Instituto Superior Técnico

Projeto MEFT

Boundedness and decay of the solutions of the wave equation in Minkowski and Schwarzschild spacetimes using energy methods

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Abstract

This work is an overview of the energy-based methods described by Luís Machado [1] to prove boundedness and decay of the solutions of the wave equation in the Minkowski spacetime and the exterior region of the Schwarzschild spacetime.

We begin by rigorously defining the concept of conservation of energy, which quickly leads to energy boundedness in the Minkowski case. To prove decay, we first introduce the Integrated Local Energy Decay estimate, which shows that energy decays in spatially compact regions. We use this to arrive at the Dafermos-Rodnianski hierarchy, which then implies energy decay.

After developing these methods in the simple setting of Minkowski, we apply them to the Schwarzschild spacetime, thus demonstrating their robust and versatile nature. Nevertheless, extra care is required at the event horizon and the photon sphere.

Finally, we make use of the Sobolev inequality to deduce pointwise estimates from the energy results in both spacetimes.

1 Introduction

We want to obtain boundedness and decay estimates for solutions of the wave equation:

$$\Box\psi = 0. \tag{1.1}$$

We begin by introducing the notion of an energy current along a vector field X:

$$J^X_\mu[\psi] := T_{\mu\nu}[\psi]X^\nu, \tag{1.2}$$

where $T_{\mu\nu}[\psi] = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\psi\partial_\alpha\psi$ is the energy-momentum tensor associated with ψ. Now, if we choose a Cauchy hypersurface Σ_0 (a space-like hypersurface which can be thought of as defining a global "moment in time"), we can define the energy at each time τ as the flux of this current through $\Sigma_\tau := \phi_t(\Sigma_0)$ (where ϕ_t is the flow of the vector field ∂_t):

$$\mathbb{E}^X[\psi](\tau) := \int_{\Sigma_\tau} J^X_\mu[\psi]n^\mu_{\Sigma_\tau}, \tag{1.3}$$

where n_{Σ_τ} is the future-pointing unit normal for Σ_τ.
Then, applying the divergence theorem in a region $R^{t_2}_{t_1} = \bigcup_{t\in[t_1,t_2]}\Sigma_\tau$, we get:

$$\int_{\partial R} J^X_\mu[\psi]n^\mu_{\partial R} = \int_R \nabla^\mu J^X_\mu[\psi], \tag{1.4}$$

where $n_{\partial R}$ is oriented according to Figure 1.1.
A quick calculation shows that:

$$\nabla^\mu J^X_\mu[\psi] = \frac{1}{2}T_{\mu\nu}(\mathcal{L}_X g)^{\mu\nu} + (X\psi)\Box\psi. \tag{1.5}$$

So, if we choose X to be a Killing vector field ($\mathcal{L}_X g = 0$), and using equation (1.1), we find that $\nabla^\mu J^X_\mu[\psi] = 0$; therefore:

$$\int_{\partial R} J^X_\mu[\psi]n^\mu_{\partial R} = \int_{\Sigma_{\tau_1}} J^X_\mu[\psi]n^\mu_{\Sigma_{\tau_1}} - \int_{\Sigma_{\tau_2}} J^X_\mu[\psi]n^\mu_{\Sigma_{\tau_2}} = 0. \tag{1.6}$$

Finally, plugging in (1.3), we arrive at energy conservation:

$$\mathbb{E}^X[\psi](\tau) = \mathbb{E}^X[\psi](0), \quad \forall\tau > 0. \tag{1.7}$$

The idea is that, by carefully constructing our hypersurfaces, we can leverage this conservation of energy to obtain the desired boundedness and decay results. But to achieve that, we must somehow go from these energy estimates to pointwise estimates of ψ itself.
From the definition of $T_{\mu\nu}[\psi]$, it is easy to guess that the energy will take the form of an integral of terms containing $(\partial_i\psi)^2$. So, we will make use of the Sobolev inequality:

$$|\psi| \lesssim \|\psi\|_{H^k(\Omega)} := \sqrt{\int_\Omega \sum_{|\alpha|\leq k} (\partial^\alpha\psi)^2}, \tag{1.8}$$

where $\psi \in C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ is a regular domain, $k > n/2$, $|\alpha| := \sum_{i=1}^n \alpha_i$, and $\partial^\alpha := \partial_1^{\alpha_1}...\partial_n^{\alpha_n}$. In other words, we can bound $|\psi|$ using the norm of ψ on the Sobolev space $H^k(\Omega)$ (see [2] for more detail), which is essentially the usual integral norm (the L^2 norm) of the sum of every combination of up to k partial derivatives of ψ (which includes ψ itself, the zeroth order derivative).

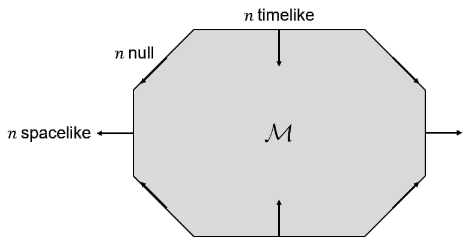

FIGURE 1.1: Normal vectors for the divergence theorem in Lorentzian manifolds [1].

2 Minkowski spacetime

In the Minkowski spacetime, the explicit solutions to (1.1) are widely known; using these, it is trivial to deduce boundedness and decay estimates. So, this section mainly serves as an opportunity to introduce and develop the methods in the simplest framework, thus setting the stage for Section 3.
We will use spherical coordinates, anticipating the treatment of the Schwarzschild spacetime; with these, the metric is given by

$$g = -dt^2 + dr^2 + r^2 d\sigma^2, \tag{2.1}$$

where $d\sigma^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the usual metric on $\mathbb{S}^2$. The volume element is $\sqrt{|\det g|}\bigwedge_i dx^i = r^2\, dt \wedge dr \wedge d\Omega$, with $d\Omega = \sin\theta\, d\theta \wedge d\varphi$.
After choosing $\Sigma_0 = \{t = 0\}$, we can set the initial conditions: $\psi\big|_{\Sigma_0} = \psi_0$ and $\partial_t\psi\big|_{\Sigma_0} = \psi_1$, where ψ_i are smooth, compactly supported functions. Notice that $\phi_\tau(\Sigma_0) = \{t = \tau\}$.
We select $T := \partial_t$ as the Killing field, which matches n_{Σ_τ}.

2.1 Boundedness

We begin by computing the energy:

$$\begin{aligned} \mathbb{E}^T[\psi](\tau) &= \int_{\Sigma_\tau} T_{\mu\nu}[\psi] X^\nu n^\mu_{\Sigma_\tau}\, d\Sigma_\tau = \int_{\Sigma_\tau} \left(\partial_\nu\psi\partial_\mu\psi - \frac{1}{2}g_{\mu\nu}(\partial^\alpha\psi\partial_\alpha\psi)\right)(\partial_t)^\nu(\partial_t)^\mu\, d\Sigma_\tau \\ &= \int_{\Sigma_\tau} \partial_t\psi\partial_t\psi - \frac{1}{2}g_{tt}(\partial^\alpha\psi\partial_\alpha\psi)\, d\Sigma_\tau = \frac{1}{2}\int_{\Sigma_\tau}\left((\partial_t\psi)^2 + (\partial_r\psi)^2 + \frac{1}{r^2}|\nabla\!\!\!/\psi|^2\right) r^2\, dr\, d\Omega, \end{aligned} \tag{2.2}$$

where $|\nabla\!\!\!/\psi|^2 = (\partial_\theta\psi)^2 + \frac{1}{\sin^2\theta}(\partial_\varphi\psi)^2$.
In order to bound the 0^{th} term in the Sobolev norm of ψ, we will now attempt to estimate ψ^2.

Fix $R>0$, and let $t_0>0$, $r_0\geq R$ and $\omega\in\mathbb{S}^2$. Using the Fundamental Theorem of Calculus and the Cauchy-Schwarz inequality:

$$\begin{aligned}\psi^2(t_0,r_0,\omega)&=\left(\int_{r_0}^{\infty}\partial_r\psi(t_0,r,\omega)\,dr\right)^2\leq\left(\int_{r_0}^{\infty}(\partial_r\psi(t_0,r,\omega))^2\,r^2\,dr\right)^2\int_{r_0}^{\infty}\frac{1}{r^2}\,dr\\&=\frac{1}{r_0}\int_{r_0}^{\infty}(\partial_r\psi(t_0,r,\omega))^2\,r^2\,dr\leq\frac{1}{R}\int_0^{\infty}(\partial_r\psi(t_0,r,\omega))^2\,r^2\,dr.\end{aligned}\tag{2.3}$$

By integrating on $\mathbb{S}^2$ and using (2.2), we can get an energy bound:

$$\int_{\mathbb{S}^2}\psi^2(t_0,r_0,\omega)\,d\Omega\leq\frac{1}{R}\int_{\Sigma_{t_0}}(\partial_r\psi(t_0,r,\omega))^2\,d\Sigma_{t_0}\lesssim_R\mathbb{E}^T[\psi](t_0).\tag{2.4}$$

To generate the remaining components of $\|\psi\|^2_{H^2(\mathbb{S}^2)}$, let $\Omega_{1,\ldots,3}$ be the three rotational Killing vector fields. Since they are Killing, $[\Omega_i,\Box]=0$ (see [3] for more information); and therefore $\Omega_i\psi$ and $\Omega_i\Omega_j\psi$ also satisfy the wave equation. So, substituting these in place of ψ in (2.4), and adding everything up:

$$\|\psi(t_0,r_0,\cdot)\|_{H^2(\mathbb{S}^2)}\lesssim_R\sqrt{\mathbb{E}^T[\psi](t_0)+\sum_i\mathbb{E}^T[\Omega_i\psi](t_0)+\sum_{i,j}\mathbb{E}^T[\Omega_i\Omega_j\psi](t_0)}:=E_0,\tag{2.5}$$

where we know that E_0 does not depend on t_0 due to conservation of energy (1.7). So, applying (1.8):

$$|\psi(t_0,r_0,\omega)|\lesssim_R E_0.\tag{2.6}$$

So, we have just proven boundedness on the spheres foliating space for $r_0\geq R$, at any time. From (2.4), it is clear that this procedure yields a useless bound for $R\to 0$, so we need a different approach for $r_0<R$. Starting with an integration by parts, where the boundary term at $r=\infty$ is zero because ψ is compactly supported, and using Young's inequality, which states that $|ab|\leq\frac{1}{2}\left((\epsilon a)^2+\left(\frac{b}{\epsilon}\right)^2\right)$:

$$\int_0^{\infty}\psi^2\,dr=\left[r\psi^2\right]_{r=0}^{r=\infty}-2\int_0^{\infty}r\psi\partial_r\psi\leq\frac{1}{2}\int_0^{\infty}\psi^2\,dr+2\int_0^{\infty}(\partial_r\psi)^2r^2\,dr.\tag{2.7}$$

Then, integrating on $\mathbb{S}^2$, we get:

$$\int_{\Sigma_\tau}\frac{\psi^2}{r^2}\,d\Sigma_\tau\leq 4\int_{\Sigma_\tau}(\partial_r\psi)^2\,d\Sigma_\tau\implies\int_{\Sigma_\tau\cap\{r<R\}}\psi^2\,d\Sigma_\tau\lesssim_R\mathbb{E}^T[\psi](\tau).\tag{2.8}$$

Finally, we repeat the procedure used in (2.5) — but now with the three translational Killing fields instead — to get the energy bound for $\|\psi\|_{H^2(\Sigma_\tau\cap\{r<R\})}$. And after invoking (1.7) and (1.8) again, we now have a global bound for $|\psi|$.

2.2 Decay

The boundedness result hinged on the fact that energy is conserved across the hypersurfaces Σ_τ. However, a constant energy will not be useful in proving decay. Intuitively, this decay takes place because the waves are radiating away to infinity along null directions; so, we should try to capture this when defining our hypersurfaces. The idea is to use the previous definition of Σ_τ for a region $\{r<R\}$, where R is such that the support of ψ at $t=0$ is contained herein, and then switch to a null surface N_τ.
We will start by introducing the null coordinates $(u,v)=(t-r,t+r)$, with $\partial_u=\frac{1}{2}(\partial_t-\partial_r)$ and $\partial_v=\frac{1}{2}(\partial_t+\partial_r)$. The metric becomes:

$$g=-du\,dv+r^2d\sigma^2.\tag{2.9}$$

We can now define our hypersurfaces as $\widetilde{\Sigma}_\tau = S_\tau \cup N_\tau$, with

$$S_\tau := \{t = \tau,\, r < R\}, \tag{2.10a}$$
$$N_\tau := \{u = \tau - R,\, r \geq R\}. \tag{2.10b}$$

Additionally, we define the region $R_{\tau_1}^{\tau_2} := \bigcup_{\tau\in[\tau_1,\tau_2]} \widetilde{\Sigma}_\tau$. On N_τ, we will use the volume element $r^2\, dv \wedge d\Omega$. These hypersurfaces can all be visualized in the Penrose diagram featured in Figure 2.1.

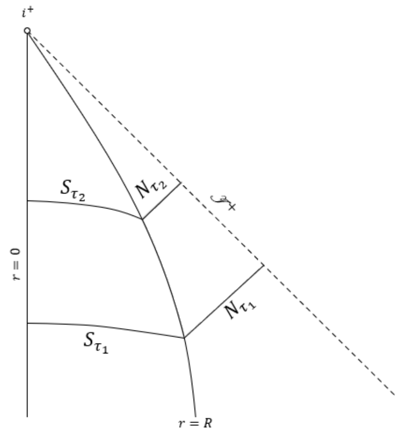

FIGURE 2.1: Hypersurfaces for the Minkowski decay estimate. Adapted from [1].

Computing the energy in this setup yields:

$$\widetilde{\mathbb{E}}^T[\psi](\tau) = \frac{1}{2}\int_{S_\tau}\left((\partial_t\psi)^2 + (\partial_r\psi)^2 + \frac{1}{r^2}|\nabla\!\!\!/\,\psi|^2\right) r^2\, dr\, d\Omega + \int_{N_\tau}\left((\partial_v\psi)^2 + \frac{1}{4r^2}|\nabla\!\!\!/\,\psi|^2\right) r^2\, dv\, d\Omega. \tag{2.11}$$

And indeed, after applying the divergence theorem in $R_{\tau_1}^{\tau_2}$, we see that it is no longer conserved:

$$\widetilde{\mathbb{E}}^T[\psi](\tau_2) = \widetilde{\mathbb{E}}^T[\psi](\tau_1) - \int_{\mathscr{I}^+\cap\{\tau_1 - R\leq u\leq \tau_2 - R\}}\left((\partial_u\psi)^2 + \frac{1}{4r^2}|\nabla\!\!\!/\,\psi|^2\right) r^2\, du\, d\Omega \leq \widetilde{\mathbb{E}}^T[\psi](\tau_1), \tag{2.12}$$

where the integral at $\mathscr{I}^+$ is defined in $\{v = v_0\}$ with $v_0 \to \infty$.
So, (2.12) implies that the energy is decaying; but crucially, we want to obtain a decay rate as well. The first step is the so-called Integrated Local Energy Decay estimate, pioneered by Morawetz in [4]:

$$\int_{\tau_1}^{\tau_2}\int_{S_t}\left((\partial_t\psi)^2 + (\partial_r\psi)^2 + \frac{1}{r^2}|\nabla\!\!\!/\,\psi|^2 + \psi^2\right) dS_t\, dt \lesssim_R \widetilde{\mathbb{E}}^T[\psi](\tau_1). \tag{2.13}$$

We obtain this by first applying the divergence theorem in the region $R_{\tau_1}^{\tau_2}$ with a modified current defined with $X = f(r)\partial_r$, whose divergence contains the terms in the integrand of (2.13). We then proceed by doing an energy bound for each of them individually, using (2.12) along with Young's inequality and other appropriate estimates.
With these ingredients, we can obtain the Dafermos-Rodnianski hierarchy, originally developed in [5]:

$$\int_{\tau_1}^{\tau_2} \widetilde{\mathbb{E}}^T[\psi](\tau)\, d\tau \lesssim \widetilde{\mathbb{E}}^T[\psi](\tau_1) + f_1(\tau_1), \tag{2.14a}$$
$$\int_{\tau_1}^{\tau_2} f_1(\tau)\, d\tau \lesssim \widetilde{\mathbb{E}}^T[\psi](\tau_1) + f_2(\tau_1), \tag{2.14b}$$

where $f_1(\tau) := \int_{N_\tau}(\partial_v\phi)^2 r\, dv\, d\Omega$ and $f_2(\tau) := \int_{N_\tau}(\partial_v\phi)^2 r^2\, dv\, d\Omega$, with $\phi := r\psi$.

Using (2.14a) and the fact that the energy is now a decreasing function, we immediately get a decay rate of $1/t$:

$$\widetilde{\mathbb{E}}^T[\psi](\tau) \leq \frac{1}{\tau}\int_0^\tau \widetilde{\mathbb{E}}^T[\psi](\tau')\,d\tau' \lesssim \frac{1}{\tau}\left(\widetilde{\mathbb{E}}^T[\psi](0) + f_1(0)\right) \lesssim \frac{1}{\tau}, \tag{2.15}$$

where $f_1(0) = 0 = f_2(0)$ because $\psi\big|_{N_0} = 0$, since the support for the initial data is contained in $\{r < R\}$. However, this can be improved. Let $\{\tau_n\}_{n\in\mathbb{N}}$ be a sequence with $\tau_n \in [2^n, 2^{n+1}[$ and such that

$$f_1(\tau_n) = \frac{1}{2^n}\int_{2^n}^{2^{n+1}} f_1(\tau)\,d\tau, \quad \forall n \in \mathbb{N}. \tag{2.16}$$

Note that this constraint generates a well-defined sequence due to the mean value theorem for integrals and the fact that (2.14b) implies that $\int_0^\infty f_1(\tau)\,d\tau \lesssim \widetilde{\mathbb{E}}^T[\psi](0) < \infty$.
So, using (2.14a) and $f_1(\tau_n) \lesssim 2^{-n}\widetilde{\mathbb{E}}^T[\psi](0) \lesssim \tau_n^{-1}$, we get an energy decay rate of $1/\tau^2$:

$$\widetilde{\mathbb{E}}^T[\psi](\tau_{n+2}) \leq \frac{\int_{\tau_n}^{\tau_{n+2}} \widetilde{\mathbb{E}}^T[\psi](\tau)\,dt}{\tau_{n+2} - \tau_n} \lesssim \frac{\widetilde{\mathbb{E}}^T[\psi](\tau_n) + f_1(\tau_n)}{2^{n+2} - 2^{n+1}} \lesssim \frac{\tau_n^{-1}}{2^n} \lesssim \frac{1}{\tau_{n+2}^2}, \quad \forall n \in \mathbb{N}. \tag{2.17}$$

Finally, for a pointwise bound, we must again split our analysis into two cases, according to the partition of $\widetilde{\Sigma}_\tau$ made in (2.10). For the region $\{r < R\}$, we can use methods similar to (2.3) and (2.4) to obtain

$$\int_{S_\tau} \psi^2\,dS_\tau \lesssim_R \widetilde{\mathbb{E}}^T[\psi](\tau) \lesssim \frac{1}{\tau^2}, \tag{2.18}$$

and then we can apply the translational Killing fields strategy to get $\|\psi\|^2_{H^2(S_\tau)} \lesssim_R \frac{1}{\tau^2}$. Lastly, we conclude $|\psi\big|_{S_\tau}| \lesssim_R \frac{1}{\tau}$ through the Sobolev inequality. For the region $\{r \geq R\}$, we similarly obtain

$$\int_{\mathbb{S}^2} \psi(\tau + r - R, r, \omega)^2\,d\Omega \lesssim_R \widetilde{\mathbb{E}}^T[\psi](\tau) \lesssim \frac{1}{\tau^2}, \tag{2.19}$$

and then arrive at $|\psi\big|_{N_\tau}| \lesssim_R \frac{1}{\tau}$ by using the rotational Killing fields and (1.8). So we end up with:

$$|\psi|_{\widetilde{\Sigma}_\tau}| \lesssim \frac{1}{\tau}. \tag{2.20}$$

3 Schwarzschild spacetime

The Schwarzschild spacetime is the simplest black hole solution to Einstein's equations; it represents a non-rotating, chargeless black hole with mass m. Using spherical coordinates, the metric is:

$$g = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma^2. \tag{3.1}$$

Notice that g has a singularity at $r = 2m$, the event horizon ($\mathscr{H}^+$). However, it turns out that we can remove it by simply choosing different coordinates. For example, by using the Lemaître coordinates (t^*, r) where $t^* = t + 2m\ln|r - 2m|$:

$$g = -\left(1 - \frac{2m}{r}\right)dt^{*2} + \frac{4m}{r}dt^* dr + \left(1 + \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma^2. \tag{3.2}$$

Notice that, although t^* is the redefined coordinate and r was kept the same, the basis vectors behave in the opposite manner: $(\partial_{t^*})_r = \partial_t$ and $Y := (\partial_r)_{t^*} = -\frac{2m/r}{1-2m/r}\partial_t + \partial_r$.

Another useful choice is the Eddington-Finkelstein coordinates (v, r), with $v = t + r + 2m \ln |r - 2m|$:

$$g = -\left(1 - \frac{2m}{r}\right) dv^2 + 2\, dv\, dr + r^2 d\sigma^2. \tag{3.3}$$

They are also singularity-free, and the relevant basis vectors are now $(\partial_v)_r = \partial_t$ and $Z := (\partial_r)_v = Y - \partial_t$. Finally, by similarly defining $u = t - r - 2m \ln |r - 2m|$, we get the new null coordinates (u, v):

$$g = -\left(1 - \frac{2m}{r}\right) du\, dv + r^2 d\sigma^2. \tag{3.4}$$

For the hypersurface setup (which we will use for both boundedness and decay), the idea is again for $\widetilde{\Sigma}_\tau$ to have a region of constant time S_τ, and then a null region N_τ along which the waves travel towards $\mathscr{I}^+$.

However, due to the singularity in (3.1), the hypersurfaces of constant time accumulate at $r = 2m$, and thus never cross $\mathscr{H}^+$. So, to capture the fact that the waves can fall into the black hole, we will use hypersurfaces L_τ which meet $\mathscr{H}^+$ transversally. We end up with $\widetilde{\Sigma}_\tau = L_\tau \cup S_\tau \cup N_\tau$ (and again $R_{\tau_1}^{\tau_2} := \bigcup_{\tau \in [\tau_1, \tau_2]} \widetilde{\Sigma}_\tau$), where

$$L_\tau := \{t^* = \tau + 2m \ln(r_0 - 2m),\, 2m \le r < r_0\}, \tag{3.5a}$$
$$S_\tau := \{t = \tau,\, r_0 \le r \le R\}, \tag{3.5b}$$
$$N_\tau := \{u = \tau - R - 2m \ln(R - 2m),\, r > R\}, \tag{3.5c}$$

and r_0, R are chosen such that the initial data is supported in S_0. These can be seen in Figure 3.1.

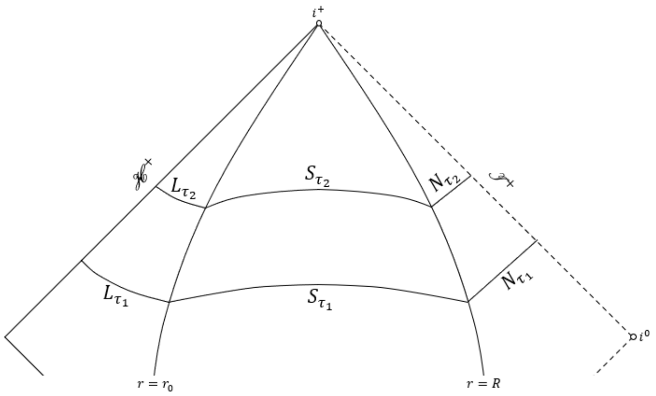

FIGURE 3.1: Hypersurfaces for the Schwarzschild spacetime. Adapted from [1].

3.1 Boundedness

Given the new setup, we will now deduce boundedness through energy decay rather than conservation. Calculating the energy yields (omitting all constants and factors of m, r_0, and R):

$$\begin{aligned} \widetilde{\mathbb{E}}^T[\psi](\tau) \sim &\int_{L_\tau} \left((\partial_t \psi)^2 + \left(1 - \frac{2m}{r}\right) (Y\psi)^2 + \frac{1}{r^2} |\nabla\!\!\!/ \psi|^2 \right) r^2\, dr\, d\Omega \\ &+ \int_{S_\tau} \left((\partial_t \psi)^2 + (\partial_r \psi)^2 + \frac{1}{r^2} |\nabla\!\!\!/ \psi|^2 \right) r^2\, dr\, d\Omega \\ &+ \int_{N_\tau} \left((\partial_v \psi)^2 + \frac{1}{r^2} |\nabla\!\!\!/ \psi|^2 \right) r^2\, dr\, d\Omega. \end{aligned} \tag{3.6}$$

And after applying the divergence theorem in $R_{\tau_1}^{\tau_2}$, we get a result similar to (2.12), but this time with an extra boundary term at $\mathscr{H}^+$. However, just like the term at $\mathscr{I}^+$, the new boundary term is also negative, so we still arrive at energy decay.
Unfortunately, there is a degeneracy present in (3.6) due to the factor $(1-2m/r)$. This is problematic because, by losing control over the $(Y\psi)^2$ term near $\mathscr{H}^+$, we can no longer estimate every term to produce the Schwarzschild version of (2.13).
It turns out that we can get around this issue by once again working with a modified current $J_\mu^N[\psi]$, where

$$N := \left(1 + \frac{5}{m}(r-2m)\right)\partial_t + \left(-3 - \frac{1}{m}(r-2m)\right)Z \tag{3.7}$$

is the so-called redshift vector field [6]. Crucially, we arrive at this definition by requiring N to satisfy

$$J_\mu^N[\psi]n_{L_\tau}^\mu \sim (\partial_t\psi)^2 + (Y\psi)^2 + \frac{1}{r^2}|\nabla\!\!\!/\psi|^2 \tag{3.8}$$

near $\mathscr{H}^+$, since this is precisely what we want for the integrand in L_τ in (3.6). Note: to simplify the calculations, we would like to retain the usual ∂_t normal in the remaining regions. So, we actually use a new vector field $\widetilde{N}$, which smoothly transitions from N to ∂_t at the end of L_τ, using a simple interpolation:

$$\widetilde{N} = \gamma(r)N + (1-\gamma(r))\partial_t, \tag{3.9}$$

where γ is a smooth function satisfying $\gamma(r) = 1$ for $r \leq 0.9r_0$ and $\gamma(r) = 0$ for $r \geq r_0$.
Calculating the new energy, we get:

$$\begin{aligned}\widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau) \sim &\int_{L_\tau}\left((\partial_t\psi)^2 + (Y\psi)^2 + \frac{1}{r^2}|\nabla\!\!\!/\psi|^2\right) r^2\,dr\,d\Omega \\ &+ \int_{S_\tau}\left((\partial_t\psi)^2 + (\partial_r\psi)^2 + \frac{1}{r^2}|\nabla\!\!\!/\psi|^2\right) r^2\,dr\,d\Omega \\ &+ \int_{N_\tau}\left((\partial_v\psi)^2 + \frac{1}{r^2}|\nabla\!\!\!/\psi|^2\right) r^2\,dr\,d\Omega.\end{aligned} \tag{3.10}$$

Crucially, comparing with (3.6), there is no degenerate factor affecting $(Y\psi)^2$.
Finally, re-applying the divergence theorem in $R_{\tau_1}^{\tau_2}$ with this modified energy (while being especially careful with the terms in the interpolation region) yields the non-degenerate version of energy decay:

$$\widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau_2) \lesssim \widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau_1) \tag{3.11}$$

We can now use this result along with the procedure in (2.3) to get:

$$\int_{\mathbb{S}^2}\psi^2(\tau_1, r_1, \omega)\,d\Omega \leq \frac{1}{2m}\int_{\mathbb{S}^2}\int_{r_1}^\infty (\partial_\rho\psi(\tau_1, r, \omega))^2 r^2\,dr\,d\Omega \lesssim \widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau_1) \lesssim \widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](0), \tag{3.12}$$

with $r_1 \geq 2m$, and where ∂_ρ is Y for $r < r_0$, ∂_ρ is ∂_r for $r_0 \leq r \leq R$, and ∂_ρ is $(\partial_r)_u$ for $r > R$.
We then apply the rotational Killing fields method and the Sobolev inequality to conclude that $|\psi|$ is bounded in the region $\{r \geq 2m\}$.

3.2 Decay

To obtain a decay rate, we begin again by deducing an Integrated Local Energy Decay estimate using methods similar to the ones employed to obtain (2.13):

$$\int_{R_{\tau_1}^{\tau_2}} \frac{1}{r}\left(\left(1-\frac{2m}{r}\right)^2(\partial_r\psi)^2+\left(1-\frac{3m}{r}\right)^2\left(\frac{1}{r^\delta}(\partial_t\psi)^2+|\nabla\!\!\!/\psi|^2\right)+\frac{\psi^2}{r}\right)dr\,d\Omega\,dt \lesssim \widetilde{\mathbb{E}}^T[\psi](\tau_1), \quad (3.13)$$

where $\delta > 0$.
Notice that we now have a degeneracy both at $\mathscr{H}^+$ and at $r = 3m$, the photon sphere; the latter happens because photons can get trapped there, orbiting the black hole in a neighborhood of this region for an arbitrarily long time, thus preventing energy decay. So, this degeneracy stems from a real physical phenomenon rather than a coordinate artifact, and is therefore unavoidable.
On the other hand, the degeneracy at $\mathscr{H}^+$ can be removed using $\widetilde{N}$ again, resulting in:

$$\int_{R_{\tau_1}^{\tau_2}} \frac{1}{r}\left((\partial_\rho\psi)^2+\left(1-\frac{3m}{r}\right)^2\left(\frac{1}{r^\delta}(\partial_t\psi)^2+|\nabla\!\!\!/\psi|^2\right)+\frac{\psi^2}{r}\right)dr\,d\Omega\,dt \lesssim \widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau_1). \quad (3.14)$$

Even though we cannot avoid the photon sphere degeneracy, we can try to make use of symmetries in the Schwarzschild spacetime to get an improved result; specifically, since the spacetime is static and spherically symmetric, we know that (3.14) is also valid for $\partial_t\psi$ and $\Omega_i\psi$. Therefore, we get:

$$\int_{R_{\tau_1}^{\tau_2}} \frac{1}{r^2}\left((\partial_t\psi)^2+(\partial_\rho\psi)^2+|\nabla\!\!\!/\psi|^2\right)dr\,d\Omega\,dt \lesssim \widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau_1)+\widetilde{\mathbb{E}}^{\widetilde{N}}[\partial_t\psi](\tau_1)+\sum_i \widetilde{\mathbb{E}}^{\widetilde{N}}[\Omega_i\psi](\tau_1), \quad (3.15)$$

where we used the fact that $\sum_i(\Omega_i\psi)^2 = |\nabla\!\!\!/\psi|^2$.
There is seemingly a tradeoff here, in the sense that we exchanged a degeneracy for a weaker estimate, due to the extra energy terms on the right-hand side; but, unsurprisingly, they behave exactly like $\widetilde{\mathbb{E}}^{\widetilde{N}}[\psi]$.
We can now obtain the Dafermos-Rodnianski hierarchy for Schwarzschild (using $f_i(\tau)$ from (2.14)):

$$\int_{\tau_1}^{\tau_2} \widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau)\,d\tau \lesssim \widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau_1)+\widetilde{\mathbb{E}}^{\widetilde{N}}[\partial_t\psi](\tau_1)+\sum_i \widetilde{\mathbb{E}}^{\widetilde{N}}[\Omega_i\psi](\tau_1)+f_1(\tau_1), \quad (3.16a)$$

$$\int_{\tau_1}^{\tau_2} f_1(\tau)\,d\tau \lesssim \widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau_1)+f_2(\tau_1), \quad (3.16b)$$

Comparing with (2.13), we have the extra energy terms from (3.15), as expected. They are also present in the counterpart to (2.15), but that does not prevent us from arriving at the $1/t$ decay rate:

$$\widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau) \le \frac{1}{\tau}\int_0^\tau \widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau')\,d\tau' \lesssim \frac{1}{\tau}\left(\widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](0)+\widetilde{\mathbb{E}}^{\widetilde{N}}[\partial_t\psi](0)+\sum_i \widetilde{\mathbb{E}}^{\widetilde{N}}[\Omega_i\psi](0)\right) \lesssim \frac{1}{\tau}, \quad (3.17)$$

Then, defining the sequence τ_n analogously to (2.16), and using the fact that the extra energies also satisfy (3.17), we get the new version of (2.17):

$$\widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau_{n+2}) \lesssim \frac{\widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau_n)+\widetilde{\mathbb{E}}^{\widetilde{N}}[\partial_t\psi](\tau_n)+\sum_i \widetilde{\mathbb{E}}^{\widetilde{N}}[\Omega_i\psi](\tau_n)+f_1(\tau_n)}{2^{n+2}-2^{n+1}} \lesssim \frac{1}{\tau_{n+2}^2}, \quad \forall n\in\mathbb{N}, \quad (3.18)$$

Finally, to obtain pointwise estimates, we mimic the procedure used in Section 3.1, except that in (3.12) we now use the fact that $\widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](\tau)$ is bounded by $1/\tau^2$ rather than by $\widetilde{\mathbb{E}}^{\widetilde{N}}[\psi](0)$.
In the end, we obtain (2.20) once again.

4 Conclusions

We were able to prove that the solutions to the wave equation are bounded and moreover decay with a rate of $1/t$ in the Minkowski and external Schwarzschild spacetimes, as long as the initial wave is compactly supported. Intuitively, one could argue that boundedness is a consequence of the conservation of energy, and decay is caused by the fact that the energy contained in any given spatial region is decreasing as the waves radiate away to infinity. After formalizing these physically intuitive concepts and using them to motivate our analysis, we obtained a very powerful and robust estimation framework. Indeed, more so than merely obtaining these well-known boundedness and decay results, we demonstrated the promising applicability of energy methods. However, we also made use of certain symmetries, so it is expected that less symmetric spacetimes will require a more elaborate approach. Even in the relatively simple Schwarzschild case, we encountered issues due to degeneracies at the event horizon and at the photon sphere. Nevertheless, the additional effort and complexity introduced while handling these issues provided some useful physical insight.

A promising avenue of future development would be to adapt these methods to the Kerr spacetime, which describes a rotating black hole. Here, we can no longer rely on spherical symmetry, and the photon-trapping phenomenon is much more complex.

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