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Metamaterials 3 (2009) 115–128

Metamaterials

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Nonlocal homogenization of an array of cubic particles made of resonant rings

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Received 19 May 2009; received in revised form 24 August 2009; accepted 28 August 2009
Available online 8 September 2009

Abstract

Here, we develop a nonlocal homogenization model to characterize the electrodynamics of an array of cubic particles made of resonant rings. The effective parameters are calculated from the microscopic fields produced by a periodic external excitation. It is confirmed that the spatial dispersion effects cannot be neglected in the regime where $\mu \approx 0$. We demonstrate that when the array of resonant rings is combined with a triple wire medium formed by connected wires, the structure may behave approximately as an isotropic left-handed material.

PACS: 42.70.Qs; 78.66.Sq

Keywords: Homogenization; Spatial dispersion; Artificial magnetism

1. Introduction

Structured materials with a strong magnetic response have been under intense research in recent years [1–10], mainly due to their potential applications in the design of imaging systems with improved resolution [2–5]. In particular, it was recently demonstrated that a metamaterial lens formed by split-ring resonators (SRRs) boosts the sensitivity of the coil used in magnetic resonance imaging when operated in the regime $\mu = -1$ [6].

In most of the studies published in the literature, it is typically assumed from the outset that the response of the metamaterial is local, and, based on that assumption the effective parameters are usually calculated using the retrieval procedure reported in Ref. [11] (inversion of the scattering parameter data). Recently, in Ref. [12] the Lorentz local field theory was used to homogenize a metamaterial formed by an array of cubic particles with tetrahedral symmetry formed by split-ring resonators (a topology similar to that considered here), and the nonlocal magnetic permeability was calculated. It was demonstrated that the spatially dispersive model provides a unified description of the transverse electromagnetic waves and of the so-called magnetoinductive waves [13], demonstrating in this manner that the latter are a short-wavelength continuation of the former. In this work, we investigate a problem closely related to

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doi:10.1016/j.metmat.2009.08.004

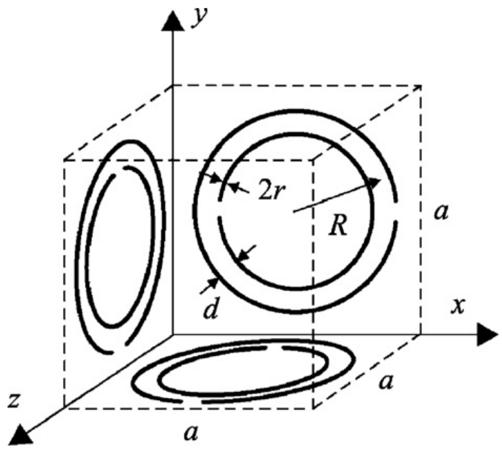

Fig. 1. Geometry of the unit cell of a structured material formed by resonant rings. The figure depicts the particular case in which the resonant particles are SRRs.

that considered in Ref. [12] but from a different perspective. Instead of applying the Lorentz's local field theory, we compute the nonlocal parameters using the homogenization method proposed in the works [14,15], which is based on the idea of exciting the metamaterial with a periodic source with suitable phase-shift. We obtain the exact solution (taking into account the interaction between all the particles in lattice) of the homogenization problem under the approximation that the response of the inclusions can be described using the dipole approximation, and an explicit formula for the nonlocal magnetic permeability is derived. The results of the analytical model are compared with the effective parameters obtained with full wave simulations that take into account all the details of the microstructure of the material. Finally, we study the electrodynamics of a system formed by the array of SRRs and a connected array of wires. It should be mentioned that other previous works (e.g. Ref. [16]) have studied the homogenization of arrays of SRRs taking into account rigorously the mutual effects and lattice ordering. The main contribution of our analysis, which extends our previous work [12], is the characterization of the spatial dispersion effects.

2. Homogenization

A representative geometry of the unit cell of the material under analysis is shown in Fig. 1. The unit cell contains three different resonant metallic rings, being each ring normal to one of the Cartesian axes. The lattice is simple cubic with lattice constant a. Clearly, each basic inclusion has an anisotropic response, but in the long wavelength limit the response of the composite material is approximately isotropic due to the spatial arrangement and orientation of the particles. The ith resonant ring in the unit cell is by definition normal to the unit vector $\hat{\mathbf{u}}_i$ ($i = x, y, z$), and is centered at the point $\mathbf{r}_{0,i} = -(a/2)\,\hat{\mathbf{u}}_i$.

The resonant rings are generic planar (or quasi-planar) inclusions, which may be produced by some lumped or distributed capacitance. We suppose that the rings can be characterized by an impedance $Z_0 = j\omega L + 1/(j\omega C)$ obtained from a circuit model (for simplicity the effect of metallic loss is neglected; see Refs. [17,18] for the particular case of SRRs). The resonant rings will be modeled as dipole-type magnetic particles characterized by a uniaxial magnetic polarizability dyadic (tensor). The magnetic polarizability dyadic of the ith ring is

$$\bar{\bar{\alpha}}_i = \alpha_m \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i, \tag{1}$$

where $\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i \equiv \hat{\mathbf{u}}_i \otimes \hat{\mathbf{u}}_i$ represents the dyadic (tensor) product of two vectors. The parameter α_m (with unities $[m^3]$) is related to the impedance Z_0 as follows (including the effect of the radiation loss)

$$\alpha_m^{-1} = \frac{Z_0}{-j\omega A^2 \mu_0} + j\frac{1}{6\pi}\left(\frac{\omega}{c}\right)^3 = \alpha_0^{-1}\left[\left(\frac{\omega_r}{\omega}\right)^2 - 1\right] + j\frac{1}{6\pi}\left(\frac{\omega}{c}\right)^3, \tag{2}$$

where $\alpha_0^{-1} = L/\mu_0 A^2$, A is the area of the ring and $\omega_r = 1/\sqrt{LC}$.

For simplicity, it will be assumed that the rings do not have an electric response (i.e. that the electric polarizability vanishes; in Section 3, we will discuss how to incorporate the electric response in the model). Thus, at a microscopic level, the magnetic dipole moment of the ith particle in the unit cell, $\mathbf{p}_i$, must verify:

$$\frac{\mathbf{p}_i}{\mu_0} = \bar{\bar{\alpha}}_i \cdot \mathbf{H}_{\text{loc}}\left(\mathbf{r}_{0,i}\right), \quad i = x, y, z, \tag{3}$$

where $\mathbf{H}_{\text{loc}}$ is the local magnetic field that polarizes the pertinent ring. The magnetic dipole $\mathbf{p}_i$ as defined above is related to the more traditional definition given in textbooks (e.g. Ref. [19]) as $\mathbf{m} = \mathbf{p}_i/\mu_0$.

2.1. Two models for the response of the rings

It is possible to model the response of the resonant rings using two alternative approaches. The first model takes into account that the rings are metallic particles, and thus that an external field induces a microscopic *electric current density* in each ring. Since it is assumed that the rings only have a magnetic response, which is necessarily caused by the vortex part of the induced electric current (artificial magnetism), the induced current is related to the magnetic dipole moments as follows (see

Ref. [15] for a related result)

$$\mathbf{J}_{e,\mathrm{dip}} = \nabla \times \left\{ \sum_{i=1,2,3} \sum_{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}_{\mathbf{I}} - \mathbf{r}_{0,i}) \frac{\mathbf{p}_i}{\mu_0} e^{-j\mathbf{k}\cdot\mathbf{r}_{\mathbf{I}}} \right\}, \tag{4}$$

where $\mathbf{r}_{\mathbf{I}} = a(i_1, i_2, i_3)$ represents a generic lattice point, $\mathbf{I} = (i_1, i_2, i_3)$ is a triple index of integers, and $\mathbf{p}_i$ is the magnetic dipole moment of the ith particle in the unit cell. Following the works [14,15], the unbounded metamaterial can be characterized by a dielectric function of the type $\bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}(\omega, \mathbf{k})$, where ω is the angular frequency and $\mathbf{k} = (k_x, k_y, k_z)$ is the wave vector. The procedure to compute the dielectric function for a given ω and $\mathbf{k}$ is to excite the metamaterial with an external Floquet-type electric current density $\mathbf{J}_{e,\mathrm{ext}} = \mathbf{J}_{e,\mathrm{av}}\, e^{-j\mathbf{k}\cdot\mathbf{r}}$, where $\mathbf{J}_{e,\mathrm{av}}$ is a constant vector [14]. Then, one needs to solve the corresponding source-driven electromagnetic problem and calculate the microscopic fields. The averaged macroscopic fields and the generalized electric polarization vector are computed using the microscopic fields. Finally, the unknown dielectric function $\bar{\bar{\varepsilon}}(\omega, \mathbf{k})$ is obtained from the macroscopic fields and from the generalized polarization vector [14]. This approach is further developed in Appendix A.

The second model is based on the observation that since the resonant rings only have a magnetic response, they may be as well regarded as pure magnetic particles which, when excited by an external field, originate a *magnetic current density* given by:

$$\mathbf{J}_{m,\mathrm{dip}} = \sum_{i=1,2,3} \left(\sum_{\mathbf{I}} \delta\left(\mathbf{r} - \mathbf{r}_{\mathbf{I}} - \mathbf{r}_{0,i}\right) j\omega \mathbf{p}_i\, e^{-j\mathbf{k}\cdot\mathbf{r}_{\mathbf{I}}} \right), \tag{5}$$

i.e. the second model for the response of the rings regards the inclusions as magnetic particles characterized by a magnetic current density (instead of an electric current density as in Eq. (4)). Such model may seem less rigorous than the model associated with Eq. (4), but as discussed below, they actually lead to equivalent results after proper homogenization.

It is important to note that the homogenization method introduced in Ref. [14] assumes that the inclusions are either dielectric or metallic materials with constant permeability ($\mu = \mu_0$), i.e. the particles must have exclusively an electric response (even though, as mentioned before, magnetic effects may occur due to the eddy part of the electric current). Thus, the method of Ref. [14] cannot be directly applied to characterize a material formed by pure magnetic particles, for which the response is characterized by a magnetic current density as in Eq. (5). However, evidently a structured material formed by magnetic-type inclusions is the electromagnetic dual of a material formed by electric-type inclusions, and thus it is trivial to modify and generalize the method of Ref. [14] to such configurations. It should be clear that such modified homogenization approach is based on the introduction of a magnetic function of the type $\bar{\bar{\mu}} = \bar{\bar{\mu}}(\omega, \mathbf{k})$, which, as detailed below, can be calculated by exciting the metamaterial with an external magnetic current density $\mathbf{J}_{m,\mathrm{ext}} = \mathbf{J}_{m,\mathrm{av}}\, e^{-j\mathbf{k}\cdot\mathbf{r}}$.

One important point is the relation between the dielectric function $\bar{\bar{\varepsilon}}(\omega, \mathbf{k})$ calculated within the framework of the first model (developed in Appendix A), and the magnetic function $\bar{\bar{\mu}} = \bar{\bar{\mu}}(\omega, \mathbf{k})$ calculated within the framework of the second model (developed in Section 2.2). From the results of Appendix A, it turns out that:

$$\frac{\bar{\bar{\varepsilon}}(\omega, \mathbf{k})}{\varepsilon_0} = \bar{\bar{\mathbf{I}}} + c^2 \frac{\mathbf{k}}{\omega} \times \left(\mu_0 \bar{\bar{\mu}}^{-1} - \bar{\bar{\mathbf{I}}} \right) \times \frac{\mathbf{k}}{\omega}, \tag{6}$$

where $\bar{\bar{\mathbf{I}}}$ is the identity dyadic and c is the speed of light in the host medium (assumed vacuum for simplicity). Interestingly, in Refs. [14,15,20,21] it is demonstrated that an unbounded material characterized by a given magnetic permeability $\bar{\bar{\mu}}$ and an electric permittivity $\varepsilon = \varepsilon_0$, can be as well characterized by a nonlocal dielectric function defined as in Eq. (6) (in the sense that the plane wave dispersion characteristic and the associated electric field polarizations are the same independent of the adopted constitutive relations). Thus, Eq. (6) demonstrates that the two models described in this section yield equivalent results. We will adopt the second model in the following sections to calculate an explicit expression for $\bar{\bar{\mu}} = \bar{\bar{\mu}}(\omega, \mathbf{k})$.

2.2. The homogenization problem

Following the discussion of Section 2.1, here we assume that the rings can be modeled as pure magnetic particles, whose response to an external field originates a magnetic current density given by Eq. (5). In order to determine the unknown $\bar{\bar{\mu}} = \bar{\bar{\mu}}(\omega, \mathbf{k})$, we excite the electromagnetic crystal with an external (magnetic-type) source such that

$$\mathbf{J}_{m,\mathrm{ext}} = j\omega \frac{\mathbf{p}_{\mathrm{ext}}}{V_{\mathrm{cell}}} e^{-j\mathbf{k}\cdot\mathbf{r}}, \tag{7}$$

where $V_{\mathrm{cell}} = a^3$ is the volume of the unit cell, and $\mathbf{p}_{\mathrm{ext}}$ is a constant vector that determines the applied current.

The microscopic fields $(\mathbf{E}, \mathbf{H})$ are the solution of the electromagnetic problem

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H} - \mathbf{J}_{m,\mathrm{ext}} - \mathbf{J}_{m,\mathrm{dip}} \tag{8a}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon_0\mathbf{E}, \tag{8b}$$

where $\mathbf{J}_{m,\mathrm{dip}}$ is given by Eq. (5), and is written in terms of the unknown magnetic dipole moments of the particles in the unit cell $\mathbf{p}_i$ (which obviously depend on the external field). Notice that since the current source $\mathbf{J}_{m,\mathrm{ext}}$ has the Floquet property and the material is periodic, it is clear that both $(\mathbf{E}, \mathbf{H})$ and $\mathbf{J}_{m,\mathrm{dip}}$ have the Floquet property as well (this actually justifies formula (5)). It is important to emphasize that system (8) is a source driven problem and not an eigenvalue problem. Notice also that in this formulation all the microscopic currents are purely magnetic, because the rings are modeled as true magnetic particles.

The solution of (8) can be written in a straightforward manner in terms of the Green dyadic $\bar{\bar{\mathbf{G}}}_p\left(\mathbf{r}|\,\mathbf{r}';\omega,\mathbf{k}\right) = \left(\bar{\bar{\mathbf{I}}} + c^2/\omega^2\nabla\nabla\right)\Phi_p\left(\mathbf{r}|\,\mathbf{r}';\omega,\mathbf{k}\right)$ introduced in Refs. [14,15], being Φ_p the lattice Green function [22]. The Green dyadic verifies $\nabla \times \nabla \times \bar{\bar{\mathbf{G}}}_p\left(\mathbf{r}|\,\mathbf{r}'\right) - (\omega/c)^2\bar{\bar{\mathbf{G}}}_p\left(\mathbf{r}|\,\mathbf{r}'\right) = \bar{\bar{\mathbf{I}}}\left(\sum_{\mathbf{I}}\delta\left(\mathbf{r}-\mathbf{r}'-\mathbf{r}_{\mathbf{I}}\right)e^{-j\mathbf{k}\cdot\mathbf{r}}\right)$. Thus, it is simple to verify that the solution of system (8) is such that:

$$\mathbf{H}(\mathbf{r}) = (-j\omega\varepsilon_0)\sum_{i=1,2,3}\bar{\bar{\mathbf{G}}}_p\left(\mathbf{r}|\,\mathbf{r}_{0,i}\right)\cdot j\omega\mathbf{p}_i + (-j\omega\varepsilon_0)\,\bar{\bar{\mathbf{G}}}_{\mathrm{av}}\cdot j\omega\mathbf{p}_{\mathrm{ext}}\,e^{-j\mathbf{k}\cdot\mathbf{r}}, \tag{9}$$

where $\bar{\bar{\mathbf{G}}}_{\mathrm{av}}$ is the dyadic

$$\bar{\bar{\mathbf{G}}}_{\mathrm{av}} = \frac{1}{V_{\mathrm{cell}}}\frac{\bar{\bar{\mathbf{I}}} - c^2/\omega^2\mathbf{k}\mathbf{k}}{k^2-(\omega/c)^2}, \tag{10}$$

with $k^2 = \mathbf{k}\cdot\mathbf{k}$ and $\mathbf{k}\mathbf{k} = \mathbf{k}\otimes\mathbf{k}$. The first term in the right-hand side of Eq. (9) corresponds to the field created by the induced magnetic dipoles, whereas the second terms corresponds to the field created by the external source $\mathbf{J}_{m,\mathrm{ext}}$. It is interesting to note that $\bar{\bar{\mathbf{G}}}_{\mathrm{av}}$ is the spatial average of the Green dyadic:

$$\bar{\bar{\mathbf{G}}}_{\mathrm{av}} = \frac{1}{V_{\mathrm{cell}}}\int_{\mathrm{cell}}\bar{\bar{\mathbf{G}}}_p\left(\mathbf{r}|\,\mathbf{r}'\right)e^{+j\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}d^3\mathbf{r}'. \tag{11}$$

To obtain the complete solution of (8) we still have to determine the magnetic dipole moments, $\mathbf{p}_i$ $(i = 1, 2, 3)$, of the rings in the unit cell. This can be done using the microscopic relations (3). It is clear from Eq. (9) that the local field on the ith ring is

$$\mathbf{H}_{\mathrm{loc},i} = \left(\frac{\omega}{c}\right)^2\overline{\overline{\mathbf{G}'_p}}(0|\,0)\cdot\frac{\mathbf{p}_i}{\mu_0} + \sum_{j\neq i}\left(\frac{\omega}{c}\right)^2\bar{\bar{\mathbf{G}}}_p(\mathbf{r}_{0,i}|\mathbf{r}_{0,j})\cdot\frac{\mathbf{p}_j}{\mu_0} + \left(\frac{\omega}{c}\right)^2\bar{\bar{\mathbf{G}}}_{\mathrm{av}}\cdot\frac{\mathbf{p}_{\mathrm{ext}}}{\mu_0}e^{-j\mathbf{k}\cdot\mathbf{r}_{0,i}}, \tag{12}$$

where by definition

$$\overline{\overline{\mathbf{G}'_p}}\left(\mathbf{r}|\,\mathbf{r}'\right) = \bar{\bar{\mathbf{G}}}_p\left(\mathbf{r}|\,\mathbf{r}'\right) - \overline{\overline{\mathbf{G}_0}}\left(\mathbf{r}|\,\mathbf{r}'\right) \tag{13}$$

and $\overline{\overline{\mathbf{G}_0}}\left(\mathbf{r}|\,\mathbf{r}'\right)$ is the free-space Green dyadic. Using now the microscopic relations (3) and the fact that $\mathbf{p}_i = p_i\hat{\mathbf{u}}_i$, it is readily found that for $i = 1, 2, 3$

$$\alpha_m^{-1}\frac{p_i}{\mu_0} = \left(\frac{\omega}{c}\right)^2\left(\hat{\mathbf{u}}_i\cdot\overline{\overline{\mathbf{G}'_p}}(0|\,0)\cdot\hat{\mathbf{u}}_i\frac{p_i}{\mu_0} + \sum_{j\neq i}\hat{\mathbf{u}}_i\cdot\bar{\bar{\mathbf{G}}}_p\left(\mathbf{r}_{0,i}|\,\mathbf{r}_{0,j}\right)\cdot\hat{\mathbf{u}}_j\frac{p_j}{\mu_0} + \hat{\mathbf{u}}_i\cdot\bar{\bar{\mathbf{G}}}_{\mathrm{av}}\cdot\frac{\mathbf{p}_{\mathrm{ext}}}{\mu_0}e^{-j\mathbf{k}\cdot\mathbf{r}_{0,i}}\right). \tag{14}$$

The above equation can be written in matrix notation as follows:

$$[a_{ij}]\cdot\begin{pmatrix}p_1e^{+j\mathbf{k}\cdot\mathbf{r}_{0,1}}\\ p_2e^{+j\mathbf{k}\cdot\mathbf{r}_{0,2}}\\ p_3e^{+j\mathbf{k}\cdot\mathbf{r}_{0,3}}\end{pmatrix} = \left(\frac{\omega}{c}\right)^2\overline{\overline{\mathbf{G}}}_{\mathrm{av}}\cdot\mathbf{p}_{\mathrm{ext}}, \tag{15}$$

where the matrix entries are defined as $a_{ii} = \alpha_m^{-1} - \hat{\mathbf{u}}_i\cdot(\omega/c)^2\overline{\overline{\mathbf{G}'_p}}(0|\,0)\cdot\hat{\mathbf{u}}_i$, and $a_{ij} = -\hat{\mathbf{u}}_i\cdot(\omega/c)^2\bar{\bar{\mathbf{G}}}_p\left(\mathbf{r}_{0,i}|\,\mathbf{r}_{0,j}\right)e^{j\mathbf{k}\cdot(\mathbf{r}_{0,i}-\mathbf{r}_{0,j})}\cdot\hat{\mathbf{u}}_j$ for $i\neq j$. Eq. (15) formally relates the amplitudes of the induced magnetic dipole moments p_i $(i = 1, 2, 3)$ with the external source $(\mathbf{p}_{\mathrm{ext}})$. Thus, the microscopic fields are completely determined by the solution of Eq. (15).

We are now in a position to determine the magnetic function $\bar{\bar{\mu}} = \bar{\bar{\mu}}(\omega,\mathbf{k})$ of the metamaterial. To begin with, we introduce the macroscopic averaged electric and magnetic fields, which for the present homogenization problem where all the particles are purely magnetic, should be defined as

$$\mathbf{E}_{\mathrm{av}} = \frac{1}{V_{\mathrm{cell}}}\int_{\mathrm{cell}}\mathbf{E}(\mathbf{r})\,e^{+j\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r},$$

$$\mathbf{H}_{\mathrm{av}} = \frac{1}{V_{\mathrm{cell}}}\int_{\mathrm{cell}}\mathbf{H}(\mathbf{r})\,e^{+j\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r}. \tag{16}$$

It is simple to verify that the Maxwell equations (8) imply that

$$\omega\mu_0\mathbf{H}_{av} - \mathbf{k}\times\mathbf{E}_{av} = -\omega\frac{\mathbf{p}_{ext}}{V_{cell}} - \omega\mu_0\mathbf{M} \tag{17a}$$

$$\mathbf{k}\times\mathbf{H}_{av} + \omega\varepsilon_0\mathbf{E}_{av} = 0, \tag{17b}$$

where the magnetization vector $\mathbf{M}$ was defined as

$$\mathbf{M} \equiv \frac{\left(\mathbf{J}_{m,\mathrm{dip}}\right)_{av}}{j\omega\mu_0} = \frac{1}{V_{cell}}\sum_{i=1,2,3}\frac{\mathbf{p}_i}{\mu_0}e^{j\mathbf{k}\cdot\mathbf{r}_{0,i}} \tag{18}$$

and we used the fact that $(\mathbf{J}_{m,\mathrm{ext}})_{av} = j\omega\mathbf{p}_{ext}/V_{cell}$, being $(\mathbf{J}_{m,\mathrm{dip}})_{av}$ and $(\mathbf{J}_{m,\mathrm{ext}})_{av}$ defined consistently with Eq. (16).

The magnetic function $\bar{\bar{\mu}}$ must ideally be such that, independent of the external excitation, one has $\mathbf{B}_{av} = \bar{\bar{\mu}}\cdot\mathbf{H}_{av}$ where by definition the macroscopic induction field is given by $\mathbf{B}_{av} \equiv \mu_0(\mathbf{H}_{av}+\mathbf{M})$, consistently with the classical formula. Thus, $\bar{\bar{\mu}}$ must be such that $\left(\bar{\bar{\mu}}/\mu_0 - \bar{\bar{\mathbf{I}}}\right)\cdot\mathbf{H}_{av} = \mathbf{M}$, for arbitrary $\mathbf{p}_{ext}$.

One interesting aspect implicit in the previous discussion is that when the metamaterial is formed by purely magnetic particles, as considered in this section, the fundamental field entity (obtained from averaging the microscopic field) is the magnetic field intensity ($\mathbf{H}$), whereas the derived field is the induction field ($\mathbf{B}$). Indeed, this a trivial consequence of the formulation of Ref. [14], noting that $\mathbf{H}$ is the electromagnetic dual of $\mathbf{E}$, and $\mathbf{B}$ is the electromagnetic dual of $\mathbf{D}$. Quite differently, when all the particles only have an electric response (e.g. standard dielectrics or metals) the fundamental field quantity is the induction field, whereas in those conditions the magnetic field should be regarded as the derived quantity [14] (see also Appendix A).

In order to calculate $\bar{\bar{\mu}}$ we substitute (17b) into (17a) and solve the resulting equation with respect to $\mathbf{p}_{ext}$. It may be easily verified that:

$$\frac{\mathbf{p}_{ext}}{\mu_0} = -\mathbf{M}V_{cell} + \frac{c^2}{\omega^2}\bar{\bar{\mathbf{G}}}_{av}^{-1}\cdot\mathbf{H}_{av}, \tag{19}$$

where $\bar{\bar{\mathbf{G}}}_{av}^{-1}$ is the inverse of the dyadic defined by Eq. (10). Now, substituting the above equation into Eq. (15), and noting that since $\mathbf{p}_i = p_i\hat{\mathbf{u}}_i$ the magnetization vector is such that $\mu_0 V_{cell}\mathbf{M} = (p_1e^{+j\mathbf{k}\cdot\mathbf{r}_{0,1}}, p_2e^{+j\mathbf{k}\cdot\mathbf{r}_{0,2}}, p_3e^{+j\mathbf{k}\cdot\mathbf{r}_{0,3}})$, it is found that:

$$\left[a_{ij}\right]\cdot\mathbf{M} = -\left(\frac{\omega}{c}\right)^2\bar{\bar{\mathbf{G}}}_{av}\cdot\mathbf{M} + \frac{1}{V_{cell}}\mathbf{H}_{av}. \tag{20}$$

Therefore, it is clear that in order that $\left(\bar{\bar{\mu}}/\mu_0 - \bar{\bar{\mathbf{I}}}\right)\cdot\mathbf{H}_{av} = \mathbf{M}$, $\bar{\bar{\mu}}$ should verify

$$\frac{\bar{\bar{\mu}}}{\mu_0} = \bar{\bar{\mathbf{I}}} + \frac{1}{V_{cell}}\bar{\bar{\chi}}^{-1}, \tag{21}$$

where the dyadic $\bar{\bar{\chi}}$ is such that (with $\chi_{ij} = \hat{\mathbf{u}}_i\cdot\bar{\bar{\chi}}\cdot\hat{\mathbf{u}}_j$),

$$\chi_{ii} = \alpha_m^{-1} - \hat{\mathbf{u}}_i\cdot\bar{\bar{\mathbf{C}}}_{int}\cdot\hat{\mathbf{u}}_i \tag{22a}$$

$$\chi_{ij} = -\hat{\mathbf{u}}_i\cdot\bar{\bar{\mathbf{D}}}_{int}\left(\mathbf{r}_{0,i}\middle|\,\mathbf{r}_{0,j}\right)\cdot\hat{\mathbf{u}}_j,\quad \text{for } i\neq j \tag{22b}$$

and $\bar{\bar{\mathbf{C}}}_{int}$ and $\bar{\bar{\mathbf{D}}}_{int}$ are defined by

$$\bar{\bar{\mathbf{C}}}_{int}(\omega,\mathbf{k}) = \left(\frac{\omega}{c}\right)^2\left(\overline{\overline{\mathbf{G}'_p}}(0|\,0) - \bar{\bar{\mathbf{G}}}_{av}\right) \tag{23a}$$

$$\bar{\bar{\mathbf{D}}}_{int}\left(\mathbf{r}|\,\mathbf{r}';\omega,\mathbf{k}\right) = \left(\frac{\omega}{c}\right)^2\left(\overline{\overline{\mathbf{G}_p}}(\mathbf{r}|\mathbf{r}')e^{j\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} - \bar{\bar{\mathbf{G}}}_{av}\right). \tag{23b}$$

Eq. (21) establishes that the magnetic function (i.e. the spatially dispersive permeability) of the material can be written exclusively in terms of magnetic polarizability of the basic particles and of the *interaction* dyadics $\bar{\bar{\mathbf{C}}}_{int}$ and $\bar{\bar{\mathbf{D}}}_{int}$. The dyadic $\bar{\bar{\mathbf{C}}}_{int}$ describes the interaction between rings with the same orientation, whereas $\bar{\bar{\mathbf{D}}}_{int}$ describes the interaction between rings with different orientations. In general, these dyadics need to be numerically evaluated (using for example the mixed-domain Green function representation of Ref. [22]). It is interesting to mention that $\bar{\bar{\mathbf{C}}}_{int}$ is precisely the same dyadic that was obtained in Ref. [15]. Specifically, $\bar{\bar{\mathbf{C}}}_{int}$ relates the local fields and the macroscopic fields for point dipole particles (Lorentz–Lorenz formulas). In particular, it was demonstrated in Ref. [15] that for a simple cubic lattice (as the one considered in this work), we have that to a first approximation:

$$C_{11}(\omega,\mathbf{k}) \equiv \hat{\mathbf{u}}_1\cdot\mathrm{Re}\left\{\bar{\bar{\mathbf{C}}}_{int}\right\}\cdot\hat{\mathbf{u}}_1 \approx \frac{1}{a^3}\left[\frac{1}{3} - 0.15\left(\frac{\omega}{c}a\right)^2 + 0.052\left(\cos\left(k_xa\right)-1\right)\right.$$
$$\left. -0.026\left(\cos\left(k_ya\right)-1\right) - 0.026\left(\cos\left(k_za\right)-1\right)\right] \tag{24}$$

The interaction constants C_{22} and C_{33} are defined similarly, by permutating the wave vector components k_x, k_y and k_z. The imaginary part of the interaction dyadic verifies $\mathrm{Im}\left\{\bar{\bar{\mathbf{C}}}_{int}\right\} = \frac{1}{6\pi}(\omega/c)^3\bar{\bar{\mathbf{I}}}$ [15]. The first term in

Eq. (24) is the familiar static interaction constant, $1/3a^3$, for a simple cubic lattice. The second term in Eq. (24) is a frequency correction of the static term. The remaining terms, whose amplitude is evidently quite small, are related to structural spatial dispersion effects [15]. In general these small corrections are negligible, except near to a resonance of the electric/magnetic response of the particles where they may play an important role [15,23].

On the other hand, it can be shown that because of symmetry reasons $\mathrm{Im}\left\{\bar{\bar{\mathbf{D}}}_{\mathrm{int}}\left(\mathbf{r}_{0,i}\middle|\,\mathbf{r}_{0,j}\right)\right\}=0,\ i\neq j$. Moreover, $D_{\mathrm{int},ij}=\hat{\mathbf{u}}_i\cdot\bar{\bar{\mathbf{D}}}_{\mathrm{int}}\left(\mathbf{r}_{0,i}\middle|\,\mathbf{r}_{0,j}\right)\cdot\hat{\mathbf{u}}_j$, can be very accurately represented by the following Taylor series in powers of k_i, k_j and (ω/c) $(i\neq j)$

$$D_{\mathrm{int},ij}(\omega,\mathbf{k})\approx\frac{k_ik_j}{a}\left[-0.1936-0.01188\left(\frac{\omega}{c}a\right)^2+0.005902\,a^2\left({k_i}^2+{k_j}^2\right)\right],\qquad(25)$$

even when the wave vector is close to the boundary of the Brillouin zone or when the frequency is moderately large. It is interesting to note that $D_{\mathrm{int},ij}=0$ (for arbitrary i,j, with $i\neq j$) when the wave vector $\mathbf{k}$ is directed along one of the coordinate axes, i.e. in these conditions the coupling between the resonant rings in the unit cell is mutually cancelled.

It is clear that Eqs. (24) and (25) imply that in the very long wavelength limit, when $|ka|\ll1$ and $|\omega a/c|\ll1$, the dyadic $\bar{\bar{\chi}}$ is approximately diagonal and verifies $\bar{\bar{\chi}}=\left(Re\left\{\alpha_m^{-1}\right\}-1/3a^3\right)\bar{\bar{\mathbf{I}}}$. Thus, within such approximations the nonlocal permeability reduces to the classical Clausius–Mossotti formula

$$\frac{\bar{\bar{\mu}}}{\mu_0}\approx\left(1+\frac{1}{V_{\mathrm{cell}}}\frac{1}{\mathrm{Re}\left\{\alpha_m^{-1}\right\}-1/3a^3}\right)\bar{\bar{\mathbf{I}}}.\qquad(26)$$

This result confirms that in the very long wavelength limit the material response is isotropic, as could be expected from the symmetry of the lattice. Other materials formed by uniaxial resonators, suitably oriented along different directions in order to yield an isotropic response, have been as well considered in Refs. [24–26]. It will be shown below that such simple result (which neglects the nonlocal effects) is in general invalid, particularly when the effective permeability is near zero.

2.3. Comparison with the model of Ref. [12]

In our previous paper (Ref. [12]) an array of cubic particles made of SRRs was homogenized using a local field approach, and the nonlocal permeability was written in terms of an impedance matrix that incorporates all the magnetoinductive effects between the rings. The case of wave propagation along one of the Cartesian axes was analyzed in particular detail and it was shown that for propagation along the x-direction ($k_y=k_z=0$), the yy-component of the permeability dyadic is [12]

$$\left.\frac{\mu_{yy}}{\mu_0}\right|_{k_y=k_z=0}\approx1+\frac{1}{a^3}\frac{1}{\mathrm{Re}\left\{\alpha_m^{-1}\right\}-\left(\alpha_0^{-1}2M_c/L\left(\cos\left(k_xa\right)-1\right)+\alpha_0^{-1}\left(2M_a/L+4M_c/L\right)\right)-1/3a^3},\qquad(27)$$

where M_a and M_c are the mutual inductances between closest rings of the same orientation, placed in the axial and the coplanar directions, respectively, and α_0 is defined as in Eq. (2). For the particular case of magnetic dipole-type inclusions, straightforward calculations (using the fact that the magnetic field created by a magnetic dipole is $\mathbf{B}=(\mu_0/4\pi)(3\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{m}-\mathbf{m}/r^3)$ [19]) show that in the quasi-static limit $M_a+2M_c=0$ and $\alpha_0^{-1}(2M_c/L)=2M_c/\mu_0A^2=(-1/2\pi)(1/a^3)=-0.16/a^3$. Therefore, Eq. (27) simplifies to:

$$\left.\frac{\mu_{yy}}{\mu_0}\right|_{k_y=k_z=0}\approx1+\frac{1}{a^3}\frac{1}{\mathrm{Re}\left\{\alpha_m^{-1}\right\}+0.16/a^3(\cos(k_xa)-1)-1/3a^3}\qquad(28)$$

It is interesting to compare the above formula with the magnetic permeability derived in Section 2.2. Since for $k_y=k_z=0$ we have $D_{\mathrm{int},ij}=0$ and it is clear that Eq. (21) implies that the magnetic permeability function is diagonal and such that $\mu_{yy}\big|_{k_y=k_z=0}=\mu_0\left(1+1/\left(a^3\chi_{22}\right)\right)$, or equivalently:

$$\left.\frac{\mu_{yy}}{\mu_0}\right|_{k_y=k_z=0}=1+\frac{1}{a^3}\frac{1}{\mathrm{Re}\left\{\alpha_m^{-1}\right\}+0.026/a^3\left(\cos\left(k_xa\right)-1\right)-1/a^3\left(1/3-0.15(\omega a/c)^2\right)}\qquad(29)$$

Comparing Eqs. (28) and (29) two differences are detected. The first difference is the frequency correction of the static interaction constant (the term $0.15(\omega a/c)^2$), which was not considered in Ref. [12]. The second difference is the coefficient that multiplies the term $(\cos(k_x a) - 1)$, which is responsible for the spatial dispersion effects. Apparently, the theory of our previous paper [12] (when applied to the particular case of magnetic dipoles) overestimates this coefficient by nearly one order of magnitude. A possible reason for this disagreement is that formula (28) is based on the nearest neighbor approximation, while the formula derived in this work takes into account the coupling between all the particles in the lattice. Indeed, the nearest neighbor approximation may be more useful when the mutual coupling between adjacent rings is large (which was the case studied in our previous work [12], where the resonant rings have larger diameters), whereas it may not be so accurate for dipole-type particles with a comparatively weaker mutual coupling.

3. Plane wave dispersion characteristic

Next, we apply the developed homogenization model to characterize the dispersion characteristic of plane waves. To begin with, we remind that the model introduced in Section 2 supposes that the rings only have a magnetic response. Actually, this is a very rough approximation, since it is well known that in general resonant rings have as well an electric response (see for example Ref. [17]). As in Ref. [12], we will assume that the electric response of the rings can be taken into account by considering that the effective permittivity is equal to ε_{SRR}. Within this approximation the metamaterial is characterized by a local effective permittivity (ε_{SRR}) and by the nonlocal magnetic function (21) ($\bar{\bar{\mu}}(\omega, \mathbf{k})$).[1] It is interesting to note that this hypothesis is nothing more than assuming that the electric and magnetic responses are decoupled. Indeed, ε_{SRR} may be easily related to the electric polarizability of the particles using a homogenization approach similar to that of Section 2 (in general ε_{SRR} may also depend on the wave vector, however since the electric resonance occurs at a frequency significantly higher than the magnetic resonance, that dependence is expected to be small for frequencies comparable or smaller than the frequency associated with the magnetic resonance, which is the case of interest in this work).

[1] Alternatively, the metamaterial could as well be characterized by the nonlocal dielectric function (6) with the first term in the right-hand side ($\bar{\bar{I}}$) replaced by $(\varepsilon_{SRR}/\varepsilon_0)\,\bar{\bar{I}}$, and by $\mu = \mu_0$.

From Eq. (17), setting $\mathbf{p}_{\text{ext}} = 0$, i.e. removing the external source, using $\mathbf{M} = \left(\bar{\bar{\mu}}/\mu_0 - \bar{\bar{\mathbf{I}}}\right) \cdot \mathbf{H}_{\text{av}}$, and replacing ε_0 by ε_{SRR}, it is readily found after some manipulations that

$$\left(\omega^2 \varepsilon_{SRR} \bar{\bar{\mu}} - k^2 \bar{\bar{\mathbf{I}}} + \mathbf{k}\mathbf{k}\right) \cdot \mathbf{H}_{\text{av}} = 0. \tag{30}$$

3.1. Propagation along the coordinate axes

First we will analyze the case of propagation along the one of the coordinate axes, let us say the x-axis ($k_y = k_z = 0$). For this case the modes can be classified as longitudinal modes and transverse modes. The longitudinal modes are such that the magnetic field is parallel to the wavevector ($\mathbf{H}_{\text{av}} \sim \mathbf{k}$), which from Eq. (30) implies that

$$\bar{\bar{\mu}}(\omega, \mathbf{k}) \cdot \mathbf{k} = 0. \tag{31}$$

As discussed in Section 2, for propagation along the coordinate axes the magnetic function is diagonal, and thus the above relation is equivalent to $\mu_{xx} = 0$. Using Eq. (21) and the fact that $D_{\text{int},ij} = 0$, we find that

$$\left.\frac{\mu_{xx}}{\mu_0}\right|_{k_y = k_z = 0} = 1 + \frac{1}{a^3} \frac{1}{\operatorname{Re}\left\{\alpha_m^{-1}\right\} - 1/a^3\left(1/3 - 0.15(\omega a/c)^2 + 0.052\left(\cos\left(k_x a\right) - 1\right)\right)} \tag{32}$$

Thus, the dispersion relation of the longitudinal mode is:

$$a^3 \operatorname{Re}\left\{\alpha_m^{-1}\right\} + \frac{2}{3} + 0.15\left(\frac{\omega}{c} a\right)^2 - 0.052\left(\cos\left(k_x a\right) - 1\right) = 0. \tag{33}$$

On the other hand, the metamaterial also supports transverse modes for which $\mathbf{H}_{\text{av}} \cdot \mathbf{k} = 0$. It is simple to verify that for propagation along the x-direction the dispersion relation of the transverse modes is

$$\omega^2 \varepsilon_{SRR} \mu_{yy}(\omega, k_x) = k_x^2, \tag{34}$$

being μ_{yy} given by Eq. (29).

3.2. Propagation along the main diagonal

It is also interesting to analyze the propagation properties along the “main diagonal” of the unit cell, namely along the ΓR direction, being Γ the origin of the Brillouin zone and $R = (\pi/a, \pi/a, \pi/a)$. It is thus clear that $k_x = k_y = k_z$, and therefore from Eqs. (22a) and (24) it follows that $\chi_{ii} = \operatorname{Re}\left\{\alpha_m^{-1}\right\} - 1/a^3\left(1/3 - 0.15(\omega a/c)^2\right)$, for $i = 1, 2, 3$. On the other hand, from Eqs. (22b) and (25) it is obvious that χ_{ij} is independent of the values of i and j for $i \neq j$. Based on these properties it can be verified that the magnetic function is such that:

$$\bar{\bar{\mu}}|_{k_x=k_y=k_z} = \left(\bar{\bar{\mathbf{I}}} - \frac{\mathbf{kk}}{k^2}\right)\mu_t + \frac{\mathbf{kk}}{k^2}\mu_l \tag{35a}$$

$$\mu_t = \mu_0\left(1 + \frac{1}{V_{\text{cell}}}\frac{1}{\chi_{11} - \chi_{12}}\right);$$

$$\mu_l = \mu_0\left(1 + \frac{1}{V_{\text{cell}}}\frac{1}{\chi_{11} + 2\chi_{12}}\right), \tag{35b}$$

with $\chi_{12} = -k^2/3a\left[-0.1936 - 0.01188(\omega a/c)^2 + 0.005902\,a^2 2k^2/3\right]$, and $k^2 = \mathbf{k}\cdot\mathbf{k} = k_x^2 + k_y^2 + k_z^2$. It is clear that the above expression implies that the electromagnetic modes can also be classified as longitudinal and transverse modes. Using Eq. (31), it follows that the dispersion characteristic of the longitudinal mode (for propagation in the ΓR direction) is $\mu_l = 0$, or equivalently

$$a^3\mathrm{Re}\left\{\alpha_m^{-1}\right\} + \frac{2}{3} + 0.15\left(\frac{\omega}{c}a\right)^2 + 2a^3\chi_{12} = 0. \tag{36}$$

On the other hand, the dispersion characteristic of the transverse modes is obviously

$$\omega^2\varepsilon_{SRR}\mu_t(\omega, k) = k^2. \tag{37}$$

4. Numerical example and discussion

In order to describe the implications of the theory developed in Section 3, we will analyze the case where the resonant rings are the edge-side coupled (EC) SRRs originally proposed by Pendry [1]. To ease the numerical and analytical modeling it is assumed that the SRRs are formed by thin wires with radius $r = 0.01a$ (see Fig. 1), instead of planar particles. The mean radius of the outer (inner) ring is $R + d/2$ ($R - d/2$), where $d = 0.1607R$ is the mean distance between rings, and $R = 0.4a$ is the average radius. Each ring has a split that covers an angular sector of $\varphi_{gap} = 10°$. It can be shown that the self-inductance and capacitance of the EC-SRRs may be estimated using the following formulas (see Refs. [12,17,23] for closely related results)

$$C = \frac{\varepsilon_0 \pi R\left(\pi - \varphi_{gap}\right)}{\cosh^{-1}\left(d^2/2r^2 - 1\right)},$$

$$L = \mu_0 R\left[\ln\left(\frac{8R}{r}\right) - 2\right]. \tag{38}$$

The SRRs are oriented as shown in Fig. 1, consistent with the proposal of Ref. [27]. Such structure, unlike an ideal lattice of point magnetic dipoles, cannot be truly regarded as an isotropic magnetic material. Indeed, as discussed in Ref. [27], to ensure that a regular arrangement of particles is isotropic, the unit cell should be invariant under the application of proper rotations of the regular tetrahedron (cubic point group T), and the structure of Fig. 1 does not have such symmetry.[2] Moreover, the material also does not have inversion symmetry since SRRs in opposite faces of the cube have the same orientation due to the translational symmetry, and thus bianisotropic effects may occur [17]. Despite these problems, it will be shown below that the metamaterial (formed by very thin and closely spaced rings) has approximately an isotropic response.

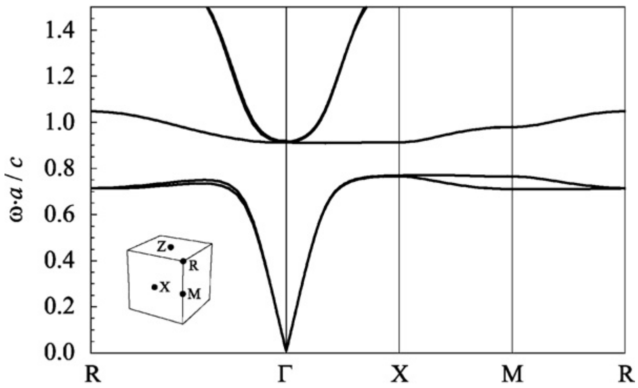

Fig. 2. Band structure of a material formed EC-SRRs with unit cell as in Fig. 1. The inset shows the first Brillouin zone and the high-symmetry points $X = (\pi/a, 0, 0)$, $M = (\pi/a, \pi/a, 0)$ and $R = (\pi/a, \pi/a, \pi/a)$.

The band structure of the array of SRR resonators was calculated using the hybrid integral equation-plane wave method described in Ref. [28], and is reported in Fig. 2. It can be seen that the structured material has a complete band gap that occurs due to the strong magnetic response of the rings. Below the magnetic resonance (normalized frequency $\omega a/c \approx 0.77$), the material supports only two electromagnetic modes (TEM waves), whose dispersion characteristic is nearly degenerated.

On the other hand, above the magnetic plasma frequency ($\omega a/c \approx 0.91$, i.e. the frequency where the propagation is resumed), the material supports three different modes. One of the modes is expected to be a longitudinal wave (the so-called longitudinal magne-

[2] The unit cell of Fig. 1 is invariant under the rotation $\mathbf{4}_x{\cdot}\mathbf{4}_z$, where $\mathbf{4}_i$ ($i = x, y, z$) represents a 4-fold (90-degree) rotation with respect to the ith axis. Thus, the metamaterial is invariant under $\pm 120°$ rotations with respect to the diagonal of the cube along $(1, -1, 1)$. However, it has not the same property along the other three diagonals, as would be required so that it would be invariant under the application of the T cubic point group.

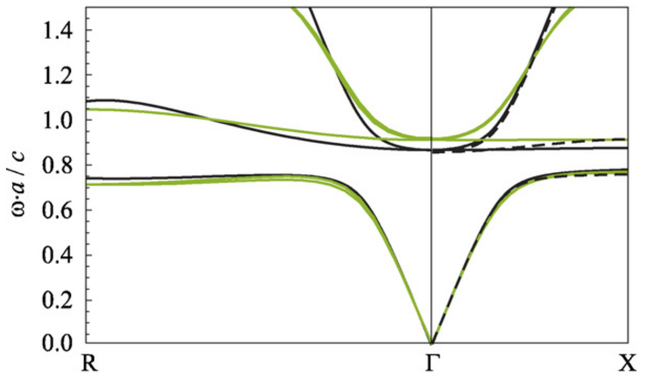

Fig. 3. Dispersion characteristic along the directions ΓR and ΓX. Black solid line: dipole based model proposed in this work. Black dashed line: model proposed in Ref. [12] (ΓX propagation), adapted for the case of point dipoles. Green (thick) lines: full wave electromagnetic simulations. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of the article.)

toinductive wave [13]) and has a nearly flat dispersion characteristic. The other two modes are approximately TEM waves and are nearly degenerated. The dispersion curves confirm that the response of the material below the magnetic resonance is to a good approximation isotropic. However, it can be seen that near the magnetic plasma frequency the dispersion characteristic of the longitudinal magnetoinductive wave depends significantly on the direction of the wave vector, having a much larger slope along the direction ΓR than along the direction ΓX. Similar results have been reported in our previous work (Ref. [12]) for the case where the inclusions are SRRs with two splits (unit cell with tetrahedral symmetry). Thus, the direction dependent properties of the longitudinal magnetoinductive wave can only be explained in terms of the emergence of nonlocal effects [21], induced by the granularity of the material.

We have calculated the theoretical dispersion characteristic using the analytical formulas reported in Sections 3.1 and 3.2, with the static effective permittivity $\varepsilon_{SRR} = 2.1\varepsilon_0$. This value was numerically computed using the homogenization approach of Ref. [14] for $\omega = 0$. The calculated dispersion curves are plotted in Fig. 3 (solid black lines) superposed on the numerical results (green thick lines). Despite the simplicity of the analytical model (which treats the SRRs as magnetic point dipoles), a fairly good agreement is revealed. In particular, below the magnetic resonance the analytical model concurs very well with the full wave simulations. Near the magnetic plasma frequency, the quantitative agreement is coarser, but the qualitative agreement remains good. Consistent with the full wave simulations, the analytical model predicts that for propagation along ΓX the dispersion characteristic of the longitudinal magnetoinductive wave is flat, whereas for propagation along ΓR the dispersion curve has a significant dispersion. This supports that indeed the lack of isotropy of the longitudinal mode is due to the emergence of nonlocal effects, more specifically due to the relatively large values of $D_{\mathrm{int},ij} = -\chi_{ij}$ (see Eq. (36)), which characterizes the interaction between magnetic particles with different orientations.

Fig. 3 also reports (black dashed line) the results yielded by the model proposed in our previous work, Ref. [12], (adapted for the case of point dipoles: see Section 2.3) for propagation along the ΓX-direction. Consistent with the discussion of Section 2.3, it is seen that the model of Ref. [12] tends to overestimate the slope of the longitudinal model along the ΓX-direction. Apart from that discrepancy, the general agreement between the model proposed here and the results of Ref. [12] is good.

It is important to underline that in a local material (i.e. in the absence of spatial dispersion) the longitudinal wave should have a completely flat dispersion characteristic, independent of the direction of the wave vector. It seems that one of the most common manifestations of nonlocal effects in structured media is that the longitudinal wave is highly dispersive. This implies that unlike in a local material the longitudinal mode can be excited by an external source and represent an additional propagation channel. In an array of cubic particles made of EC-SRRs the nonlocal effects are dominant for propagation along the ΓR direction. Similar nonlocal effects have also been reported for arrays of uniaxial SRRs [23,29], for connected wire media [30–33], and for plasmonic nanorods [34].

The effects of spatial dispersion in the longitudinal mode can be tamed by decreasing the electrical size of the particles at the resonance, for example by increasing the capacitance of the rings. Indeed, from Eqs. (2) and (36) it is found that to a first order approximation (neglecting powers of both $(\omega a/c)^2$ and $(ka)^4$) the dispersion of the longitudinal mode along ΓR is

$$\left(\frac{\omega}{\omega_{mp}}\right)^2 \approx \frac{1}{1+\left(1-\omega_{mp}^2/\omega_r^2\right)0.1936(ka)^2}, \qquad (39)$$

where the magnetic plasma frequency (which defines the onset of propagation of the longitudinal mode) is approximately $\omega_{mp} \approx \omega_r \Big/ \sqrt{1-2\Big/\left(3a^3\alpha_0^{-1}\right)}$. Let us estimate the range of wave vectors for which $\left|\left(\omega-\omega_{mp}\right)/\omega_{mp}\right| < \delta$, where δ is some small number. Using Eq. (39) it can be easily seen that the solution is of the form $k < C/a$, where C is some constant that only

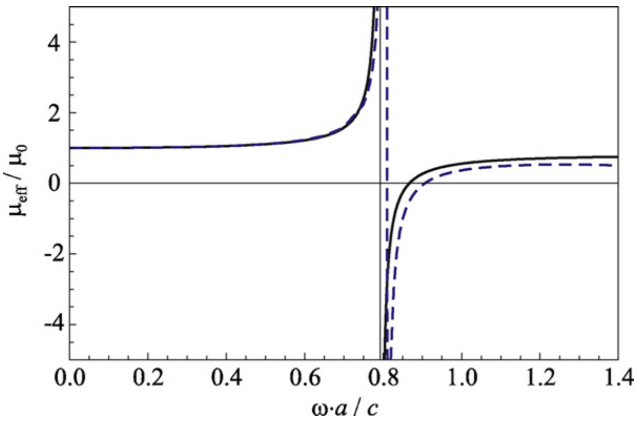

Fig. 4. Effective permeability of the array of cubic particles made of EC-SRRs as a function of frequency. Dashed blue line: full wave result extracted using the method reported in Ref. [14]. Solid black line: analytical result, $\mu(\omega, \mathbf{k}=0)$, calculated using formula (29). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of the article.)

depends on δ and on ω_{mp}/ω_r. But, for a fixed fill fraction (i.e. for R/a and r/a fixed) the ratio ω_{mp}/ω_r is independent of a. Moreover, the electrical size of the unit cell at the plasma frequency $\omega_{mp}a/c$ may be decreased by increasing the capacitance of the rings (e.g. by decreasing the inter-ring distance d). Thus, the range of wave vectors for which the condition $\left|(\omega-\omega_{mp})/\omega_{mp}\right| < \delta$ is observed verifies $kc/\omega_{mp} < C/(\omega_{mp}\, a/c)$, where C only depends on the fill fraction and δ, and $\omega_{mp}a/c$ is determined by the electrical size of the unit cell. Therefore, the values of k (normalized to the frequency of operation) for which the dispersion of the longitudinal mode is below some given threshold is broader when the electrical cell size of the unit cell is smaller, showing that in these conditions the nonlocal effects are less important.

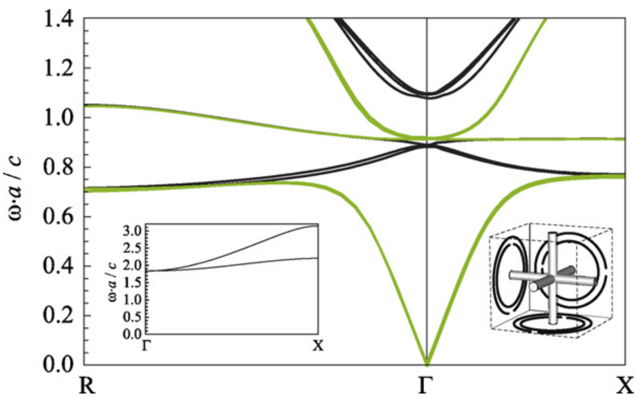

Fig. 5. Band structure of an array of cubic particles made of SRRs and connected wires (solid black lines; see geometry of the unit cell in the small right-hand side inset), superposed on the band structure of the array of SRRs (green lighter lines). The left-hand side inset shows the dispersion characteristic of the connected wire medium (without SRRs). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of the article.)

In order to further reveal other possible nonlocal effects, we have numerically calculated (with a full wave simulation) the effective permeability of the array of EC-SRRs using the approach described in Ref. [14]. The effective permeability is written in terms of the second order derivatives of the nonlocal dielectric function with respect to the wave vector [14]. The extraction procedure is only meaningful if the material's response is approximately local. The computed results are represented in Fig. 4 (dashed blue line), superposed on the results obtained with the analytical model (29), $\mu(\omega, \mathbf{k}=0)$ (solid black line). It is seen that the curves are nearly coincident, except for a small shift in frequency above the resonance. Given that the dashed line was extracted under the hypothesis that the material's response is local, these results suggest that, except for the dispersive longitudinal mode, the spatial dispersion effects are relatively weak, especially for $\mathbf{k}$ near the origin of the Brillouin zone.

5. Array of SRRs combined with a triple wire medium

Given the contemporary interest in materials with simultaneously negative permittivity and permeability [2,26,35], it is pertinent to study the electromagnetic response of a metamaterial formed by an array of SRRs combined with an array of connected wires [30–33,36,37] (see the geometry of the unit cell in the right-hand side inset of Fig. 5). Based on the results of the seminal work [35], it seems plausible that such structure may behave to some approximation as a nearly isotropic left-handed material. It is expected that the magnetic response of the composite material will be determined by the SRRs, whereas the electric response is mainly determined by the array of connected wires.

It was shown in Refs. [30,31] that the triple wire medium (with no SRRs) can be accurately characterized by the following nonlocal dielectric function (assuming perfectly conducting wires and that the host material is air):

$$\bar{\bar{\varepsilon}}(\omega, \mathbf{k}) = \varepsilon_{t,WM}(\omega)\left(\bar{\bar{\mathbf{I}}} - \frac{\mathbf{kk}}{k^2}\right) + \varepsilon_{l,WM}(\omega, k)\frac{\mathbf{kk}}{k^2}, \tag{40a}$$

where the transverse and longitudinal components of the dielectric function are

$$\frac{\varepsilon_{t,WM}}{\varepsilon_0}(\omega) = 1 - \frac{\beta_p^2 c^2}{\omega^2} \tag{40b}$$

$$\frac{\varepsilon_{l,\text{WM}}}{\varepsilon_0}(\omega, k) = 1 - \frac{\beta_p^2}{\omega^2/c^2 - k^2/l_0}. \quad (40c)$$

Above, $\beta_p = \left[2\pi/\left(\ln\left(a/2\pi r_w\right) + 0.5275\right)\right]^{1/2}/a$ is the plasma wavenumber, r_w is the radius of the wires, and l_0 is a constant defined in Refs. [30,31] that depends on the radius of the wires. The connected wire medium imitates to some extent a plasma characterized by a Drude type dispersion model, except for the longitudinal mode, which similar to the results of Section 4, may be highly dispersive [30,31].

If one assumes that the array of wires interacts weakly with the SRRs, as suggested by the symmetry of the material, by the results of Ref. [35] and also discussed in our previous work [12], it is straightforward to obtain a homogenization model for the composite material formed by SRRs and metallic wires. In fact, since the connected wire medium does not have a magnetic response, it follows that the magnetic function of the composite medium is the same as that of the array of cubic particles formed by EC-SRRs [Eq. (21)]. On the other hand, the effective permittivity of the composite medium should verify $\bar{\bar{\varepsilon}}_{\text{WM+SRR}} = \varepsilon_0\bar{\bar{\mathbf{I}}} + (\varepsilon_{\text{SRR}} - \varepsilon_0)\bar{\bar{\mathbf{I}}} + \left(\bar{\bar{\varepsilon}}_{\text{WM}} - \varepsilon_0\bar{\bar{\mathbf{I}}}\right)$, where ε_{SRR}, already defined in section III, is the effective permittivity of the array of EC-SRRs. Thus, within the considered hypothesis it is found that

$$\bar{\bar{\varepsilon}}(\omega, \mathbf{k}) = \varepsilon_{t,\text{WM+SRR}}(\omega)\left(\bar{\bar{\mathbf{I}}} - \frac{\mathbf{kk}}{k^2}\right) + \varepsilon_{l,\text{WM+SRR}}(\omega, k)\frac{\mathbf{kk}}{k^2} \quad (41a)$$

$$\frac{\varepsilon_{t,\text{WM+SRR}}}{\varepsilon_0}(\omega) = \varepsilon_{\text{SRR},r} - \frac{\beta_p^2 c^2}{\omega^2} \quad (41b)$$

$$\frac{\varepsilon_{l,\text{WM+SRR}}}{\varepsilon_0}(\omega, k) = \varepsilon_{\text{SRR},r} - \frac{\beta_p^2}{\omega^2/c^2 - k^2/l_0}. \quad (41c)$$

where $\varepsilon_{\text{SRR},r} = \varepsilon_{\text{SRR}}/\varepsilon_0$. It is important to emphasize that the magnetic effects of the composite medium are not included in the nonlocal dielectric function, and are described separately by $\bar{\bar{\mu}}(\omega, \mathbf{k})$, given by Eq. (21) [the magnetic effects could be easily incorporated into the dielectric function using a formula similar to (6); however, here it is preferable to separate the electric and magnetic responses since such framework is necessary to study the longitudinal magnetoinductive wave for which $\mu_l = 0$ [21].

Using the hybrid integral equation-plane wave method of Ref. [28], we have computed the band structure of the composite material formed by connected wires and EC-SRRs. The geometry of the EC-SRRs is the same as that considered in section IV, and the triple wire medium is formed by wires with radius $r_w = 0.05a$. The calculated band structure is represented in Fig. 5 (solid black lines) for the directions ΓR and ΓX of the Brillouin zone. It is seen that the material supports backward waves for $0.77 < \omega a/c < 0.89$, i.e. roughly in the same frequency band where the array of EC-SRRs is characterized by a complete band gap (the band structure of the array of EC-SRRs is represented with green lines in Fig. 5). This supports the hypothesis that the composite material formed by wires and SRRs behaves as a left-handed medium. Quite interesting, the upper frequency of the backward wave regime is slightly below the magnetic plasma frequency of the array of EC-SRRs. Most likely this is a consequence of some residual bianisotropic effects, consistent with the theory of Ref. [17].

Besides the two backward wave (TEM) modes, the composite material also supports a longitudinal mode at the magnetic plasma frequency. Remarkably, except very near to the Γ point, the dispersion of the longitudinal mode cannot be distinguished from the dispersion of the longitudinal magnetoinductive wave identified in Section 4 (nearly flat green line in Fig. 5). This behavior completely supports the hypothesis that the triple wire medium interacts weakly with the array of SRRs. Indeed, the dispersion of the longitudinal mode is determined by condition (31), $[\bar{\bar{\mu}}(\omega, \mathbf{k}) \cdot \mathbf{k} = 0]$, which is completely independent of the dielectric function $\bar{\bar{\varepsilon}}(\omega, \mathbf{k})$ of the composite material (because the magnetoinductive longitudinal mode is associated with a trivial macroscopic electric field: $\mathbf{E}_{\text{av}} = 0$). This explains why the dispersion characteristic of this mode remains nearly invariant when the triple wire medium is added to the array of EC-SRRs.

The composite material formed by SRRs and wires has a band gap above the magnetic plasma frequency. The propagation is resumed at $\omega a/c \approx 1.08$, which supposedly corresponds to the electric plasma frequency for which $\varepsilon_{\text{WM+SRR}} \approx 0$. The theoretical value of the electric plasma frequency may be estimated using Eq. (41b) and is $\omega_{ep}/c = \beta_p/\sqrt{\varepsilon_{\text{SRR},r}}$. Since, for wires with $r_w = 0.05a$ we have that $\beta_p = 1.93/a$, using $\varepsilon_{\text{SRR},r} = 2.1$ (see Section 4) we obtain the theoretical value, $\omega_{ep} a/c = 1.33$, which is slightly larger than the more precise value obtained from the band structure of the material. Most likely the reason for the discrepancy is that $\varepsilon_{\text{SRR},r} = 2.1$ is the static permittivity of the array of SRRs, and thus our model may underestimate $\varepsilon_{\text{SRR},r}$ at the electric plasma frequency.

Above the electric plasma frequency the metamaterial supports three electromagnetic modes. Two of the modes are expected to be associated with TEM waves, whereas the other mode is expected to be associated with an electric-type longitudinal mode. Curiously, it can be seen in Fig. 5 that the dispersion characteristic of these three modes is nearly coincident. This contrasts markedly with the case where the metallic wires stand alone in the lattice (see the left-hand side inset of Fig. 5), for which the dispersion of the longitudinal mode is much smaller than that of the transverse modes. To explain this curious phenomenon, we note that near the plasma frequency the magnetic response of the SRRs is expected to be relatively weak (see Fig. 4) [i.e. $\bar{\bar{\mu}}(\omega, \mathbf{k}) \approx \mu_0 \bar{\bar{\mathbf{I}}}$], and thus to a first a approximation the composite metamaterial may be described using only the nonlocal dielectric function $\bar{\bar{\varepsilon}}(\omega, \mathbf{k})$. It should be clear that Eq. (41) implies that the metamaterial supports two transverse modes with dispersion characteristic $k^2 = \varepsilon_{t,WM+SRR}\,\mu_0\,\omega^2$, i.e.

$$\frac{\omega^2}{c^2} = \frac{\beta_p^2}{\varepsilon_{SRR,r}} + \frac{k^2}{\varepsilon_{SRR,r}} \tag{42a}$$

and a longitudinal mode with dispersion characteristic $\varepsilon_{l,WM+SRR} = 0$, i.e.

$$\frac{\omega^2}{c^2} = \frac{\beta_p^2}{\varepsilon_{SRR,r}} + \frac{k^2}{l_0}. \tag{42b}$$

Comparing Eqs. (42a) and (42b), it is seen that the dispersion of the transverse and longitudinal modes is the same when $\varepsilon_{SRR,r} \approx l_0$. But for a triple wire medium with $r_w = 0.05a$ we have that $l_0 \approx 2.03$ (see Refs. [30,31]), and thus the condition $\varepsilon_{SRR,r} \approx l_0$ seems to be verified in our problem. This coincidence explains the similarity of the dispersion characteristics of the transverse and longitudinal modes above the electric plasma frequency.

6. Conclusion

Using nonlocal homogenization methods we have calculated theoretically the magnetic function of an array of cubic magnetic resonators, under the approximation that the inclusions can be modeled using the magnetic dipole approximation. The proposed model complements the analysis of our previous work [12], and takes into account all interactions between the particles (i.e. magnetic dipoles), and both frequency and spatial dispersion. It was shown that the properties of the longitudinal magnetoinductive wave are determined by nonlocal effects, which are caused by the interaction of the magnetic resonators with different orientations. Our results suggest that except in the regime where $\mu \approx 0$, the effects of spatial dispersion may be relatively weak, especially if $\mathbf{k}$ is near the origin of the Brillouin zone. In addition, we studied the propagation properties in a composite material formed by EC-SRRs and a triple wire medium. It was shown that such material supports a backward wave regime, where it may behave as a nearly isotropic left-handed material. In addition, it also supports both magnetic-type and electric-type longitudinal modes, and two TEM modes above the electric plasma frequency. It is hoped that the present study contributes for the understanding of nonlocal homogenization techniques, and stimulates the study of truly isotropic local left-handed metamaterials.

Acknowledgements

This work was partially funded by Fundação para Ciência e a Tecnologia under project PDTC/EEA-TEL/71819/2006, by the Universidad Nacional de Colombia under project DIB-8003310, by the Spanish Ministerio de Educación y Ciencia and European Union FEDER funds (projects TEC2007-65376, TEC2007-68013-C02-01, and CSD2008-00066), Junta de Andalucía (project TIC-253), and by the Czech Grant Agency (project no. 102/09/0314).

Appendix A.

In this appendix we briefly describe the solution of the homogenization problem when the resonant rings are modeled as particles with an electric response. As discussed in Section 2.1, in these circumstances the magnetic effects are related to the vortex part of the induced electric current density.

Following Ref. [14], the nonlocal dielectric function is calculated by exciting the metamaterial with an external electric current density, $\mathbf{J}_{e,\text{ext}} = j\omega\left(\mathbf{p}_{\text{ext}}^{(e)}/V_{\text{cell}}\right)e^{-j\mathbf{k}\cdot\mathbf{r}}$, where $\mathbf{p}_{\text{ext}}^{(e)}$ is a constant vector. Thus, the microscopic electromagnetic fields, $(\mathbf{E}, \mathbf{B})$, verify:

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} \tag{A1a}$$

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = j\omega\varepsilon_0\mathbf{E} + \mathbf{J}_{e,\text{ext}} + \mathbf{J}_{e,\text{dip}}, \tag{A1b}$$

where the microscopic electric current density, $\mathbf{J}_{e,\text{dip}}$, is given by Eq. (4), and is written in terms of the unknown magnetic dipole moments of the particles in the unit cell $\mathbf{p}_i$. It is important to emphasize that when the response of the particles is characterized by an electric current density, as considered here, the microscopic fields are the electric field intensity ($\mathbf{E}$) and the magnetic induction ($\mathbf{B}$). Quite differently, from duality, when the response of the particles is characterized by a magnetic current density the microscopic fields are the electric field intensity (**E**) and the magnetic field intensity (**H**) (see Section 2.2).

It may be easily shown that the solution of Eq. (A1) is:

$$\mathbf{E}(\mathbf{r}) = (-j\omega\mu_0) \sum_{i=1,2,3} \nabla \times \bar{\bar{\mathbf{G}}}_p \left(\mathbf{r}|\,\mathbf{r}_{0,i}\right) \cdot \frac{\mathbf{p}_i}{\mu_0} + (-j\omega\mu_0)\,\bar{\bar{\mathbf{G}}}_{\text{av}} \cdot j\omega \mathbf{p}_{\text{ext}}^{(e)}\, e^{-j\mathbf{k}\cdot\mathbf{r}}. \quad \text{(A2)}$$

Thus, the local electric field in the immediate vicinity of the *i*th ring is

$$\mathbf{E}_{\text{loc},i}(\mathbf{r}) = -j\omega\nabla \times \overline{\overline{\mathbf{G}'_p}} \left(\mathbf{r}|\,\mathbf{r}_{0,i}\right) \cdot \mathbf{p}_i - j\omega \sum_{j \neq i} \nabla \times \bar{\bar{\mathbf{G}}}_p \left(\mathbf{r}|\,\mathbf{r}_{0,j}\right) \cdot \mathbf{p}_j + \omega^2 \mu_0 \bar{\bar{\mathbf{G}}}_{\text{av}} \cdot \mathbf{p}_{\text{ext}}^{(e)} e^{-j\mathbf{k}\cdot\mathbf{r}}, \quad \text{(A3)}$$

whereas the local induction field, $\mathbf{B}_{\text{loc},i} = \nabla \times \mathbf{E}_{\text{loc},i}/(-j\omega)$, is

$$\mathbf{B}_{\text{loc},i}(\mathbf{r}) = \left(\frac{\omega}{c}\right)^2 \overline{\overline{\mathbf{G}'_p}} \left(\mathbf{r}|\,\mathbf{r}_{0,i}\right) \cdot \mathbf{p}_i + \sum_{j \neq i} \left(\frac{\omega}{c}\right)^2 \bar{\bar{\mathbf{G}}}_p \left(\mathbf{r}|\,\mathbf{r}_{0,j}\right) \cdot \mathbf{p}_j + \omega\mu_0 \mathbf{k} \times \bar{\bar{\mathbf{G}}}_{\text{av}} \cdot \mathbf{p}_{\text{ext}}^{(e)}\, e^{-j\mathbf{k}\cdot\mathbf{r}}, \quad \text{(A4)}$$

being $\overline{\overline{\mathbf{G}'_p}}$ defined as in Eq. (13). It should be clear that in the present context $\mathbf{H}_{\text{loc}}$ in Eq. (3) should be identified with $\mathbf{B}_{\text{loc}}/\mu_0$. Thus, using the auxiliary relation $\mathbf{k} \times \bar{\bar{\mathbf{G}}}_{\text{av}} = \bar{\bar{\mathbf{G}}}_{\text{av}} \times \mathbf{k} = \bar{\bar{\mathbf{G}}}_{\text{av}} \cdot (\mathbf{k} \times \bar{\bar{\mathbf{I}}})$, [see Eq. (10)], and the property $\mathbf{p}_i = p_i \hat{\mathbf{u}}_i$, it is found that for $i = 1, 2, 3$

$$\alpha_m^{-1} \frac{p_i}{\mu_0} = \left(\frac{\omega}{c}\right)^2 \left(\hat{\mathbf{u}}_i \cdot \overline{\overline{\mathbf{G}'_p}}(0|\,0) \cdot \hat{\mathbf{u}}_i \frac{p_i}{\mu_0} + \sum_{j \neq i} \hat{\mathbf{u}}_i \cdot \bar{\bar{\mathbf{G}}}_p \left(\mathbf{r}_{0,i}\middle|\,\mathbf{r}_{0,j}\right) \cdot \hat{\mathbf{u}}_j \frac{p_j}{\mu_0} + \hat{\mathbf{u}}_i \cdot \bar{\bar{\mathbf{G}}}_{\text{av}} \cdot \left(\frac{c^2\mathbf{k}}{\omega} \times \mathbf{p}_{\text{ext}}^{(e)}\right) e^{-j\mathbf{k}\cdot\mathbf{r}_{0,i}} \right). \quad \text{(A5)}$$

Interestingly, the above result is equivalent to Eq. (14), provided we make the identification $\mathbf{p}_{\text{ext}} = \mathbf{k} \times \mathbf{p}_{\text{ext}}^{(e)}/\omega\varepsilon_0$. Therefore, making manipulations similar to those of Section 2.2, we conclude that

$$\bar{\bar{\chi}} \cdot \mathbf{M} = \left(\frac{\omega}{c}\right)^2 \bar{\bar{\mathbf{G}}}_{\text{av}} \cdot \left(\mathbf{M} + \frac{1}{V_{\text{cell}}} \frac{c^2\mathbf{k}}{\omega} \times \mathbf{p}_{\text{ext}}^{(e)} \right), \quad \text{(A6)}$$

where the magnetization vector **M** is defined as in Eq. (18), and the dyadic $\bar{\bar{\chi}}$ is defined as in Eq. (22).

The dielectric function must verify $\left(\bar{\bar{\varepsilon}}(\omega, \mathbf{k}) - \varepsilon_0 \bar{\bar{\mathbf{I}}}\right) \cdot \mathbf{E}_{\text{av}} = \mathbf{P}_g$, independent of the applied current density, where $\mathbf{P}_g$ is the generalized electric polarization vector given by [14]:

$$\mathbf{P}_g = \frac{1}{V_{\text{cell}}\, j\omega} \int_{\text{cell}} \mathbf{J}_{e,\text{dip}}(\mathbf{r})\, e^{+j\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{r} = -\frac{\mathbf{k}}{\omega} \times \mathbf{M}. \quad \text{(A7)}$$

By averaging the microscopic Maxwell's equations (A1) [see Eqs. (9) and (14) of Ref. [14]], it can be easily shown that the macroscopic electric field must be such that:

$$\mathbf{E}_{\text{av}} = \omega^2 \mu_0 V_{\text{cell}} \bar{\bar{\mathbf{G}}}_{\text{av}} \cdot \left(\mathbf{P}_g + \frac{\mathbf{p}_{\text{ext}}^{(e)}}{V_{\text{cell}}} \right). \quad \text{(A8)}$$

Thus, using again the result $\mathbf{k} \times \bar{\bar{\mathbf{G}}}_{\text{av}} = \bar{\bar{\mathbf{G}}}_{\text{av}} \times \mathbf{k}$ and Eq. (A7), it is found after some algebra that:

$$\frac{\mathbf{k}}{\mu_0\omega} \times \mathbf{E}_{\text{av}} = \mathbf{M} + \frac{\omega^2}{c^2} V_{\text{cell}} \bar{\bar{\mathbf{G}}}_{\text{av}} \cdot \left(\mathbf{M} + \frac{1}{V_{\text{cell}}} \frac{c^2\mathbf{k}}{\omega} \times \mathbf{p}_{\text{ext}}^{(e)} \right). \quad \text{(A9)}$$

Substituting now the above formula into Eq. (A6), it follows the magnetization vector is related to the macroscopic electric field as

$$\mathbf{M} = -\left(\mu_0 \bar{\bar{\mu}}^{-1} - \bar{\bar{\mathbf{I}}}\right) \cdot \left(\frac{\mathbf{k}}{\mu_0\omega} \times \mathbf{E}_{\text{av}}\right), \quad \text{(A10)}$$

where $\bar{\bar{\mu}}$ is defined as in Eq. (21), and we have used the property $\bar{\bar{\mu}}^{-1} = \bar{\bar{\mathbf{I}}} - \left(\bar{\bar{\mathbf{I}}} + V_{\text{cell}} \bar{\bar{\chi}}\right)^{-1}$. Finally, using the property $\left(\bar{\bar{\varepsilon}}(\omega, \mathbf{k}) - \varepsilon_0 \bar{\bar{\mathbf{I}}}\right) \cdot \mathbf{E}_{\text{av}} = \mathbf{P}_g$ and Eqs. (A7) and (A10), it is found that the nonlocal dielectric function of the metamaterial verifies, indeed, Eq. (6), as we wanted to prove.

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