **P**, and **M**, represent the electric field, the magnetic field, the polarization vector, and the magnetization vector, respectively, and η_0 is the intrinsic impedance in free space. The operator $\hat{G}_{\rm p}(\omega,\mathbf{k}_{||})$ is defined as in Ref. [10], and satisfies $\hat{G}_{\rm p} = (\nabla\times\nabla\times -\frac{\omega^2}{c^2})^{-1}$ subject to Bloch-periodic boundary conditions in the transverse directions. Noting that $\nabla\times\nabla\times\hat{G}_{\rm p} = \frac{\omega^2}{c^2}\hat{G}_{\rm p} + \hat{I}$, and taking into account that the curl operator, $\nabla\times$, commutes with $\hat{G}_{\rm p}$ because it evidently commutes with $\hat{G}_{\rm p}^{-1}$, we can rewrite Eq. (8) as follows:

$$\begin{pmatrix}\mathbf{E}\\ -\eta_0\mathbf{B}/\mu_0\end{pmatrix} - \frac{\omega^2}{c^2}\hat{G}_{\rm EB}\begin{pmatrix}\mathbf{P}/\varepsilon_0\\ \eta_0\mathbf{M}\end{pmatrix} = 0, \qquad (9)$$

where $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$ is the induction field. The operator $\hat{G}_{\rm EB}$ is given by $\hat{G}_{\rm EB} = \hat{U}\hat{G}_{\rm p}\hat{V}$ with $\hat{U}$ and $\hat{V}$ defined as

$$\hat{U} = \begin{pmatrix}\hat{I}\\ \frac{ic}{\omega}\nabla\times\end{pmatrix}, \quad \hat{V} = \left(\hat{I}\ \ \frac{ic}{\omega}\nabla\times\right). \qquad (10)$$

Using the bianisotropic constitutive relations it is possible to relate polarization and magnetization vectors to the macroscopic electric and induction fields as follows: $\binom{\mathbf{P}/\varepsilon_0}{\eta_0\mathbf{M}} = \hat{\chi}_{\rm EB}\binom{\mathbf{E}}{-\eta_0\mathbf{B}/\mu_0}$, where the (multiplication) operator $\hat{\chi}_{\rm EB} = \hat{\chi}_{\rm EB}(\omega)$ is represented by a 6×6 matrix defined as in Eq. (5). Thus, it follows that the Bloch eigenmodes of the structure satisfy the homogeneous system

$$\left(\hat{I} - \frac{\omega^2}{c^2}\hat{G}_{\rm EB}\hat{\chi}_{\rm EB}\right)\mathbf{F} = 0, \quad \mathbf{F} = \begin{pmatrix}\mathbf{E}\\ -\eta_0\mathbf{B}/\mu_0\end{pmatrix}. \qquad (11)$$

All the operators are defined on $H_{\Omega,\mathbf{k}_{||}}\times H_{\Omega,\mathbf{k}_{||}} \to H_{\Omega,\mathbf{k}_{||}}\times H_{\Omega,\mathbf{k}_{||}}$, and the six-vector field **F** is defined over the unit cell Ω.[2] Proceeding as in Ref. [10], it is possible to obtain an integral equation whose unknown is $\mathbf{F}_{\rm A}$, defined as the restriction of **F** to the region A. In this manner, it is found that the dispersion characteristic of the eigenmodes can be written as $D(\omega,\mathbf{k}_{||}) = 0$, with

$$D(\omega,\mathbf{k}_{||}) = \det(\hat{I}_{\rm A} - \hat{T}_{\rm A}\hat{G}_{\rm AB}\hat{T}_{\rm B}\hat{G}_{\rm BA}), \qquad (12)$$

where the operator inside brackets is defined on $H_{{\rm A},\mathbf{k}_{||}}\times H_{{\rm A},\mathbf{k}_{||}} \to H_{{\rm A},\mathbf{k}_{||}}\times H_{{\rm A},\mathbf{k}_{||}}$, and $\hat{T}_\alpha = -\frac{\omega^2}{c^2}\hat{\chi}_{{\rm EB},\alpha}(\hat{I}_\alpha - \frac{\omega^2}{c^2}\hat{G}_{\alpha\alpha}\hat{\chi}_{{\rm EB},\alpha})^{-1}$ with $\alpha = A,B$. By definition $\hat{G}_{\alpha\beta} \equiv \hat{G}_{{\rm EB},\alpha\beta}$ is the restriction of $\hat{G}_{\rm EB}$ on $H_{\beta,\mathbf{k}_{||}}\times H_{\beta,\mathbf{k}_{||}} \to H_{\alpha,\mathbf{k}_{||}}\times H_{\alpha,\mathbf{k}_{||}}$.

The previously mentioned analysis shows that the Casimir interaction energy can be formally calculated exactly in the same manner as in Ref. [10], except that in the bianisotropic case the operators $\hat{\chi}$ and $\hat{G}_{\rm p}$ of Ref. [10] must be replaced by $\hat{\chi}_{\rm EB}$ and $\hat{G}_{\rm EB}$. Based on this observation it is straightforward to verify that the results of Ref. [10] also hold in the bianisotropic case, provided the operators $\hat{\chi}_{\rm EB}$ and $\hat{G}_{\rm EB}$ are nonnegative for imaginary frequencies ($\omega = i\xi$) (i.e., $\hat{\chi}_{\rm EB} \geqslant 0$ and $\hat{G}_{\rm EB} \geqslant 0$). Indeed, these are the conditions necessary to transform the $TGTG$ structure into the $MM^\dagger$ structure [10].

The condition $\hat{G}_{\rm EB}(i\xi) \geqslant 0$ does not pose any difficulties. Indeed, for imaginary frequencies $\hat{V}(i\xi) = \hat{U}^\dagger(i\xi)$, and thus $\hat{G}_{\rm EB} = \hat{U}\hat{G}_{\rm p}\hat{U}^\dagger$. Thus, it is evident that for $\omega = i\xi$ we have $\hat{G}_{\rm EB} \geqslant 0$ because $\hat{G}_{\rm p} > 0$ [10]. On the other hand, since $\hat{\chi}_{\rm EB}$ is a multiplication operator, it is clear that the condition $\hat{\chi}_{\rm EB} \geqslant 0$ is satisfied if the matrix (5) is nonnegative for every point **r** fixed. Hence, it is sufficient to prove that for a uniform (homogeneous) material $\hat{\chi}_{\rm EB}(i\xi) \geqslant 0$. But, as demonstrated in the first part of this work, when the metamaterials consist of dielectric or metallic inclusions (with no intrinsic magnetism) such condition is necessarily observed when the effective medium is either bi-isotropic or nongyrotropic.

Therefore, using the theory of Ref. [10], we can conclude that in these circumstances the interaction between two structured bianisotropic slabs is attractive at all macroscopic distances (i.e., for $d > d_0$ where d_0 is some distance comparable to the transverse period of the nonuniform slabs). Moreover, in case the effective parameters ($\overline{\overline{\varepsilon}}$, $\overline{\overline{\mu}}$, $\overline{\overline{\vartheta}}$, and $\overline{\overline{\zeta}}$) of the slabs are independent of the transverse coordinates x and y the Casimir force is attractive at *all* distances!

Clearly, our findings oppose Ref. [9], which claimed that the interaction between two isotropic chiral metamaterials may be repulsive for a sufficiently large chirality parameter. Indeed, as demonstrated in [22], the material parameters considered in Ref. [9] are incompatible with the passivity and causality of the metamaterials. Other works have also studied scenarios where Casimir repulsion could occur [3–6], particularly when the magnetic permeability of the metamaterials follows either a Drude or a Lorentz characteristic. However, both the Drude and the Lorentz dispersion models are incompatible with the condition $\mu(\omega = 0^+) = 1$, and thus cannot describe accurately the magnetic response of metal-dielectric metamaterials near $\omega = 0$. A much more realistic model is the one described in Pendry's seminal work [23], which as shown in Ref. [3] is consistent with our theory that the magnetic response at imaginary frequencies is diamagnetic.

IV. CONCLUSION

We have studied the physical restrictions on the effective parameters of metal-dielectric metamaterials at imaginary frequencies, and demonstrated using effective-medium theory that the causality and passivity of the materials impose rather severe restrictions on the possibility of "quantum levitation." An important conclusion from our study is that the magnetic response of a metal-dielectric metamaterial is necessarily diamagnetic at imaginary frequencies.

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APPENDIX: THE CASE WHERE $\omega\overline{\overline{\chi}}$ HAS A POLE AT THE ORIGIN

In this appendix we generalize the results of Secs. II and III to the case where $\overline{\overline{\chi}}(\omega,\mathbf{k}) = \overline{\overline{\varepsilon}}_{\rm eff}(\omega,\mathbf{k})/\varepsilon_0 - \overline{\overline{\mathbf{I}}}$ has a pole of

[2] $H_{\Omega,\mathbf{k}_{||}}$ is the space of (square integrable) vector fields defined over the unit cell that satisfy Bloch-Floquet boundary conditions determined by $\mathbf{k}_{||}$ in the transverse (x and y) coordinates.

order two at the origin. As discussed before, this situation can only occur in case of natural materials with intrinsic magnetism (and eventually also in the case of superconducting metamaterials). It can be easily shown that Eq. (1) remains valid provided the term $\overline{\overline{\chi}}(i\xi,\mathbf{k})$ in the left-hand side of the equation is replaced by $\overline{\overline{\chi}}(i\xi,\mathbf{k}) + \frac{1}{\xi^2}[\lim_{\omega\to 0}\overline{\overline{\chi}}(\omega,\mathbf{k})\omega^2]$. Thus, it follows that

$$\overline{\overline{\varepsilon}}(i\xi,\mathbf{k}) - \varepsilon_0\overline{\overline{\mathbf{I}}} + \frac{1}{\xi^2}\left[\lim_{\omega\to 0}\overline{\overline{\varepsilon}}(\omega,\mathbf{k})\omega^2\right] > 0. \quad \text{(A1)}$$

In particular, let us consider the case where the nonlocal dielectric function satisfies Eq. (3) (bianisotropic model). To simplify, we also assume that there is no magnetoelectric coupling ($\overline{\overline{\zeta}} = \overline{\overline{\vartheta}} = 0$). In this case it is clear that for arbitrary real valued $\mathbf{k}$ we have

$$\overline{\overline{\varepsilon}}(i\xi) - \overline{\overline{\mathbf{I}}} - \frac{1}{\xi^2}\mathbf{k}\times[\overline{\overline{\mu}}^{-1}(i\xi) - \overline{\overline{\mu}}^{-1}(0)]\times\mathbf{k} > 0. \quad \text{(A2)}$$

Thus, it is evident that the local permittivity and permeability satisfy $\overline{\overline{\varepsilon}}(i\xi) > \overline{\overline{\mathbf{I}}}$ and $\overline{\overline{\mu}}^{-1}(i\xi) - \overline{\overline{\mu}}^{-1}(0) \geqslant 0$, where $\overline{\overline{\mu}}(0)$ is the static permeability tensor.

Suppose now that the material is diamagnetic in the static limit: $\overline{\overline{\mu}}^{-1}(0) > \overline{\overline{\mathbf{I}}}$. Then it is clear that $\overline{\overline{\mu}}^{-1}(i\xi) - \overline{\overline{\mathbf{I}}} \geqslant 0$, and thus it follows that $\hat{\chi}_{\text{EB}}$ defined as in Sec. II is non-negative: $\hat{\chi}_{\text{EB}} \geqslant 0$. Hence, as could be expected, the restrictions on Casimir repulsion derived in Sec. III also apply to media with an intrinsic diamagnetic response in the static limit.

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